Available online at www.refaad.com
Gen. Lett. Math., 9(2) (2020), 106- 128
Research Article

General Letters in Mathematics (GLM)

Journal Homepage: http://www.refaad.com/views/GLM/home.aspx

ISSN: 2519-9277 (Online) 2519-9269 (Print)

Periodic solutions for nonlinear systems of multiple integro-integral differential equations of (V F) and (F V) type with isolated singular kernels

Raad N. Butris [a*], Hewa Selman Faris [b]

[a, b] University of Duhok, College of basic education, Department of mathematics, Kurdistan region, Iraq

Abstract

In this paper, the numerical-analytic method has been used to study the existence and approximation of the periodic solutions for the vector T-system of new nonlinear multiple integro-differential equations of mixed (Volterra-Fredholm) and (Fredholm-Volterra) types. Our main task provided sufficient conditions for investigating the family continuation theorems (numerical-analytic method and Banach fixed point theorem) in compact spaces for the existence of periodic solutions for the vector T-system of nonlinear multiple integro-differential equations. All functions satisfies a Hölder condition (Hölder inequality) of orders α, β and γ where 0<α, β, γ<1.

Keywords: *Existence; uniqueness and stability; periodic nonlinear T-system; numerical-analytic methods; Banach Fixed Point Theorem; Volterra and Fredholm; multiple integro-differential equations; Hölder inequality.*

1. Introduction

Integro differential equations, that have useful applications in dynamics, biology, mechanics, physics, chemistry and e.t.c., have wide utilizations of real life in Mathematical modeling. Recently, many types of Integro and Integral differential equations have been used to approximate the periodic solution of various different differential equations such as (Volterra-Fredholm) and (Fredholm-Volterra) and mixed Volterra and Fredholm [1,2,4,5,7,8,9,12,13,14].

Method of successive periodic approximations or numerical analytic method [10,11,20] due to the simplicity and possibilities clears to approximate construction of periodic solutions of integro-differential equations. The numerical-analytic method was introduced by Samoilenko A. M. [20], to investigate the periodic solutions for almost ordinary differential equations. Thus, the integro-differential equations which depending on the (Volterra and Fredholm) that we have extended in this study, becomes more general and detailed than those introduced by Butris R. N. [6].

In this work, we concentrate on finding periodic and approximate solutions of vector systems of first-order periodic differential equations, which are a combination of periodic system of first order Volterra and Fredholm integral equations. It has been successfully convened to solve different kinds of nonlinear problems in science which are including Volterra and Fredholm's integro-differential equations.

The problems to be studied are as follows:

$$\frac{dx}{dt} = \big(A_1 + B_1(t)\big)x + \big(A_2 + B_2(t)\big)y + f(t, x, y, u), \qquad \text{(V F)}$$

$$\frac{dy}{dt} = \big(C_1 + D_1(t)\big)x + \big(C_2 + D_2(t)\big)y + g(t, x, y, v), \qquad \text{(F V)}$$

where, $0 < \tau \le t \le T, x \in G_0, y \in G_1, u \in G_u, v \in G_v$ and G_0, G_1 are closed and bounded domains subset of R^n. Also G_v and G_u are bounded domains subset of R^m.

[*]Corresponding author
Email address: raad.butris@uod.ac (Raad N. Butris)
doi: 10.31559/glm2020.9.2.7
Received: 2 Sep 2019: Revised: 11 Nov 2020: Accepted 25 Nov 2020

Suppose that the vector functions $f(t,x,y,u)$ and $g(t,x,y,v)$ are defined on domains:

$$\begin{aligned}(t,x,y,u) &\in R^n \times G_0 \times G_1 \times G_u = (-\infty,\infty) \times R^{2n} \times R^m,\\ (t,x,y,v) &\in R^n \times G_0 \times G_1 \times G_v = (-\infty,\infty) \times R^{2n} \times R^m,\end{aligned} \tag{1}$$

where, $G_0: \|x - x_0\| \le r_x$, $G_1: \|y - y_0\| \le r_y$, $G_u: \|u\| \le d_u$ and $G_v: \|v\| \le d_v$ are continuous functions in x, y, u and v which are periodic in t of period T, that is $f(t+T,x,y,u) = f(t,x,y,u)$ and $g(t+T,x,y,v) = g(t,x,y,v)$.

