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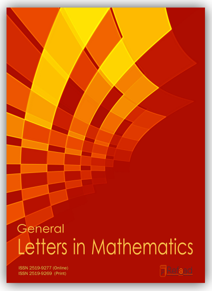

Regularized Minimization of Convex Functions in Radon Nikodym Space

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Abstract. Convex non linear optimization problems may not have a solution in infinite dimension spaces. The aim of this paper is to formulate some new results in this topic by using "technical regularization" of the objective function. The first result shows that a non linear convex proper lower semi-continuous function, on a Banach space which have the Radon-Nikodym property, could be minimized by using a small regularization. while the second one shows that this regularization can be chosen as small as required. In addition, application tracks are presented and illustrated by elementary examples.

Keywords: *Convex functions, strongly exposed point, strong minimum, regularized minimum.*
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1 Introduction

Convex non-linear optimization problems are at the core of convex and non-linear analysis. This subject has always been the subject of several developments with a large number of variations concerning objective functions, working spaces or hypothesis. Many convex optimization problems have no solutions in infinite dimensional Banach spaces. On the other hand, there is a very rich literature dealing, in the Banach spaces, with the geometric properties of convex sets and the relationships between convexity, differentiability and optimization. A large number of renowned mathematicians have contributed to the development of research in this area and around these themes. We can mention, among others, Asplund, Collier, Godefroy, Pearce, Bourgin, Bourgain, Namioka, Phelps, Castin, Fabien, Maynard, Lee, ..., see for instance: [3], [4], [6], [7], [8], [9], [12], [15], [16], [18], [20], [23] And the references cited therein.

Furthermore, the differentiability (Fréchet and Gâteau) of the convex functions has been of great interest. In particular, the relation of duality, between the differentiability and geometric properties in Asplund and Radon Nikodym spaces, has been highlighted in several works, for more details refer to [3] and [16].

In this work, we use some of these characterizations, some technical calculations and the results of our previous work, to show that the convex functions, defined on a Banach space having the property of Radon-Nikodym, which have no minimum on their domains, can be disturbed in order to have a (regularized) minimum in a well-defined

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area. The key idea is to regularize the convex function by adding a linear function, the new disturbed function will have a minimum. However, authors can not generalize the result to all Banach spaces (For example, the dual of l^2 is a Banach space without the property of Radon-Nikodym, see [16] for more details). Therefore, in infinite dimensional spaces, the Radon Nikodym property is a necessary condition to use the technical methods of this work and to obtain the main result.

2 Definitions, Preliminaries and Main Results

Le X be a Banach space, X' its dual and $\langle .,. \rangle$ the dual product. For each function f defined on X with values in $]-\infty,+\infty]$, we define:

$$dom f = \{x \in X \quad \text{such that} \quad f(x) < +\infty\}$$

$$epi f = \{(x,r) \in X \times \mathbb{R} \quad \text{such that} \quad f(x) \leq r \text{ and } x \in dom f\}$$

• A function f is said proper if it has some finite values. The conjugate of f, noted f^*, is defined on X' with values in $]-\infty,+\infty]$ such that:

$$f^*(x') = \sup_{x \in X}\{\langle x'.x \rangle - f(x)\}$$

Note that this conjugate function is a proper lower semi continuous and convex by respect to the $\sigma(X',X)$ topology.

• Let C be a nonempty convex subset of X, the indicator function of C is defined by:

$$\delta_C(x) = 0 \text{ if } x \in C \qquad \text{and} \qquad \delta_C(x) = +\infty \text{ if not}$$

The conjugate of $\delta_C(.)$ is given by:

$$\delta_C^*(x') = \sup_{x' \in C}\{\langle x'.x \rangle\}$$

• A function f is said to have a strong minimum at a if:

i) $f(x) \leq f(a)$ for every $x \in X$;

ii) $f(x_n) \to f(a)$ for every sequence $(x_n)_n$ such that $x_n \to a$.

• A function f has a regularised minimum, by a linear functional $a' \in X'$, if and only if $f - a'$ has a strong minimum in X.

• An element $e \in C$ is said to be a strongly exposed point (by $x' \in X'$) if and only:

i) $\delta_C^*(x') = \sup_{x' \in C}\{\langle x'.x \rangle\} = \langle x', e \rangle$;

ii) $\langle x', x_n \rangle \to \langle x', e \rangle$ in $\mathbb{R}$ implies that $x_n \to e$ in the norm topology of X.

