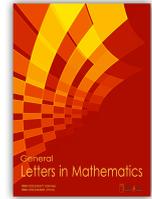




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Multifractal dimensions of vector-valued non-Gibbs measures

Bilel Selmi^{a,*}

^aAnalysis, Probability & Fractals Laboratory: LR18ES17
University of Monastir, Faculty of Sciences of Monastir
Department of Mathematics, 5000-Monastir, Tunisia.

Abstract

In the present work, we are concerned with some multifractal dimensions estimations of vector-valued measures in the framework of the so-called mixed multifractal analysis. We precisely consider some Borel probability measures that are no longer Gibbs and introduce some mixed multifractal generalizations of Hausdorff and packing dimensions of measures in a framework of relative mixed multifractal analysis. As an application, we are interested in the φ -unidimensionality of those measures and to the calculus of its mixed multifractal Hausdorff and packing dimensions. In particular, we give a necessary and sufficient condition for the existence of the φ -mixed multifractal Hausdorff and packing dimensions of a Borel probability measure. Finally, concrete examples satisfying the above property are developed.

Keywords: Hausdorff and packing measures, Hausdorff and packing dimensions, Multifractal formalism, Mixed cases.
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1. Introduction

Recently, multifractal analysis has taken an enormous interest in the mathematical literature. Many authors were interested in such analysis and its applications especially in financial time series, econophysical data, where the most used models are based on the past and for a long period are linear and multi-linear models. These models have shown some inefficiencies especially by the discovery and the inclusion of stochastic, chaotic and fractal factors in the mathematical models. This was one strong cause and motivation that have led researchers to develop more sophisticated approaches. Multifractal analysis has appeared firstly and has shown some success in overcoming many problems. However, in some others, the efforts have to be more and more developed especially for simultaneous time series behavior. One simple example is the financial crisis that appear. Such a crisis like the recent American one did not affect only the local national market but was spread and thus affected the worldwide markets, leading thus to a simultaneous or a worldwide crisis.

The multifractal analysis of a single measure passes through its local dimension or its Hölder exponent. For a measure μ eventually Borel and finite on \mathbb{R}^d and $x \in \mu$, the local dimension of μ at the point x is defined by

$$\alpha_{\mu}(x) = \lim_{r \downarrow 0} \frac{\log(\mu(B(x, r)))}{\log r}$$

*Corresponding author

Email address: bilel.selmi@fsm.rnu.tn (Bilel Selmi)

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when such a limit exists. The next step concerns the geometric study of the α -singularity set of the measure μ defined by

$$X_\mu(\alpha) = \left\{ x \in \mu \mid \lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}.$$

by means of its Hausdorff dimension

$$f_\mu(\alpha) = \dim_H X_\mu(\alpha)$$

which defines the so-called spectrum of singularities. This means that the study of the behavior of the measure is transformed into a study of sets where the focuses may somehow forget about the measure and its point-wise character and falls in set theory and the suitable coverings that permits the computation of the Hausdorff dimension. Multifractal analysis, as in the case of single one, studies both measures and functions (signals, time series, images, ...). Indeed, some geometric sets are essentially known by means of measures that are supported on, i.e., given a set X and a measure μ , the quantity $\mu(X)$ may be computed as the maximum value $\mu(F)$ for all subsets $F \subset X$. This means that, we somehow forget the geometric structure of X and focus, instead, on the properties of the measure μ . The set X is thus partitioned into α -level sets $X_\mu(\alpha)$ relatively to the regularity exponent of μ .

This makes the including of the measure μ into the computation of the fractal dimension and thus into the definition of the Hausdorff measure a necessity to understand more the geometry of the set simultaneously with the behavior of the measure that is supported on. One step ahead in this direction has been conducted by Olsen in [38] where introduced multifractal generalizations of the fractal dimensions such as Hausdorff, packing and Bouligand ones by considering general variants of measures. In [38], the characteristics of these functions such as monotony, convexity, lower and upper bounds have been studied. Next, to come back to the essential problem in multifractal formalism which consists of the computation of the spectrum of singularities $f_\mu(\alpha)$, the author proved that such generalizations may lead to a multifractal formalism but when some restriction for the single measure μ is assumed to hold. By assuming that μ belongs to the whole class of Gibbs-like measures the multifractal formalism has been proved to hold. Ben Nasr et al. in [7, 12, 13, 14] improved Olsen's result and proposed a new sufficient condition that gives the lower bound. For more details and backgrounds on multifractal analysis as well as their applications the readers may be referred also to the following essential references [1, 2, 3, 4, 17, 19, 20, 21, 22, 23, 40, 41, 43, 46, 47, 51, 52, 53, 54, 55]. Later, the lower and upper multifractal Hausdorff dimensions of a measure ν were studied by M. Dai in [16]. A similar approach, concerning the lower and upper multifractal packing dimensions, was developed by Li and Selmi in [34, 44, 45]. They proved, when the upper multifractal Hausdorff (resp. packing) dimension of the measure is small, it means that the measure ν is *very singular* with respect to the generalized multifractal Hausdorff (resp. packing) measure. In the same way, when the lower multifractal (resp. packing) dimension of the measure is large, then the measure ν is *quite regular* with respect to the generalized multifractal Hausdorff (resp. packing) measure. Note that, in those works the authors proved that the lower and upper multifractal Hausdorff and packing dimensions are related to the asymptotic behavior of the function

$$\alpha_{\mu, \nu}^q(x, r) = \frac{\log \nu(B(x, r)) - q \log \mu(B(x, r))}{\log r}.$$

Next, a first step in the mixed multifractal analysis has been developed by Olsen for one very restrictive class of measures known as the self-affine measures [42] dealing precisely with Rényi dimensions for finitely many self-affine measures. Then, motivated by this study, a mixed multifractal analysis has been developed in [10] for vector-valued measures in some more general contexts. By assuming a restrictive hypothesis looking like Gibbs-type measures and by proving a general mixed large deviation formalism a mixed multifractal formalism has been proved. In [10] and [11] a mixed multifractal analysis inspired from the one for measures has been developed in the functional case. By concentrating a vector valued Gibbs-like measure on the singularities set of finitely and simultaneously many functions, a mixed multifractal formalism for functions has been developed. General results for almost all functions have been proved

and a mixed multifractal formalism has been proved for self similar quasi self-similar functions as well as their superpositions. For more details and backgrounds on multifractal analysis as well as the mixed generalizations the readers may be referred also to the following essential references [8, 9, 40, 49, 50, 56, 57, 58, 59, 60, 61, 62].