Investigating the equations $(V\ F)$ and $(F\ V)$ verifies the periodic continuous vectors functions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ which are defined as follows:

$$x(t,x_0,y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big) - \Delta^1(t,x_0,y_0)\Big)\,ds, \tag{2}$$

where,

$$\Delta^1(t,x_0,y_0) = A_1 x_0 + \frac{A_1}{e^{A_1 T - I}}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big)\Big)\,ds, \tag{3}$$

and

$$y(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)}\Big(\big(C_1 + D_1(s)\big)x(s) + D_2(s)y(s) + g\big(s,x(s),y(s),v(s)\big) - \Delta^2(t,x_0,y_0)\Big)\,ds, \tag{4}$$

where,

$$\Delta^2(t,x_0,y_0) = C_2 y_0 + \frac{C_2}{e^{C_2 T - I}}\int_0^T e^{C_2(T-s)}\Big(\big(C_1 + D_1(s)\big)x(s) + D_2(s)y(s) + g\big(s,x(s),y(s),v(s)\big)\Big)\,ds. \tag{5}$$

In addition to the equations (2)-(5) we also have the following:

$$\begin{aligned}u(t) &= \int_{-\infty}^t \int_a^b K_1(t,s)\psi_1\big(t,s,x(s),y(s),\rho(s)\big)\,dt\,ds,\\ v(t) &= \int_a^b \int_{-\infty}^t K_2(t,s)\psi_2\big(t,s,x(s),y(s),\omega(s)\big)\,ds\,dt,\end{aligned} \tag{6}$$

$$\begin{aligned}\rho(s) &= \int_{h_1(s)}^{h_2(s)} \big(x(\tau) - y(\tau)\big)\,d\tau,\\ \omega(s) &= \int_{h_3(s)}^{h_4(s)} \big(x(\tau) - y(\tau)\big)\,d\tau.\end{aligned} \tag{7}$$

Assume that the vector functions, $f(t,x,y,u), g(t,x,y,v), \psi_1(t,s,x,y,\rho)$ and $\psi_2(t,s,x,y,\omega)$ satisfy the following inequalities

$$\|f(t,x,y,u)\| \le \vartheta_1, \|g(t,x,y,v)\| \le \vartheta_2, \tag{8}$$

$$\|f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2)\| \le \Gamma_1\|x_1 - x_2\|^\alpha + \Gamma_2\|y_1 - y_2\|^\beta + \Gamma_3\|u_1 - u_2\|^\gamma, \tag{9}$$

$$\|g(t,x_1,y_1,v_1) - g(t,x_2,y_2,v_2)\| \le \Sigma_1\|x_1 - x_2\|^\alpha + \Sigma_2\|y_1 - y_2\|^\beta + \Sigma_3\|v_1 - v_2\|^\gamma, \tag{10}$$

$$\|\psi_1(t,s,x_1,y_1,\rho_1) - \psi_1(t,s,x_2,y_2,\rho_2)\| \le h_1\|x_1 - x_2\|^\alpha + h_2\|y_1 - y_2\|^\beta + h_3\|\rho_1 - \rho_2\|^\gamma, \tag{11}$$

$$\|\psi_2(t,s,x_1,y_1,\omega_1) - \psi_2(t,s,x_2,y_2,\omega_2)\| \le l_1\|x_1 - x_2\|^\alpha + l_2\|y_1 - y_2\|^\beta + l_3\|\omega_1 - \omega_2\|^\gamma, \tag{12}$$

for all $t \in [0,T]$, $x, x_1, x_2 \in G_0$, $y, y_1, y_2 \in G_1$, $u, u_1, u_2 \in G_u$ and $v, v_1, v_2 \in G_v$, where $\vartheta_1, \vartheta_2, \Gamma_1, \Gamma_2, \Gamma_3, \Sigma_1, \Sigma_2, \Sigma_3, h_1, h_2, h_3, l_1, l_2$ and l_3 are positive constants and $0 < \alpha, \beta, \gamma < 1$.

The positive matrices $K_1(t,s)$ and $K_2(t,s)$ are the isolated singular kernels for the equations (V F) and (F V) such that

$$\begin{aligned}\|K_1(t,s)\| &\le \delta_1 e^{-\gamma_1(t-s)},\\ \|K_2(t,s)\| &\le \delta_2 e^{-\gamma_2(t-s)},\end{aligned} \tag{13}$$

Where, $\delta_1, \delta_2, \gamma_1$ and γ_2 are positive constants.
Also $A_1 = (A_{1ij})$, $A_2 = (A_{2ij})$, $B_1 = (B_{1ij})$, $B_2 = (B_{2ij})$, $C_1 = (C_{1ij})$, $C_2 = (C_{2ij})$, $D_1 = (D_{1ij})$ and $D_2 = (D_{2ij})$ are non-negative matrices, where $i, j = 1,2,\ldots,n$ and $\|\cdot\| = \max_{t\in[0,T]}|\cdot|$.