Recall the following important results which are useful to check out our main results.

(See [8], [16]).

If X is a space with the Radon Nikodym property then each closed bounded convex subset is the convex hall of its strongly exposed points.

(See [6]).

Let f be a proper lower semicontinuous function, X is a space with the Radon Nikodym property and $x_0 \in dom f$ then $(x_0, f(x_0)) \in epi f$ is strongly exposed in $X' \times \mathbb{R}$, by $(x', -1)$, if and only if $f - x'$ has a strong minimum in X reached at x_0.

Using previous definitions, preliminaries and the above theorems, we are able to formulate main results of this work. Note that these results are in the light of some results in [6] and [12].

In a Radon Nikodym space X, every proper convex lower semicontinuous functions has a strong minimum with a small regularization given by $a' \in X'$ and, the set of all linear functions regularizing f is dense in X'

Proof. For the first statement, suppose that X has the Radon Nikodym property and let f be a non linear convex lower semicontinuous function. Let $C \subset Domf$ be a closed bounded set in the interior of $Domf$. Introduce the function g such that $g(x) = f(x)$ if $x \in C$ and $g(x) = +\infty$ otherwise. Notice that g is a proper bounded lower semicontinuous convex function on C. Without loss of generality, we can assume that there exists a $M > 0$ such that $-M \leq g(x) \leq M$ for every $x \in C$. g is (X, X') lower semicontinuous on X than the set B given by:

$$B = \{x \in X \quad \text{such that } -M \leq g(x) \leq M\}$$

is closed bounded and non empty in X.
For another hand, let H and N be the sets given by:

$$H = \{(x, r) \in X \times \mathbb{R} \quad \text{such that} \quad r \leq M\}$$

$$N = H \cap epig = H \cap \{(x, r) \in X \times \mathbb{R} \quad \text{such that} \quad g(x) \leq r\}$$

Remark that $N \subset B \times [-M, M]$ is a closed bounded convex subset in $X \times \mathbb{R}$. Recall that X has the Radon Nikodym property and using result in [4], we can deduce that $X \times \mathbb{R}$ verify the same property and N is the convex hull of its $\sigma(X, X')$-strongly exposed points. Thus, there exists a point a in the interior of C such that $g(a) \leq M$ and $(a, g(a))$ is strongly exposed, by $(a', -1)$, in N. But $g(a) \leq M$ which alow us to conclude that $(a, g(a))$ is also strongly exposed in $epig$ (see [8]).
By using theorem 2, we conclude that $g - a'$ admits a strong minimum reached at a and therefore f admits a strong minimum regularized by the linear function $a' \in X'$. Finally, as proved by Huff and Moris in [12] the set of all functionals strongly exposing N is dense in X', which implies that the set of all functionals regularizing the minimization of f is also dense in X'.

□

The second main result shows that regularizing linear function can be chosen as small as we need (for instance, we can find a' such that $||a'|| \leq \varepsilon$).

Let f be a proper lower semi continuos convex function in a Radon Nikodym Banach space X, then for every $\varepsilon \leq 1$ we can find a regularizing function $a' \in X'$ such that $||a'|| \leq \varepsilon$.

Proof. Let $\varepsilon \leq 1$, as the set S of all functions exposing N is dense in X' (see [12]), we have:

$$B_{X'}(0, \frac{\varepsilon}{2}) \subset X' = \overline{S}$$

Where $\overline{S}$ is the adherence of S. Now, let $b' \in B$ then there exists a $a' \in S$ such that

$$||b' - a'|| \leq \frac{\varepsilon}{2}$$

Recall that $a' \in S$ is a regularizing function of f and, the last inequality implies that:

$$||a'|| \leq ||y'|| + \frac{\varepsilon}{2} = \varepsilon \leq 1$$

This ends the proof and show that we can find a regularizing functional of f with smaller norm as we like. □

3 Application Opportunities

Optimization deal with the problem of identifying extreme values (maxima or minima) of a function on a given domain. Usually, engineering problems are solved using combinatorial optimization, with a finite set of possible solutions. However, the complexity prohibitively increases, and approximation algorithms have to be developed. This issues occur for instance in telecommunications [17] and [19], IT security [24], and data analysis [5] problems. The classical approach to tackle these problems is discretization of the search domain [1] and [22]. However, the problem becomes a non-deterministic polynomial time hard (NP hard) that cannot be solved in polynomial time, a heuristic approach/algorithm (i.e. Bat algorithm, Cuckoo search algorithm, Swarm algorithm or Genetic algorithm) with a solvable discretization has to be adopted, which, may result in inaccurate results.