In an other context, and to overcome the problem of being non doubling and non Hölderian measure, the authors introduced, in [15, 18, 48], a relative multifractal analysis by comparing the original measure μ to an appropriate other one ν . The singularity decomposition sets $X_\mu(\alpha)$ are replaced by two-parameters ones

$$X_{\mu,\nu}(\alpha) = \left\{ x \in X \mid \lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} = \alpha \right\}$$

or

$$X_{\mu,\nu}(\alpha, \beta) = \left\{ x \in X \mid \lim_{r \downarrow 0} \frac{\log \nu(B(x, r))}{\log r} = \alpha \text{ and } \lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} = \beta \right\}.$$

In [6], a relative multifractal analysis has been developed by considering pressure-like quantities instead of Hausdorff measures. Besides, instead of evaluating or studying the local behavior of measures $\mu(B(x, r))$ by means of diameter power lows r^α , the decomposition sets $X_\mu(\alpha)$ is replaced for any Hölder continuous functions φ and ψ by

$$X_{\varphi,\psi}(\alpha) = \left\{ x \in X \mid \lim_{n \rightarrow +\infty} \left(\frac{\sum_{k=0}^n \varphi(\sigma^k(x))}{\sum_{k=0}^n \psi(\sigma^k(x))} \right) = \alpha \right\},$$

which yields a relative analysis of general density-like measures. Here σ denotes the one-sided sub-shift of finite type. A similar example will be explicitly developed in the next. The reader can also be referred to [5, 26, 27, 31, 32, 33].

In [36, 37], the authors introduced a multifractal analysis in a mixed case (which can be already adapted to single cases) where the hypothesis of the existence of Gibbs-like and/or doubling measures supported by the singularities sets is relaxed. The main aim to consider some cases of simultaneous behaviors of measures where the local Hölder behavior is controlled by special and suitable function that allows the extra-hypothesis of Gibbs-like measures not to be necessary. In the present work, we are based on the mixed multifractal formalism developed by Menceur et al., in [36], especially in [37]. We first give a φ -mixed multifractal generalization of the results about the lower and upper Hausdorff and packing dimension of some Borel probability measures that are no longer Gibbs and in a more general framework. As an application, we are interested in the φ -unidimensionality of those measures and to the calculus of its mixed multifractal Hausdorff and packing dimensions. In particular, we give a necessary and sufficient condition for the existence of the φ -mixed multifractal Hausdorff and packing dimensions of a Borel probability measure. Finally, we give some examples in this new framework.

2. Preliminaries

The purpose of this section is to present our ideas about mixed multifractal generalizations of Hausdorff and packing measures. Consider a vector valued measure $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ composed of probability measures on \mathbb{R}^d . We aim to study the simultaneous scaling behavior of μ relatively to an exponential density function. Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be such that

$$\varphi \text{ is non-decreasing and } \varphi(r) < 0 \text{ for } r \text{ small enough.}$$

For $x \in \mathbb{R}^d$ and $r > 0$ we denote $B(x, r)$ the ball of radius r and center x . We denote next

$$\mu(B(x, r)) \equiv (\mu_1(B(x, r)), \dots, \mu_k(B(x, r)))$$

and the product

$$\mu(B(x, r))^q \equiv \prod_{i=1}^k \mu_i(B(x, r))^{q_i}.$$

Let $E \subseteq \mathbb{R}^d$ be a nonempty set and $\epsilon > 0$. Let also $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$ and $t \in \mathbb{R}$ and consider the quantity

$$\overline{\mathcal{H}}_{\mu, \varphi, \epsilon}^{q, t}(E) = \inf \left\{ \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \right\},$$

where the inf is taken over the set of all centered ϵ -coverings of E , and for the empty set, $\overline{\mathcal{H}}_{\mu, \epsilon}^{q, t}(\emptyset) = 0$. It consists of a non increasing function of the variable ϵ . We denote thus

$$\overline{\mathcal{H}}_{\mu, \varphi}^{q, t}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathcal{H}}_{\mu, \varphi, \epsilon}^{q, t}(E) = \sup_{\epsilon > 0} \overline{\mathcal{H}}_{\mu, \varphi, \epsilon}^{q, t}(E).$$

Let finally

$$\mathcal{H}_{\mu, \varphi}^{q, t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu, \varphi}^{q, t}(F).$$

The quantity $\mathcal{H}_{\mu, \varphi}^{q, t}$ is an outer metric measure on \mathbb{R}^d for which Borel sets are measurable.

Now, we define similarly the mixed generalized multifractal packing measure. Let

$$\overline{\mathcal{P}}_{\mu, \varphi, \epsilon}^{q, t}(E) = \sup \left\{ \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \right\}$$

where the sup is taken over the set of all centered ϵ -packings of E . For the empty set, we set as usual $\overline{\mathcal{P}}_{\mu, \varphi, \epsilon}^{q, t}(\emptyset) = 0$. Next, we consider the limit as $\epsilon \downarrow 0$,

$$\overline{\mathcal{P}}_{\mu, \varphi}^{q, t}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathcal{P}}_{\mu, \varphi, \epsilon}^{q, t}(E) = \inf_{\epsilon > 0} \overline{\mathcal{P}}_{\mu, \varphi, \epsilon}^{q, t}(E)$$

and finally,

$$\mathcal{P}_{\mu, \varphi}^{q, t}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu, \varphi}^{q, t}(E_i).$$

As previously, $\mathcal{P}_{\mu, \varphi}^{q, t}$ is an outer metric measure on \mathbb{R}^d for which Borel sets are measurable.

Definition 2.1. The restriction of $\mathcal{H}_{\mu, \varphi}^{q, t}$ and $\mathcal{P}_{\mu, \varphi}^{q, t}$ on Borel sets are called the mixed generalized Hausdorff measure on \mathbb{R}^d and the mixed generalized packing measure on \mathbb{R}^d respectively.

It holds that these measures assign a dimension to every set $E \subseteq \mathbb{R}^d$.

Proposition 2.2. Given a subset $E \subseteq \mathbb{R}^d$,

1. There exists a unique number $b_{\mu, \varphi}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{H}_{\mu, \varphi}^{q, t}(E) = \begin{cases} +\infty & \text{for } t < b_{\mu, \varphi}^q(E), \\ 0 & \text{for } t > b_{\mu, \varphi}^q(E). \end{cases}$$

2. There exists a unique number $B_{\mu, \varphi}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{P}_{\mu, \varphi}^{q, t}(E) = \begin{cases} +\infty & \text{for } t < B_{\mu, \varphi}^q(E), \\ 0 & \text{for } t > B_{\mu, \varphi}^q(E). \end{cases}$$

3. There exists a unique number $\Delta_{\mu,\varphi}^q(E) \in [-\infty, +\infty]$ such that

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E) = \begin{cases} +\infty & \text{for } t < \Delta_{\mu,\varphi}^q(E), \\ 0 & \text{for } t > \Delta_{\mu,\varphi}^q(E). \end{cases}$$

Next, we set

$$b_{\mu,\varphi}^q(E) = \inf \left\{ t \in \mathbb{R}; \mathcal{H}_{\mu,\varphi}^{q,t}(E) = 0 \right\},$$

$$B_{\mu,\varphi}^q(E) = \inf \left\{ t \in \mathbb{R}; \mathcal{H}_{\mu,\varphi}^{q,t}(E) = 0 \right\}$$

and

$$\Delta_{\mu,\varphi}^q(E) = \inf \left\{ t \in \mathbb{R}; \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E) = 0 \right\}.$$

Definition 2.3. The quantities $b_{\mu,\varphi}^q(E)$, $B_{\mu,\varphi}^q(E)$ and $\Delta_{\mu,\varphi}^q(E)$ define the so-called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set E .