We define the non-empty sets by the following:

$$G_f = G_0 - r_x = G_0 - \varrho_1(t)\mathrm{R}_1 H_1^*, \tag{14}$$

$$G_g = G_1 - r_y = G_1 - \varrho_2(t)\mathrm{R}_2 H_2^*. \tag{15}$$

And

$$\|x_0\|\|e^{A_1 t} - I\| = Q_1, \|y_0\|\|e^{C_2 t} - I\| = Q_2, \tag{16}$$

$$\left\|e^{A_1(t-s)}\right\| \le \mathrm{R}_1, \left\|e^{C_2(t-s)}\right\| \le \mathrm{R}_2, \tag{17}$$

$$H_1 = \|h_2(t) - h_1(t)\|, H_2 = \|h_4(t) - h_3(t)\|, \tag{18}$$

$$H_1^* = \|B_1(t)\|\|x(t)\| + \|A_2 + B_2(t)\|\|y(t)\| + \vartheta_1, \tag{19}$$

$$H_2^* = \|C_1 + D_1(t)\|\|x(t)\| + \|D_2(t)\|\|y(t)\| + \vartheta_2. \tag{20}$$

Suppose that the sequences of periodic continuous vectors functions $\{x_m(t,x_0,y_0)\}_{m=1}^\infty$ and $\{y_m(t,x_0,y_0)\}_{m=1}^\infty$ are defined by the following:

$x_{m+1}(t,x_0,y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} \Big(B_1(s)x_m(s) + \big(A_2 + B_2(s)\big)y_m(s) + f\big(s,x_m(s),y_m(s),u_m(s)\big) - \Delta_m^1(t,x_0,y_0)\Big)\, ds$

With, $x(0,x_0,y_0) = x_0,\ m = 0,1,2,\ldots,$ (21)
where,

$$\Delta_m^1(t,x_0,y_0) = A_1 x_0 + \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)} \Big(B_1(s)x_m(s) + \big(A_2 + B_2(s)\big)y_m(s) + f\big(s,x_m(s),y_m(s),u_m(s)\big)\Big)\, ds, \quad (22)$$

$u_m(t,x_0,y_0) = \int_{-\infty}^t \int_a^b K_1(t,s)\psi_1\big(t,s,x_m(s),y_m(s),\rho_m(s)\big)\, dt\, ds$, $\rho_m(s) = \int_{h_1(s)}^{h_2(s)} \big(x_m(\tau) - y_m(\tau)\big)\, d\tau, m = 1,2,\ldots$ and
$y_{m+1}(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \Big(\big(C_1 + D_1(s)\big)x_m(s) + D_2(s)y_m(s) + g\big(s,x_m(s),y_m(s),v_m(s)\big) - \Delta_m^2(t,x_0,y_0)\Big)\, ds$
With, $y(0,x_0,y_0) = y_0,\ m = 0,1,2,\ldots,$ (23)
where,

$$\Delta_m^2(t,x_0,y_0) = C_2 y_0 + \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(T-s)} \Big(\big(C_1 + D_1(s)\big)x_m(s) + D_2(s)y_m(s) + g\big(s,x_m(s),y_m(s),v_m(s)\big)\Big)\, ds, \quad (24)$$

$v_m(t) = \int_a^b \int_{-\infty}^t K_2(t,s)\psi_2\big(t,s,x_m(s),y_m(s),\omega_m(t)\big)\, ds\, dt$, $\omega_m(t) = \int_{h_3(s)}^{h_4(s)} \big(x_m(\tau) - y_m(\tau)\big)\, d\tau, m = 1,2,\ldots$.