An alternative approach is to formulate the problem as a continuous optimization problem. However, the complexity of the problem depends on the characteristics of the objective function, and the shape of the search domain [21]. Often authors resort to a min-type function, which prevents the problem from being convex or smooth. Moreover, global optimization might be solved using envelope representation of the objective function. The crux of the latter approach is the assumption that there is a set H of elementary functions h such as the objective function f is the upper envelope of a subset U of this set (i.e. H), that is

$$f(x) = \sup_{h \in H}\{h(x)\} \qquad \text{for every } x \in X \tag{1}$$

a function f adhering to (1) defined on X is an abstract convex with respect to H or it could also be called a H-convex function. For more details about abstract activity and algorithms based on it refer to [2], [13] and [14]. The proposed theorem in this article, tackle optimization problems with nonlinear and not necessary has a minimum objective functions. The perturbation of the objective function using smaller linear functions, transform the problem to a solvable one as discussed above.

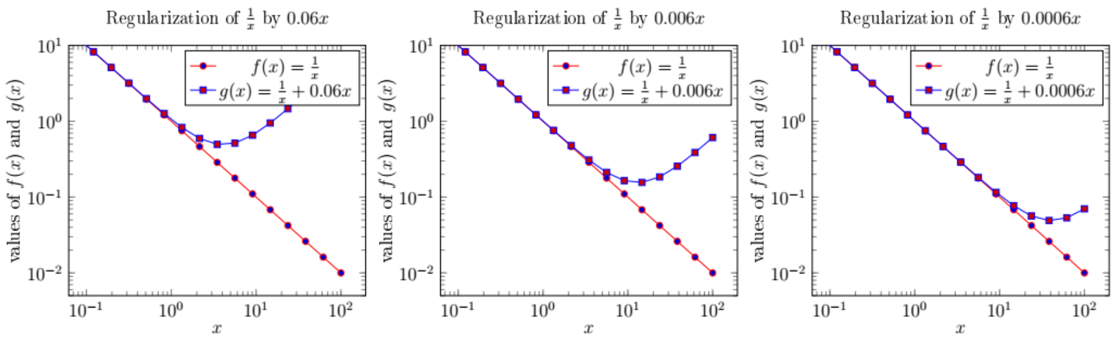
To illustrate, with elementary examples, the use mains results in optimization problems, consider the following situation: let $X = \mathbb{R}$ and:

$$f(x) = \frac{1}{x}, \quad \text{if } x \in]0, +\infty] \qquad \text{and} \qquad f(x) = +\infty \quad \text{if not}$$

Notice that f has no minimum. However, f verifies to the conditions of theorem 2, there exists multiple linear functions g, such that the function $h(x) = f(x) + g(x)$ has a minimum (perturbed minimum). The linear function g takes the form $g(x) = ax + c$ such that $\|g(x)\| \leq \epsilon$ which imply that:

$$h(x) = \frac{1}{x} + ax + c$$

The second main result shows that there exists an infinite set of linear functions, that perturb the original objective function in order to reach a perturbed minimum as shown in the following illustrations.



The optimization problem is reduced to selecting the best perturbation function. However, by limiting the solutions space $D \subset X$, the solution becomes obvious and the minimum is in the following set:

$$S = \{x_i \in D / \qquad \|h(x_i) - f(x_i)\| \leq \varepsilon\}$$

A second illustration can be shown by the following problem:

$$\text{minimize } f(x, y) = e^{(x+y)}, \quad where \quad (x, y) \in \mathbb{R}^2$$