Remark 2.4. Notice that for $k = 1$ and φ the log function $\varphi(r) = \log(r)$, we come back to the classical definitions of the Hausdorff and packing measures and dimensions in their original forms (by taking $q = 0$) and their generalized multifractal variants for q being arbitrary. The mixed case studied here may be also applied for a single measure and thus the results and characterizations outpointed in the present work remains valid for a single measure. Indeed, denote $Q_i = (0, 0, \dots, q_i, 0, \dots, 0)$ the vector with zero coordinates except the i th one which equals q_i , we obtain the multifractal generalizations of the Hausdorff φ -measure and φ -dimension, the packing φ -dimension and the logarithmic φ -index of the set E for the single measure μ_i , $b_{\mu_i,\varphi}^{Q_i}(E) = b_{\mu_i,\varphi}^{q_i}(E)$, $B_{\mu_i,\varphi}^{Q_i}(E) = B_{\mu_i,\varphi}^{q_i}(E)$ and $\Delta_{\mu_i,\varphi}^{Q_i}(E) = \Delta_{\mu_i,\varphi}^{q_i}(E)$. Similarly, for the null vector of \mathbb{R}^k , we obtain $b_{\mu,\varphi}^0(E) = b_\varphi(E)$, $B_{\mu,\varphi}^0(E) = B_\varphi(E)$ and $\Delta_{\mu,\varphi}^0(E) = \Delta_\varphi(E)$.

We may obtain further $b_{\mu,\log}^{Q_i}(E) = b_{\mu_i,\log}^{q_i}(E)$, $B_{\mu,\log}^{Q_i}(E) = B_{\mu_i,\log}^{q_i}(E)$ and $\Delta_{\mu,\log}^{Q_i}(E) = \Delta_{\mu_i,\log}^{q_i}(E)$. Similarly, for the null vector of \mathbb{R}^k , we obtain

$$b_{\mu,\log}^0(E) = b_{\log}(E) = \dim_H(E),$$

$$B_{\mu,\log}^0(E) = B_{\log}(E) = \dim_P(E)$$

and

$$\Delta_{\mu,\log}^0(E) = \Delta_{\log}(E) = \Delta(E).$$

For more details of these functions, we can see [37].

3. The main results

In the following, we introduce the φ -mixed multifractal analogous of the Hausdorff and packing dimensions of a Borel probability measure. Also, we give a φ -mixed multifractal generalization of the results about the lower and upper Hausdorff and packing dimension of some Borel probability measures that are no longer Gibbs.

Definition 3.1. The lower and upper φ -mixed multifractal Hausdorff dimensions of a measure ν with respect to a measure μ are defined by

$$\underline{b}_{\mu,\varphi}^q(\nu) = \inf \left\{ b_{\mu,\varphi}^q(E); E \subseteq \mathbb{R}^d \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{b}_{\mu,\varphi}^q(\nu) = \inf \left\{ b_{\mu,\varphi}^q(E); E \subseteq \mathbb{R}^d \text{ and } \nu(E) = 1 \right\}.$$

We denote by $b_{\mu,\varphi}^q(\nu)$ their common value, if the equality $\underline{b}_{\mu,\varphi}^q(\nu) = \overline{b}_{\mu,\varphi}^q(\nu)$ is satisfied.

Definition 3.2. The lower and upper φ -mixed multifractal packing dimensions of a measure ν with respect to a measure μ are defined by

$$\underline{B}_{\mu,\varphi}^q(\nu) = \inf \left\{ B_{\mu,\varphi}^q(E); E \subseteq \mathbb{R}^d \text{ and } \nu(E) > 0 \right\}$$

and

$$\overline{B}_{\mu,\varphi}^q(\nu) = \inf \left\{ B_{\mu,\varphi}^q(E); E \subseteq \mathbb{R}^d \text{ and } \nu(E) = 1 \right\}.$$

When $\underline{B}_{\mu,\varphi}^q(\nu) = \overline{B}_{\mu,\varphi}^q(\nu)$, we denote by $B_{\mu,\varphi}^q(\nu)$ their common value.

Definition 3.3. Let μ, ν be two Borel probability measures on \mathbb{R}^d .

1. We say that μ is absolutely continuous with respect to ν and write $\mu \ll \nu$ if, for any set $A \subset^d$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$.
2. μ and ν are said to be mutually singular and we write $\mu \perp \nu$ if there exists a set $A \subset^d$, such that $\mu(A) = 0 = \nu(\mathbb{R}^d \setminus A)$.

In the following, we will prove that the quantities $\underline{b}_{\mu}^q(\nu)$ and $\overline{b}_{\mu}^q(\nu)$ (resp. $\underline{B}_{\mu}^q(\nu)$ and $\overline{B}_{\mu}^q(\nu)$) allow to compare the measure ν with the mixed generalized multifractal Hausdorff (resp. packing) measure.

Theorem 3.4. Let ν be a Borel probability measure on \mathbb{R}^d and $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$. We have,

1. $\underline{b}_{\mu,\varphi}^q(\nu) = \sup \left\{ t \in; \nu \ll \mathcal{H}_{\mu,\varphi}^{q,t} \right\}$ and $\overline{b}_{\mu,\varphi}^q(\nu) = \inf \left\{ t \in; \nu \perp \mathcal{H}_{\mu,\varphi}^{q,t} \right\}$.
2. $\underline{B}_{\mu,\varphi}^q(\nu) = \sup \left\{ t \in; \nu \ll \mathcal{P}_{\mu,\varphi}^{q,t} \right\}$ and $\overline{B}_{\mu,\varphi}^q(\nu) = \inf \left\{ t \in; \nu \perp \mathcal{P}_{\mu,\varphi}^{q,t} \right\}$.