Suppose that the greatest Eigen-value of the matrix $\varphi_\gamma(T) = \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}$ less than one, that is

$$\max_\gamma \big(\varphi_\gamma(T)\big) = \frac{(\varphi_1(T)+\varphi_4(T)) + \sqrt{(\varphi_1(T)+\varphi_4(T))^2 - 4(\varphi_1(T)\varphi_4(T) - \varphi_2(T)\varphi_3(T))}}{2} < 1, \quad (25)$$

where,

$$\varphi_1(T) = \mathrm{R}_1 T\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\big(h_1 + h_3 H_1^\gamma\big)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^\gamma\right),$$

$$\varphi_2(t) = \mathrm{R}_1 T\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\big(h_2 + h_3 H_1^\gamma\big)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^\gamma\right),$$

$$\varphi_3(t) = \mathrm{R}_2 T\left(\|C_1 + D_1(t)\| + \Sigma_1 + \Sigma_3\left(\big(l_1 + l_3 H_2^\gamma\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma\right), \quad (26)$$

And $\varphi_4(t) = \mathrm{R}_2 T\left(\|D_2(t)\| + \Sigma_2 + \Sigma_3\left(\big(l_2 + l_3 H_2^\gamma\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma\right)$.

Also we obtain that the following:

$$\varrho_1(t) = \frac{\big(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|\big)t + \big(e^{\|A_1\|t} - \|I\|\big)T}{\big(e^{\|A_1\|T} - \|I\|\big)} \le T, \quad (27)$$

$$\varrho_2(t) = \frac{\big(e^{\|C_2\|T} - 2e^{\|C_2\|t} + \|I\|\big)t + \big(e^{\|C_2\|t} - \|I\|\big)T}{\big(e^{\|C_2\|T} - \|I\|\big)} \le T. \quad (28)$$

Definition 1 [3]: Let $\{\mathrm{f}_n\}$ be a sequence of real valued functions on a set, S. We say that $\{\mathrm{f}_n\}$ converges uniformly on S to the function, f and we write $\mathrm{f}_n \to \mathrm{f}$ uniformly, if for each, $\varepsilon > 0$ there exists N in $\mathbb{N}$ such that, $|\mathrm{f}_{N+p}(x) - \mathrm{f}(x)| < \varepsilon, \forall\, x \in S$ and all $p \in \mathbb{N}$.

Definition 2 [3]: Let $\mathrm{f}: \mathrm{S} \to \mathbb{R}$ is a function. We say that f satisfies a Lipschitz condition, if there is a constant $C > 0$ such that, $|\mathrm{f}(x_1) - \mathrm{f}(x_2)| \le C|x_1 - x_2|, \forall\, x_1, x_2 \in S$, where C is a Lipschitz constant.

Definition 3 [3]: Let $\mathrm{f}: \mathrm{S} \to \mathbb{R}$ be a function. We say that f is a contraction mapping, if it is Lipschitz with a Lipschitz constant, $0 < C < 1$.

Definition 4 [19]: A function f satisfies a Hölder condition (Hölder inequality) of order, β where, $0 < \beta < 1$, on $[a,b] \in \mathbb{R}$,if there is a constant $K > 0$, so that for all $x_1, x_2 \in [a,b]$, $|\mathrm{f}(x_1) - \mathrm{f}(x_2)| \le K|x_1 - x_2|^\beta$.

Definition 5 [19]: Suppose that $x(t)$ be a continuous solution of differential equation and $\hat{x}(t)$ any other solution. If for each, $\varepsilon > 0$, there exists $\delta > 0$, such that $\|x(t_0) - \hat{x}(t_0)\| < \delta$ for some t_0 that satisfies $\|x(t) - \hat{x}(t)\| < \varepsilon\ \forall\ t \ge t_0$.

Lemma 1 [19]: Let $a_i \in \mathbb{R}$ and $q \in (0,\infty)$, then we obtained that:

(1) If $a_i \ge 0$ and $q \ge 1$, then for $1 \le i \le m$, $\sum_{i=1}^m a_i^q \le \left(\sum_{i=1}^m a_i\right)^q \le m^{q-1}\sum_{i=1}^m a_i^q$.

The reverse holds if $0 < q \leq 1$. Hence, for $1 \leq i \leq m$, $(\sum_{i=1}^{m} a_i)^q \leq \sum_{i=1}^{m} a_i^q$.

(2) If $a_i, b_i \in \mathbb{R}$ and $0 < q \leq 1$, then for $1 \leq i \leq m$, $\|a_i - b_i\|^q \leq \|a_i - b_i\|$.

Lemma 2 [20]: Suppose that $f(t,x,y)$ be a continuous vector function on $[0,T]$. Then $\left\| \int_0^t \left(f(s,x(s),y(s)) - \frac{1}{T}\int_0^T f(s,x(s),y(s))\, ds \right) ds \right\| \leq \alpha(t) M$ holds, where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$ and $M = \max_{t\in[0,T]} \|f(t,x,y)\|, \forall\, t \in [0,T]$.