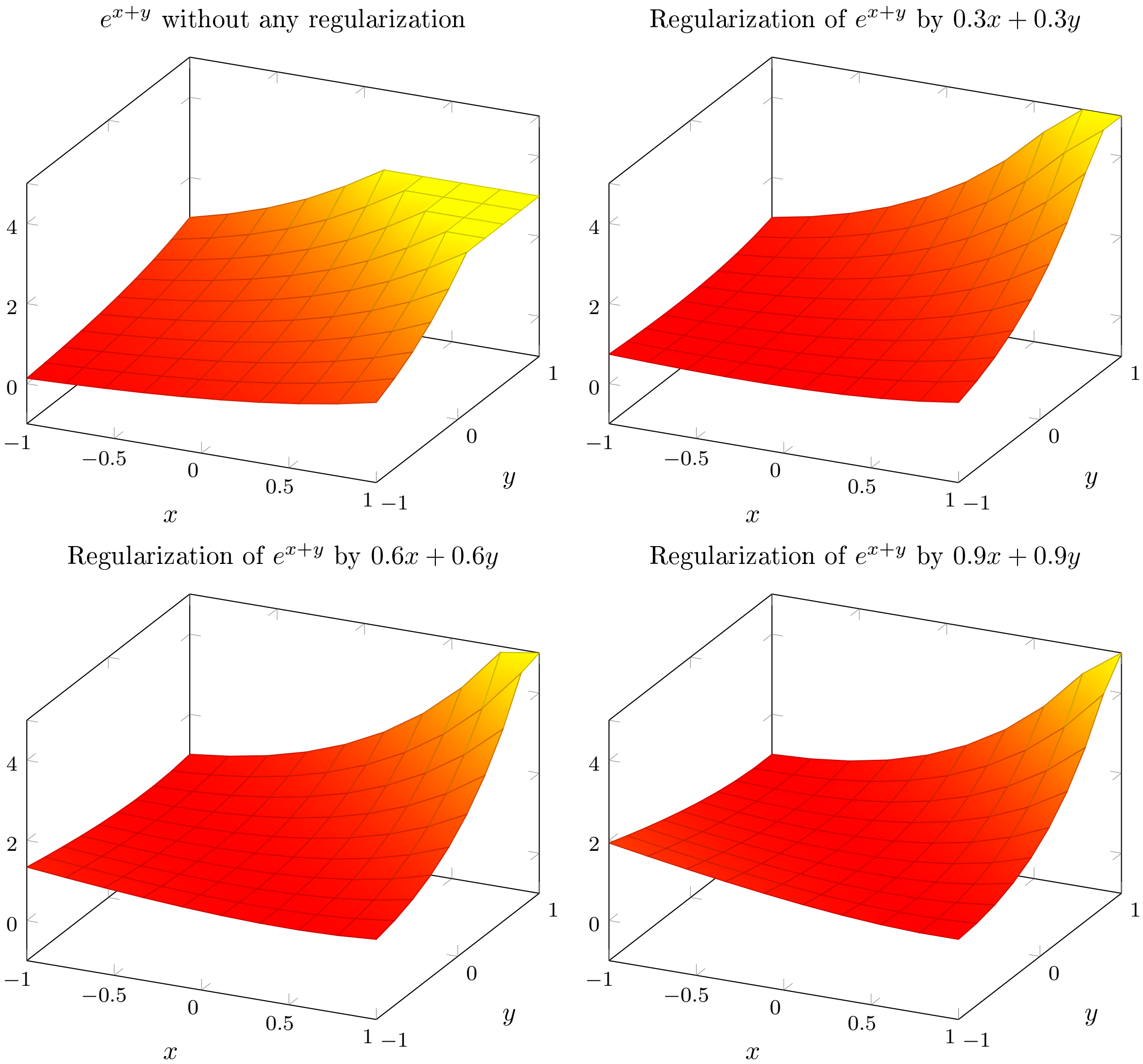
Assuming that $X = \mathbb{R}^2$, note that f has no minimum. However, f verifies conditions of theorem 2, there exist bilinear functions g, such as the function $h(x, y) = f(x, y) + g(x, y)$ has a minimum (perturbed minimum). The bilinear function can be written: $g(x, y) = ax + by$ such that

$$\|g(x, y)\| \leq \varepsilon$$

Without loss of generality, we can take $a = b$ and h is given by:

$$h(x, y) = e^{(x+y)} + ax + ay$$

Using the second main result, the exists an infinite set of linear functions, that perturb the original objective function in order to reach a perturbed minimum as illustrate in the following:



The optimization problem is reduced to selecting the best perturbation function. However, by limiting the solutions space $D \subset X$, the solution becomes obvious and the minimum is in the following set:

$$S = \{(x_i, y_i) \in D / \qquad \|h(x_i, y_i) - f(x_i, y_i)\| \leq \varepsilon\}$$

The examples above have just an illustrative role in very simple and elementary cases. they show the possible applications of main results even if the assumptions are very restrictive, in particular the finite dimension. The authors, which are from several specialties, are studying more important applications.

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