Remark 3.5. When the upper φ -mixed multifractal Hausdorff (resp. packing) dimension of the measure is small, it means that the measure ν is *very singular* with respect to the φ -mixed generalized multifractal Hausdorff (resp. packing) measure. In the same way, when the lower φ -mixed multifractal (resp. packing) dimension of the measure is large, then the measure ν is “quite regular” with respect to the φ -mixed generalized multifractal Hausdorff (resp. packing) measure.

The next result proves that the quantities $\underline{b}_{\mu,\varphi}^q(\nu)$, $\overline{b}_{\mu,\varphi}^q(\nu)$, $\underline{B}_{\mu,\varphi}^q(\nu)$ and $\overline{B}_{\mu,\varphi}^q(\nu)$ are related to the asymptotic behavior of the function $\alpha_{\mu,\nu}^{q,\varphi}(x, r)$, where

$$\alpha_{\mu,\nu}^{q,\varphi}(x, r) = \frac{\log \nu(B(x, r)) - \sum_{i=1}^k q_i \log \mu_i(B(x, r))}{\varphi(r)}.$$

Let

$$\underline{\alpha}_{\mu,\nu}^{q,\varphi}(x) = \liminf_{r \rightarrow 0} \alpha_{\mu,\nu}^{q,\varphi}(x, r) \quad \text{and} \quad \overline{\alpha}_{\mu,\nu}^{q,\varphi}(x) = \limsup_{r \rightarrow 0} \alpha_{\mu,\nu}^{q,\varphi}(x, r).$$

Theorem 3.6. Let ν be a Borel probability measures on \mathbb{R}^d and $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$. We have,

1. $\underline{b}_{\mu,\varphi}^q(\nu) = \underline{\alpha}_{\mu,\nu}^{q,\varphi}(x)$ and $\overline{b}_{\mu,\varphi}^q(\nu) = \overline{\alpha}_{\mu,\nu}^{q,\varphi}(x)$.
2. $\underline{B}_{\mu,\varphi}^q(\nu) = \underline{\alpha}_{\mu,\nu}^{q,\varphi}(x)$ and $\overline{B}_{\mu,\varphi}^q(\nu) = \overline{\alpha}_{\mu,\nu}^{q,\varphi}(x)$,

where the essential bounds being related to the measure ν .

Corollary 3.7. Let ν be a Borel probability measure on \mathbb{R}^d and take $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$. We have,

1. $\underline{b}_{\mu,\varphi}^q(\nu) \geq \alpha$ if and only if $\underline{\alpha}_{\mu,\nu}^{q,\varphi}(x) \geq \alpha$ for ν -a.e. x .
2. $\overline{b}_{\mu,\varphi}^q(\nu) \leq \alpha$ if and only if $\overline{\alpha}_{\mu,\nu}^{q,\varphi}(x) \leq \alpha$ for ν -a.e. x .
3. $\underline{B}_{\mu,\varphi}^q(\nu) \geq \alpha$ if and only if $\underline{\alpha}_{\mu,\nu}^{q,\varphi}(x) \geq \alpha$ for ν -a.e. x .
4. $\overline{B}_{\mu,\varphi}^q(\nu) \leq \alpha$ if and only if $\overline{\alpha}_{\mu,\nu}^{q,\varphi}(x) \leq \alpha$ for ν -a.e. x .

4. Proofs

4.1. Proof of Theorem 3.4

We only prove the assertion (1). Assertion (2) follows the same ideas.

Let's prove that $\underline{b}_{\mu,\varphi}^q(\nu) = \sup \{t \in; \nu \ll \mathcal{H}_{\mu,\varphi}^{q,t}\}$. Define $s = \sup \{t \in; \nu \ll \mathcal{H}_{\mu,\varphi}^{q,t}\}$. Then, for any $t < s$ and $E \subseteq^d$, such that $\nu(E) > 0$, we have $\mathcal{H}_{\mu,\varphi}^{q,t}(E) > 0$. It follows that $\underline{b}_{\mu,\varphi}^q(E) \geq t$ and then, $\underline{b}_{\mu,\varphi}^q(\nu) \geq t$. We deduce that $\underline{b}_{\mu,\varphi}^q(\nu) \geq s$.

On the other hand, for any $t > s$, there exists a set $E \subseteq^d$, such that $\nu(E) > 0$ and $\mathcal{H}_{\mu,\varphi}^{q,t}(E) = 0$. Consequently, $\underline{b}_{\mu,\varphi}^q(E) \leq t$ and so, $\underline{b}_{\mu,\varphi}^q(\nu) \leq t$. This leads to $\underline{b}_{\mu,\varphi}^q(\nu) \leq s$.

Now, we prove that $\overline{b}_{\mu,\varphi}^q(\nu) = \inf \{t \in; \nu \perp \mathcal{H}_{\mu,\varphi}^{q,t}\}$. For this, we define $s' = \inf \{t \in; \nu \perp \mathcal{H}_{\mu,\varphi}^{q,t}\}$. For $t > s'$, there exists a set $E \subseteq^d$, such that $\mathcal{H}_{\mu,\varphi}^{q,t}(E) = 0 = \nu(d \setminus E)$. Then, $\underline{b}_{\mu,\varphi}^q(E) \leq t$. Since $\nu(E) = 1$, then $\underline{b}_{\mu,\varphi}^q(\nu) \leq t$ and $\underline{b}_{\mu,\varphi}^q(\nu) \leq s'$.

Now, for $t < s'$, take $E \subseteq^d$, such that $\mathcal{H}_{\mu,\varphi}^{q,t}(E) > 0$ and $\nu(E) = 1$. It can immediately seen that $\underline{b}_{\mu,\varphi}^q(E) \geq t$. Then, $\underline{b}_{\mu,\varphi}^q(\nu) \geq t$. It follows that $\underline{b}_{\mu,\varphi}^q(\nu) \geq s'$. This ends the proof of assertion (1).

4.2. Technical lemma

We present the following technical lemma, which will be used in the proof of our main result.

Lemma 4.1. *Let ν be a Borel probability measure on d , $E \subseteq^d$ and $0 < c < \infty$ be a constant.*

1. *If $\liminf_{r \rightarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q e^{t\varphi(r)}} < c$, for all $x \in E$, then $\mathcal{P}_{\mu,\varphi}^{q,t}(E) \geq \frac{\nu(E)}{c}$.*
2. *If $\liminf_{r \rightarrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q e^{t\varphi(r)}} > c$, for all $x \in E$, then $\mathcal{P}_{\mu,\varphi}^{q,t}(E) \leq \frac{1}{c}$.*

Proof of lemma 4.1.