Theorem 1 [3]: Let S be a complete subset of, $\mathbb{R}$. A contraction mapping $I: S \to S$ has a unique fixed point.

Theorem 2 [3]: Suppose S be a Banach space and T is a contraction mapping in S. Then T has only one unique fixed point in S.

Lemma 3 [19]: Under the conditions and hypothesis for above, then the following inequalities hold:

i) $\|\rho_m(t) - \rho_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_1^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_1^\gamma$, (29)

and

ii) $\|\omega_m(t) - \omega_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_2^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_2^\gamma$. (30)

Theorem 3: Suppose that, $u(t)$, $v(t)$, $\psi_1(t,s,x,y,w)$ and $\psi_2(t,s,x,y,v)$ be continuous vector functions in the domain (1) and satisfy the inequalities (7), (8) and (9) and the relations (11) and (12). Then the following inequalities hold:

i) $\|u_m(t) - u_{m-1}(t)\| \leq \left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|x_m(t) - x_{m-1}(t)\| + \left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|y_m(t) - y_{m-1}(t)\|$,

ii) $\|v_m(t) - v_{m-1}(t)\|\ \left(l_1 + l_3 H_2^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|x_m(t) - x_{m-1}(t)\| + \left(l_2 + l_3 H_2^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|y_m(t) - y_{m-1}(t)\|$,

for all $t \in [0,T]$ and $m = 1,2,3, \dots$.

Lemma 4: Suppose that the vector functions $f(t,x,y,u)$ and $g(t,x,y,v)$ are defined and continuous on $[0,T]$ in t of T period. Then from the inequalities (27) and (28) also from lemma 1 we obtain that $\begin{pmatrix} \|E_1(t,x_0,y_0)\| \\ \|E_2(t,x_0,y_0)\| \end{pmatrix} \leq \begin{pmatrix} \varrho_1(t) \mathrm{R}_1 H^*_1 \\ \varrho_2(t) \mathrm{R}_2 H^*_2 \end{pmatrix}$ holds, where the equations (2)-(5) have derived to get the following

$$E_1(t,x_0,y_0) = \int_0^t e^{A_1(t-s)}\left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f\left(s,x(s),y(s),u(s)\right) - \left(\frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f\left(s,x(s),y(s),u(s)\right)\right) ds\right)\right) ds, \tag{31}$$

$$E_2(t,x_0,y_0) = \int_0^t e^{C_2(t-s)}\left(\left(C_1 + D_1(s)\right)x(s) + D_2(s)y(s) + g\left(s,x(s),y(s),v(s)\right) - \left(\frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(T-s)}\left(\left(C_1 + D_1(s)\right)x(s) + D_2(s)y(s) + g\left(s,x(s),y(s),v(s)\right)\right) ds\right)\right) ds, \tag{32}$$

for $0 \leq t \leq T$.

Proof: Depending on (2), (3) and (31) also by the inequalities (8) and (27) with condition (19) we have

$$\|E_1(t,x_0,y_0)\| \leq \left\| \int_0^t e^{A_1(t-s)}\left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f\left(s,x(s),y(s),u(s)\right)\right) ds - \int_0^t e^{A_1(t-s)}\left(\frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f\left(s,x(s),y(s),u(s)\right)\right) ds\right) ds \right\|,$$

$$\leq \left\| \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right.$$

$$- \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right.$$

$$\left. \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \right.$$

$$\left. \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \right\|$$

$$\leq \left\| \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right.$$

$$- \left[\frac{e^{A_1t}}{A_1} \right.$$

$$\left. \left. - \frac{I}{A_1} \right] \left(\frac{A_1}{(e^{A_1T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \right.$$

$$\left. \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \right\|$$

$$\leq \left\| \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right.$$

$$- \left[\frac{e^{A_1t}}{A_1} \frac{A_1}{(e^{A_1T} - I)} \right.$$

$$\left. \left. - \frac{I}{A_1} \frac{A_1}{(e^{A_1T} - I)} \right] \left(\int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \right.$$

$$\left. \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \right\|$$

$$\leq \left\| \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right.$$

$$\left. - \left[\frac{e^{A_1t} - I}{e^{A_1T} - I} \right] \left(\int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{e^{A_1T} - I} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \right.$$