1. Let $\eta, \epsilon > 0$ and $F \subset E$. Choose $\delta > 0$ such that

$$\overline{\mathcal{P}}_{\mu,\varphi,\delta}^{q,t}(F) \leq \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(F) + \frac{\epsilon}{c + \eta}.$$

Consider the set

$$\Omega_\delta = \left\{ B(x,r), x \in F, r < \delta \text{ and } \nu(B(x,r)) < (c + \eta) \prod_{p=1}^k \mu_p(B(x,r))^{q_p} e^{t\varphi(r)} \right\}.$$

By using Vitali covering theorem (see [35]), there exists a δ -packing $(B(x_i, r_i))_i \subset \Omega_\delta$ of F satisfying

$$\nu\left(F \setminus \cup_i B(x_i, r_i)\right) = 0.$$

We now have the following

$$\begin{aligned} \nu(F) &= \nu\left(\cup_i (F \cap B(x_i, r_i))\right) \\ &\leq \sum_i \nu(B(x_i, r_i)) \\ &\leq (c + \eta) \sum_i \prod_{p=1}^k \mu_p(B(x_i, r_i))^{q_p} e^{t\varphi(r_i)} \\ &\leq (c + \eta) \overline{\mathcal{P}}_{\mu,\varphi,\delta}^{q,t}(F) \\ &\leq (c + \eta) \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(F) + \epsilon. \end{aligned}$$

Making $\eta \rightarrow 0$ and $\epsilon \rightarrow 0$, we obtain $\nu(F) \leq c \overline{\mathcal{P}}_{\mu, \varphi}^{q,t}(F)$. If we consider a covering $(E_i)_i$ of E we obtain

$$\nu(E) = \nu\left(\cup_i (E \cap E_i)\right) \leq \sum_i \nu(E \cap E_i) \leq c \sum_i \overline{\mathcal{P}}_{\mu, \varphi}^{q,t}(E \cap E_i) \leq c \sum_i \overline{\mathcal{P}}_{\mu, \varphi}^{q,t}(E_i).$$

Then, the result follows immediately when taking the infimum for $E \subset \cup_i E_i$.

2. Fix $\epsilon > 0$ and $0 < \eta < c$. We will prove that, for any closed subset F of E ,

$$\mathcal{P}_{\mu, \varphi}^{q,t}(F)(c - \eta) \leq \nu(E) + \epsilon.$$

Let F be a closed subset E . Recall that, if $\delta \searrow 0$, then $D_F(\delta) \searrow F$ where

$$D_F(\delta) = \{x \in \mathbb{R}^d, \text{dist}(x, F) \leq \delta\},$$

where $\text{dist}(x, F) = \inf_{y \in F} \|x - y\|$ and $\|\cdot\|$ is the usual norm on \mathbb{R}^d .

So, there exists δ_0 satisfying

$$\nu(D_F(\delta)) \leq \nu(F) + \epsilon, \quad \forall 0 < \delta < \delta_0.$$

For $s \in \mathbb{R} \setminus \{0\}$, consider the set

$$F_s = \left\{ x \in F, \nu(B(x, r)) \geq (c - \eta) \prod_{p=1}^k \mu_p(B(x, r))^{q_p} e^{t\varphi(r)}, \text{ for } 0 < r < \frac{1}{s} \right\}.$$

Fix $s \in \mathbb{R} \setminus \{0\}$ and $0 < \delta < \sup\{\frac{1}{s}, \delta_0\}$. Let $(B(x_i, r_i))_i$ be a centered δ -packing of F_s . Note that

$$\begin{aligned} (c - \eta) \sum_i \prod_{p=1}^k \mu_p(B(x_i, r_i))^{q_p} e^{t\varphi(r_i)} &\leq \sum_i \nu(B(x_i, r_i)) \\ &= \nu\left(\cup_i B(x_i, r_i)\right) \leq \nu(D_F(\delta)) \leq \nu(F) + \epsilon \leq \nu(E) + \epsilon. \end{aligned}$$

which implies that

$$\mathcal{P}_{\mu, \varphi}^{q,t}(F_s)(c - \eta) \leq \overline{\mathcal{P}}_{\mu, \varphi}^{q,t}(F_s)(c - \eta) \leq \overline{\mathcal{P}}_{\mu, \varphi, \delta}^{q,t}(F_s)(c - \eta) \leq \nu(E) + \epsilon.$$

Since $F_s \nearrow F$, we obtain

$$\mathcal{P}_{\mu, \varphi}^{q,t}(F)(c - \eta) \leq \nu(E) + \epsilon.$$

Making $\eta \rightarrow 0$ and $\epsilon \rightarrow 0$ gives the desired result $\mathcal{P}_{\mu, \varphi}^{q,t}(F) \leq \frac{1}{c}$.

4.3. Proof of Theorem 3.6

1. We prove that $\underline{b}_{\mu, \varphi}^q(\nu) = \text{ess inf } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x)$.

Let $\alpha < \text{ess inf } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x)$. Then, for ν -almost every x , we can choose $r_0 > 0$, such that $0 < r < r_0$ and

$$\nu(B(x, r)) < \prod_{i=1}^k \mu_i(B(x, r))^{q_i} e^{\alpha\varphi(r)}.$$

Denote by

$$F_n = \left\{ x; \nu(B(x, r)) < \prod_{i=1}^k \mu_i(B(x, r))^{q_i} e^{\alpha\varphi(r)}, \text{ for } 0 < r < \frac{1}{n} \right\}.$$

Let $F = \cup_n F_n$. It is clear that $\nu(F) = 1$. Take E be a Borel subset of \mathbb{R}^d satisfying $\nu(E) > 0$. We have $\nu(E \cap F) > 0$ and we can therefore choose an integer n , such that $\nu(E \cap F_n) > 0$.

Now, let $\delta > 0$ and $(B(x_j, r_j))_j$ be a centered δ -covering of $E \cap F_n$. We have

$$\sum_j \nu(B(x_j, r_j)) \leq \sum_j \prod_{i=1}^k \mu_i(B(x_j, r_j))^{q_i} e^{\alpha \varphi(r_j)}.$$

This shows that

$$\nu(E \cap F_n) \leq \overline{\mathcal{H}}_{\mu, \varphi, \delta}^{q, \alpha}(E \cap F_n).$$

Letting $\delta \rightarrow 0$ gives that

$$\nu(E \cap F_n) \leq \overline{\mathcal{H}}_{\mu, \varphi}^{q, \alpha}(E \cap F_n) \leq \mathcal{H}_{\mu, \varphi}^{q, \alpha}(E \cap F_n).$$

It follows that

$$\mathcal{H}_{\mu, \varphi}^{q, \alpha}(E) \geq \mathcal{H}_{\mu, \varphi}^{q, \alpha}(E \cap F_n) > 0 \Rightarrow \underline{b}_{\mu, \varphi}^q(E) \geq \alpha.$$

We conclude that

$$\underline{b}_{\mu, \varphi}^q(\nu) \geq \text{ess inf } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x).$$

On the other hand, if $\text{ess inf } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) = \alpha$. For $\varepsilon > 0$, we will write

$$E_\varepsilon = \left\{ x \in \nu; \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) < \alpha + \varepsilon \right\}.$$

It is clear that $\nu(E_\varepsilon) > 0$. This means that $\underline{b}_{\mu, \varphi}^q(\nu) \leq \underline{b}_{\mu, \varphi}^q(E_\varepsilon)$. We will prove that

$$\underline{b}_{\mu, \varphi}^q(E_\varepsilon) \leq \alpha + \varepsilon, \quad \forall \varepsilon > 0.$$

Let $E \subset E_\varepsilon$ and $x \in E$. Then, for all $\delta > 0$ we can find $0 < r_x < \delta$, such that

$$\nu(B(x, r_x)) > \prod_{i=1}^k \mu_i(B(x, r_x))^{q_i} e^{(\alpha + \varepsilon)\varphi(r_x)}.$$

Take $\delta > 0$. The family $(B(x, r_x))_{x \in E}$ is a centered δ -covering of E . Using Besicovitch's covering theorem, we can construct ξ finite or countable sub-families

$$(B(x_{1j}, r_{1j}))_j, \dots, (B(x_{\xi j}, r_{\xi j}))_j,$$

such that each E satisfies

$$E \subseteq \bigcup_{i=1}^{\xi} \bigcup_j B(x_{ij}, r_{ij}) \quad \text{and} \quad (B(x_{ij}, r_{ij}))_j \quad \text{is a } \delta\text{-packing of } E.$$

It follows immediately that

$$\sum_{i,j} \prod_{p=1}^k \mu_p(B(x_{ij}, r_{ij}))^{q_p} e^{(\alpha + \varepsilon)\varphi(r_{ij})} \leq \xi \sum_j \nu(B(x_{ij}, r_{ij})) \leq \xi \nu(\mathbb{R}^d) < +\infty.$$

This shows that

$$\overline{\mathcal{H}}_{\mu, \varphi, \delta}^{q, \alpha + \varepsilon}(E) < +\infty \Rightarrow \overline{\mathcal{H}}_{\mu, \varphi}^{q, \alpha + \varepsilon}(E) < +\infty.$$

We obtain thus

$$\mathcal{H}_{\mu, \varphi}^{q, \alpha + \varepsilon}(E_\varepsilon) < +\infty.$$

We therefore conclude that

$$\underline{b}_{\mu, \varphi}^q(E) \leq \alpha + \varepsilon \quad \text{and} \quad \underline{b}_{\mu, \varphi}^q(\nu) \leq \text{ess inf } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x).$$

We prove in a similar way that $\overline{b}_{\mu, \varphi}^q(\nu) = \text{ess sup } \underline{\alpha}_{\mu, \nu}^{q, \varphi}(x)$.

2. We next prove that $\underline{B}_{\mu,\varphi}^q(\nu) = \text{ess inf } \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x)$, and the proof of $\bar{B}_{\mu,\varphi}^q(\nu) = \text{ess sup } \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x)$ is similar. Let $\alpha = \text{ess inf } \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x)$ and

$$E = \{x \in \mu; \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) \geq \alpha\}.$$

It is clear that $\nu(E) = 1$. Now, let $F \subset \mathbb{R}^d$ such that $\nu(F) > 0$ so $\nu(E \cap F) > 0$. For $\beta < \alpha$, we have

$$\bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) > \beta, \quad \text{for all } x \in E \cap F.$$

We can choose a sequence $r_i \rightarrow 0$ if $i \rightarrow +\infty$ such that

$$\lim_{i \rightarrow +\infty} \frac{\log \nu(B(x, r_i)) - \sum_{p=1}^k q_p \log \mu_p(B(x, r_i))}{\varphi(r_i)} > \beta.$$

Then, there exist $i_0 \in \mathbb{N}$ such that for all $i > i_0$,

$$\frac{\nu(B(x, r_i))}{\prod_{p=1}^k \mu_p(B(x, r_i))^{q_p} e^{\beta \varphi(r_i)}} < 1.$$

Thus,

$$\liminf_{r \rightarrow 0} \frac{\nu(B(x, r))}{\prod_{p=1}^k \mu_p(B(x, r))^{q_p} e^{\beta \varphi(r)}} < 1.$$

By using Lemma 4.1, we obtain

$$\mathcal{P}_{\mu,\varphi}^{q,\beta}(F) \geq \mathcal{P}_{\mu,\varphi}^{q,\beta}(E \cap F) \geq \nu(E \cap F) > 0.$$

We conclude from this that

$$\underline{B}_{\mu,\varphi}^q(\nu) \geq \beta \quad \text{for all } \beta < \alpha.$$

By the arbitrariness of α , we have

$$\underline{B}_{\mu,\varphi}^q(\nu) \geq \text{ess inf } \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x).$$

Now we will show that $\underline{B}_{\mu,\varphi}^q(\nu)$ is the least upper bound of $\text{ess inf } \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x)$. In fact, if not, there is $\alpha < \underline{B}_{\mu,\varphi}^q(\nu)$ such that $\nu(E) > 0$, where

$$E = \{x \in \mu; \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) < \alpha\}.$$

Let $F \subset \mathbb{R}^d$ satisfying $\nu(F) > 0$ and so $\nu(E \cap F) > 0$. Let $x \in E \cap F$, then

$$\bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) < \alpha.$$

Similar to the above discussion, we get

$$\liminf_{r \rightarrow 0} \frac{\nu(B(x, r))}{\prod_{p=1}^k \mu_p(B(x, r))^{q_p} e^{\alpha \varphi(r)}} > 1.$$

Using Lemma 4.1, we have

$$\mathcal{P}_{\mu,\varphi}^{q,\alpha}(F) \leq 1.$$

Hence $\underline{B}_{\mu,\varphi}^q(\nu) \leq \alpha$, which contradicts the choice of α . Which achieves the proof of Theorem 3.6.

5. Application

As an application of Theorems 3.4 and 3.6, we are interested to the φ -unidimensionality of a probability measure ν and to the calculus of its mixed multifractal Hausdorff and packing dimensions. In particular, we tried through this section to suggest a necessary and sufficient condition for the existence of the φ -mixed multifractal Hausdorff and packing dimensions of a Borel probability measure ν .