$$\left. \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \right\|,$$

$$\leq \left\| \left(I - \frac{e^{A_1 t} - I}{e^{A_1 T} - I} \right) \int_0^t e^{A_1(t-s)} \left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f\left(s, x(s), y(s), u(s)\right) \right) ds \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{e^{A_1 T} - I} \int_t^T e^{A_1(T-s)} \left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) \right. \right. \right.$$

$$\left. \left. \left. + f\left(s, x(s), y(s), u(s)\right) \right) ds \right) ds \right\|,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} \right) t\mathrm{R}_1 H_1^* + \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} (T - t)\mathrm{R}_1 H_1^*,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} \right) t\mathrm{R}_1 H_1^* + \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} T\mathrm{R}_1 H_1^* - \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} t\mathrm{R}_1 H_1^*,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} - \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} \right) t\mathrm{R}_1 H_1^* + \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} T\mathrm{R}_1 H_1^*,$$

$$\leq \left(\|I\| - 2\frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} \right) t\mathrm{R}_1 H_1^* + \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} T\mathrm{R}_1 H_1^*,$$

$$\leq \left(\left(\|I\| - 2\frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} \right) t + \frac{e^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - \|I\|)} T \right) \mathrm{R}_1 H_1^*,$$

$$\leq \left(\frac{(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|)t + (e^{\|A_1\|t} - \|I\|)T}{e^{\|A_1\|T} - \|I\|} \right) \mathrm{R}_1 H_1^* \leq \varrho_1(t)\mathrm{R}_1 H_1^*.$$

Also base on (4), (5) and (32) and by the inequalities (8) and (28) with condition (20) we have

$$\|E_2(t, x_0, y_0)\| = \left\| \int_0^t e^{C_2(t-s)} \left(\left(C_1 + D_1(s)\right)x(s) + D_2(s)y(s) + g\left(s, x(s), y(s), v(s)\right) \right. \right.$$

$$\left. \left. - \left(\frac{C_2}{(e^{C_2 T} - I)} \int_0^T e^{C_2(T-s)} \left(\left(C_1 + D_1(s)\right)x(s) + D_2(s)y(s) + g\left(s, x(s), y(s), v(s)\right) \right) ds \right) \right) ds \right\|,$$

$$\leq \left(\frac{(e^{\|C_2\|T} - 2e^{\|C_2\|t} + \|I\|)t + (e^{\|C_2\|t} - \|I\|)T}{(e^{\|C_2\|T} - \|I\|)} \right) \mathrm{R}_2 H_2^* \leq \varrho_2(t)\mathrm{R}_2 H_2^*.$$

Forming the results of $\|E_1(t, x_0, y_0)\|$ and $\|E_2(t, x_0, y_0)\|$ in a vector form we obtain the inequality,

$$\begin{pmatrix} \|E_1(t, x_0, y_0)\| \\ \|E_2(t, x_0, y_0)\| \end{pmatrix} \leq \begin{pmatrix} \varrho_1(t)\mathrm{R}_1 H_1^* \\ \varrho_2(t)\mathrm{R}_2 H_2^* \end{pmatrix}.$$

2. Approximation of periodic solutions of equations (V F) and (F V)

The approximation of periodic solution for system (V F), (V F) have introduced by the following theorem:

Theorem 4: Let the vector functions $f(t, x, y, u)$ and $g(t, x, y, v)$ are defined and continuous on the domain (1) and satisfy the inequalities (4)-(11) and the conditions (18)-(20) and (25). Then the sequences of functions (21) and (23) in t of period T are uniformly converge as $m \to \infty$ to the limit functions $x^o(t, x_0, y_0)$ and $y^o(t, x_0, y_0)$ respectively and satisfying the equations (2)-(4) which are unique solutions of (V F) and (F V) from (26), (27) and (28) by the following inequality:

$$\begin{pmatrix} \|x^o(t, x_0, y_0) - x_0(t, x_0, y_0)\| \\ \|y^o(t, x_0, y_0) - y_0(t, x_0, y_0)\| \end{pmatrix} \leq \begin{pmatrix} \varrho_1(t)\mathrm{R}_1 H_1^* \\ \varrho_2(t)\mathrm{R}_2 H_2^* \end{pmatrix}, \tag{33}$$

$$\begin{pmatrix} \|x^o(t, x_0, y_0) - x_m(t, x_0, y_0)\| \\ \|y^o(t, x_0, y_0) - y_m(t, x_0, y_0)\| \end{pmatrix} \leq \varphi_\gamma^m(T) \left(E - \varphi_\gamma(T) \right)^{-1} \Omega_1(T), \tag{34}$$

Where, $\Omega_1(T) = \begin{pmatrix} \varrho_1(t)\mathrm{R}_1 H_1^* \\ \varrho_2(t)\mathrm{R}_2 H_2^* \end{pmatrix}$ and E is an identity matrix.