Let ν be a probability measure on \mathbb{R}^d and $\alpha \in \mathbb{R}$. ν is said to have α as a φ -mixed multifractal exact Hausdorff dimension with respect to a measure μ and q if

$$\alpha_{\mu,\nu}^{q,\varphi}(x) = \alpha, \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R}^d .$$

Theorem 5.1. *Let ν be a Borel probability measure on \mathbb{R}^d and take $\alpha \in \mathbb{R}$. Then the following conditions are equivalent*

1. ν has α as a φ -mixed multifractal exact Hausdorff dimension with respect to a measure μ and q .
2. We have,
 - (a) there exist a set $E \subset \mathbb{R}^d$ with $b_{\mu,\varphi}^q(E) = \alpha$, such that $\nu(E) = 1$,
 - (b) if $E \subset \mathbb{R}^d$ satisfies $b_{\mu,\varphi}^q(E) < \alpha$, then $\nu(E) = 0$.
3. We have,
 - (a) $\nu \ll \mathcal{H}_{\mu,\varphi}^{q,\alpha-\epsilon}$, for all $\epsilon > 0$.
 - (b) $\nu \perp \mathcal{H}_{\mu,\varphi}^{q,\alpha+\epsilon}$, for all $\epsilon > 0$.

Remark 5.2. Notice that the measure has α as a φ -mixed multifractal exact Hausdorff dimension with respect to a measure μ and q if and only if the φ -mixed multifractal Hausdorff dimension of ν exists and equals to α .

Let ν be a probability measure on \mathbb{R}^d and $\alpha \in \mathbb{R}$. ν is said to have α as a φ -mixed multifractal exact paking dimension with respect to a measure μ and q if

$$\bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) = \alpha, \quad \text{for } \nu\text{-a.e. } x \in \mathbb{R}^d .$$

The symmetrical results for the mixed multifractal packing measure and dimension are true as well, the proof of which is quite analogous to that in the above theorem.

Theorem 5.3. *Let ν be a Borel probability measure on \mathbb{R}^d and take $\alpha \in \mathbb{R}$. Then the following conditions are equivalent*

1. ν has α as a φ -mixed multifractal exact packing dimension with respect to a measure μ and q .
2. We have,
 - (a) there exist a set $E \subset \mathbb{R}^d$ with $B_{\mu,\varphi}^q(E) = \alpha$, such that $\nu(E) = 1$,
 - (b) if $E \subset \mathbb{R}^d$ satisfies $B_{\mu,\varphi}^q(E) < \alpha$, then $\nu(E) = 0$.
3. We have,
 - (a) $\nu \ll \mathcal{P}_{\mu,\varphi}^{q,\alpha-\epsilon}$, for all $\epsilon > 0$.
 - (b) $\nu \perp \mathcal{P}_{\mu,\varphi}^{q,\alpha+\epsilon}$, for all $\epsilon > 0$.

5.1. Proof of Theorem 5.1

We can deduce from Theorem 3.6 and Corollary 3.7 that if ν has α as a φ -multifractal exact Hausdorff dimension, then

$$b_{\mu,\varphi}^q(\nu) = \bar{b}_{\mu,\varphi}^q(\nu) = \alpha.$$

By using Theorem 3.4, we have the following assertions

1. ν is absolutely continuous with respect to $\mathcal{H}_{\mu,\varphi}^{q,\alpha-\epsilon}$, for all $\epsilon > 0$.
2. ν and $\mathcal{H}_{\mu,\varphi}^{q,\alpha+\epsilon}$ are mutually singular, for all $\epsilon > 0$.

Conversely, by using Theorem 3.4 and under the hypothesis (a) and (a) of (3) we have

$$\alpha - \varepsilon \leq \underline{b}_{\mu, \varphi}^q(\nu) \leq \overline{b}_{\mu, \varphi}^q(\nu) \leq \alpha + \varepsilon, \text{ for all } \varepsilon > 0.$$

The arbitrary in ε implies that

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r)) - \sum_{i=1}^k q_i \log \mu_i(B(x, r))}{\varphi(r)} = \alpha, \quad \text{for } \nu\text{-a.e. } x.$$

Now, we only need to prove the equivalence of the assertions (2) and (3). Assume that the measure ν satisfies the hypothesis (a) and (b) of (2). Let $E \subset \mathbb{R}^d$ and suppose that $\mathcal{H}_{\mu, \varphi}^{q, \alpha - \varepsilon}(E) = 0$, for all $\varepsilon > 0$. Then, we have that $b_{\mu, \varphi}^q(E) \leq \alpha - \varepsilon < \alpha$. It follows immediately from condition (b) of (2) that $\nu(E) = 0$. Also, we deduce that $\nu \ll \mathcal{H}_{\mu, \varphi}^{q, \alpha - \varepsilon}$, for all $\varepsilon > 0$.

Thanks to condition (a) of (2), there exists a set $E \subset \mathbb{R}^d$ of mixed multifractal Hausdorff dimension α , such that $\nu(E) = 1$ and $b_{\mu, \varphi}^q(E) = \alpha < \alpha + \varepsilon$, for all $\varepsilon > 0$. Which implies that $\mathcal{H}_{\mu, \varphi}^{q, \alpha + \varepsilon}(E) = 0$. We conclude from this that $\nu \perp \mathcal{H}_{\mu, \varphi}^{q, \alpha + \varepsilon}$, for all $\varepsilon > 0$.

Now, assume that ν satisfies conditions (a) and (b) of (3). This means that $\nu \ll \mathcal{H}_{\mu, \varphi}^{q, \alpha - \varepsilon}$, for all $\varepsilon > 0$. Taking a Borel set E with $b_{\mu, \varphi}^q(E) = \beta < \alpha$ and $\varepsilon = \frac{\alpha - \beta}{2}$, we get $\mathcal{H}_{\mu, \varphi}^{q, (\alpha + \beta)/2}(E) = 0$. Then, $\nu(E) = 0$.

Since $\nu \perp \mathcal{H}_{\mu, \varphi}^{q, \alpha + \varepsilon}$, for all $\varepsilon > 0$, we can therefore choose a set F_ε with $\mathcal{H}_{\mu, \varphi}^{q, \alpha + \varepsilon}(F_\varepsilon) = 0$ and $\nu(F_\varepsilon) = 1$. Hence, $b_{\mu, \varphi}^q(F_\varepsilon) \leq \alpha + \varepsilon$. Choose a sequence $(\varepsilon_k)_k$ such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$ and consider the set $F = \bigcap_{k \geq 1} F_{\varepsilon_k}$. It is clear that $\nu(F) = 1$ and

$$b_{\mu, \varphi}^q(F) \leq \liminf_{k \rightarrow \infty} b_{\mu, \varphi}^q(F_{\varepsilon_k}) \leq \liminf_{k \rightarrow \infty} (\alpha + \varepsilon_k) = \alpha.$$

If $b_{\mu, \varphi}^q(F) = \alpha$, then the condition (a) of (2) is satisfied for $E = F$. But if $b_{\mu, \varphi}^q(F) < \alpha$, then putting $E = F \cup G$, for some Borel set G of mixed multifractal Hausdorff dimension α which implies that $\nu(E) = 1$ and $b_{\mu, \varphi}^q(E) = \max \{ b_{\mu, \varphi}^q(F), b_{\mu, \varphi}^q(G) \} = \alpha$.

6. Examples

Remark 6.1. Notice that for $k = 1$ and φ the log function $\varphi(r) = \log(r)$, we come back to the classical definitions of the Hausdorff and packing dimensions (see [29, 30, 24, 25] in their original forms (by taking $q = 0$) and their generalized multifractal variants for q being arbitrary (see [16, 34, 45]).

In the next parts, more examples will be discussed related to our concepts.

6.1. Example 1

Let $\mu_1, \mu_2, \dots, \mu_k$ be probability measures on \mathbb{R}^d and $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$ such that, for all $x \in \mathbb{R}^d$,

$$\prod_{i=1}^k \mu_i(B(x, r))^{q_i} \sim e^{\varphi(r)}, \text{ as } r \rightarrow 0.$$

Let ν be a compactly supported Borel probability measure on \mathbb{R}^d and $T : \nu \rightarrow \nu$ a lipschitz function. Suppose that ν is T -invariant and ergodic on ν . Then,

$$\overline{b}_{\mu, \varphi}^q(\nu) = \underline{b}_{\mu, \varphi}^q(\nu) \quad \text{and} \quad \overline{B}_{\mu, \varphi}^q(\nu) = \underline{B}_{\mu, \varphi}^q(\nu).$$

In fact, since T is a Lipschitz function, then $T(B(x, r)) \subseteq B(T(x), r)$. Since ν is T -invariant, then we can deduce that

$$\nu(B(x, r)) \leq \nu(T^{-1}(T(B(x, r)))) \leq \nu(T^{-1}(B(T(x), r))) = \nu(B(T(x), r)).$$

It follows that,

$$\begin{aligned} \frac{\log \nu(B(x, r)) - \sum_{i=1}^k q_i \log \mu_i(B(x, r))}{\varphi(r)} &\sim \frac{\log \nu(B(x, r))}{\varphi(r)} - 1 \\ &\geq \frac{\log \nu(B(T(x), r))}{\varphi(r)} - 1 \\ &\sim \frac{\log \nu(B(T(x), r)) - \sum_{i=1}^k q_i \log \mu_i(B(T(x), r))}{\varphi(r)}, \end{aligned}$$

which proves that

$$\underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) \geq \underline{\alpha}_{\mu, \nu}^{q, \varphi}(T(x)).$$

Since ν is ergodic, then the function $\underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) - \underline{\alpha}_{\mu, \nu}^{q, \varphi}(T(x))$ is positive and satisfies

$$\int \left(\underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) - \underline{\alpha}_{\mu, \nu}^{q, \varphi}(T(x)) \right) d\nu(x) = 0.$$

We can conclude that,

$$\underline{\alpha}_{\mu, \nu}^{q, \varphi}(x) = \underline{\alpha}_{\mu, \nu}^{q, \varphi}(T(x)) \quad \text{for } \nu \text{-a.e. } x$$

and that the functions $\underline{\alpha}_{\mu, \nu}^{q, \varphi}$ is T -invariant. It follows that $\underline{\alpha}_{\mu, \nu}^{q, \varphi}$ is ν -almost everywhere constant, which says that

$$\underline{b}_{\mu, \varphi}^q(\nu) = \underline{b}_{\mu, \varphi}^q(\nu).$$

We prove in a similar way that

$$\overline{B}_{\mu, \varphi}^q(\nu) = \overline{B}_{\mu, \varphi}^q(\nu).$$

6.2. Example 2

In this example, we will consider an explicit gauge function φ and/or a measure case μ where the constructed measure supported on such set due to Olsen is not Gibbs, not Hölderian and not doubling. In the present case, a suitable way is to choose, for $\gamma > 0$, the function

$$\varphi(r) = \log r (|\log r|)^{\gamma-1}.$$

Remark here that if we denote

$$\nu_{q, \alpha}(B(x, r)) = \prod_{i=1}^k (\mu_i(B(x, r)))^{q_i} e^{\alpha \varphi(r)},$$

we get immediately

$$\frac{\nu_{q, \alpha}(B(x, 2r))}{\nu_{q, \alpha}(B(x, r))} \sim \prod_{i=1}^k \left[\frac{\mu_i(B(x, 2r))}{\mu_i(B(x, r))} \right]^{q_i} e^{\alpha(\log 2)|\log r|^{\gamma-1}}, \text{ as } r \rightarrow 0.$$

So, even if μ is a doubling measure, which is the case in Olsen's original work, we get here

$$\frac{\nu_{q, \alpha}(B(x, 2r))}{\nu_{q, \alpha}(B(x, r))} \sim C_q e^{\alpha(\log 2)|\log r|^{\gamma-1}} \rightarrow +\infty \text{ (or } 0), \text{ as } r \rightarrow 0.$$

This means that the measure $\nu_{q,t}$ which plays the role of the Gibbs measure supported on the singularity set of μ is not Gibbs in the present case. In particular, we have

$$\frac{\log \nu_{q,\alpha}(B(x,r)) - \sum_{i=1}^k q_i \log \mu_i(B(x,r))}{\varphi(r)} = \frac{\log \left(\prod_{i=1}^k (\mu_i(B(x,r)))^{q_i} e^{\alpha \varphi(r)} \right) - \sum_{i=1}^k q_i \log \mu_i(B(x,r))}{\varphi(r)} = \alpha.$$

which implies that

$$\alpha_{\mu, \nu_{q,\alpha}}^{q,\varphi}(x) = \alpha \text{ for } \nu_{q,\alpha} \text{-a.e. } x$$

and then

$$b_{\mu,\varphi}^q(\nu_{q,\alpha}) = \alpha = B_{\mu,\varphi}^q(\nu_{q,\alpha}).$$

Question: Let $\mu_1, \mu_2, \dots, \mu_k$ be probability measures on \mathbb{R}^d , $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$ and $E \subseteq \mathbb{R}^d$. Then, the following problem remains open:

$$b_{\mu,\varphi}^q(E) = \sup \left\{ \bar{b}_{\mu,\varphi}^q(\nu); \nu \in \mathcal{P}(E) \right\} = \sup \left\{ \int \alpha_{\mu,\nu}^{q,\varphi}(x) d\nu(x); \nu \in \mathcal{P}(E) \right\}$$

and

$$B_{\mu,\varphi}^q(E) = \sup \left\{ \bar{B}_{\mu,\varphi}^q(\nu); \nu \in \mathcal{P}(E) \right\} = \sup \left\{ \int \bar{\alpha}_{\mu,\nu}^{q,\varphi}(x) d\nu(x); \nu \in \mathcal{P}(E) \right\}?$$

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