Proof: The sequences of functions $\{x_m(t, x_0, y_0)\}_{m=1}^{\infty}$ and $\{y_m(t, x_0, y_0)\}_{m=1}^{\infty}$ are defined on (21) and (23), continuous on the domain (1) and periodic in t of period T. Firstly, by lemma (4) and from (21) when, $m = 0$, we have

$$\|x_1(t,x_0,y_0)-x_0\|$$

$$=\Bigg\|x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\Bigg(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)-\Bigg(A_1x_0+\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds\Bigg)\Bigg)ds-x_0\Bigg\|,$$

$$=\Bigg\|x_0e^{A_1t}-A_1x_0\int_0^t e^{A_1(t-s)}\,ds-x_0+\int_0^t e^{A_1(t-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds-\int_0^t e^{A_1(t-s)}\Bigg(\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds\Bigg)ds\Bigg\|,$$

$$=\Bigg\|x_0e^{A_1t}-A_1x_0\left[\frac{e^{A_1t}}{A_1}-\frac{I}{A_1}\right]-x_0+\int_0^t e^{A_1(t-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds-\int_0^t e^{A_1(t-s)}\Bigg(\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds\Bigg)ds\Bigg\|,$$

$$\leq\Bigg\|\int_0^t e^{A_1(t-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds-\int_0^t e^{A_1(t-s)}\Bigg(\frac{A_1}{(e^{A_1T}-I)}\int_0^t e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds\Bigg)ds\Bigg\|$$

$$+\Bigg\|\int_0^t e^{A_1(t-s)}\Bigg(\frac{A_1}{(e^{A_1T}-I)}\int_t^T e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds\Bigg)ds\Bigg\|,$$

$$\leq \left\| \left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)x_0(s) + \big(A_2 + B_2(s)\big) y_0(s) + f\big(s, x_0(s), y_0(s), u_0(s)\big) \Big) ds \right\|$$

$$+ \left\| \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x_0(s) + \big(A_2 + B_2(s)\big) y_0(s) \right.\right.$$

$$\left.\left. + f\big(s, x_0(s), y_0(s), u_0(s)\big) \Big) ds \right) ds \right\|,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) \int_0^t \|e^{A_1(t-s)}\| \big\| B_1(t)x(t) + \big(A_2 + B_2(t)\big) y(t) + f\big(t, x(t), y(t), u(t)\big) \big\| ds$$

$$+ \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \int_t^T \|e^{A_1(T-s)}\| \big\| B_1(t)x(t) + \big(A_2 + B_2(t)\big) y(t) + f\big(t, x(t), y(t), u(t)\big) \big\| ds,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) t\mathrm{R}_1 H^*_1 + \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} (T - t)\mathrm{R}_1 H^*_1,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) t\mathrm{R}_1 H^*_1 + \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} T\mathrm{R}_1 H^*_1 - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} t\mathrm{R}_1 H^*_1,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} - \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) t\mathrm{R}_1 H^*_1 + \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} T\mathrm{R}_1 H^*_1,$$

$$\leq \left(\|I\| - 2\frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) t\mathrm{R}_1 H^*_1 + \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} T\mathrm{R}_1 H^*_1,$$

$$\leq \left(\left(\|I\| - 2\frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} \right) t + \frac{e^{\|A_1\| t} - \|I\|}{(e^{\|A_1\| T} - \|I\|)} T \right) \mathrm{R}_1 H^*_1,$$

$$\leq \left(\frac{(e^{\|A_1\| T} - \|I\|)t - 2(e^{\|A_1\| t} - \|I\|)t + (e^{\|A_1\| t} - \|I\|)T}{(e^{\|A_1\| T} - \|I\|)} \right) \mathrm{R}_1 H^*_1,$$

$$\leq \left(\frac{(e^{\|A_1\| T} - 2e^{\|A_1\| t} + \|I\|)t + (e^{\|A_1\| t} - \|I\|)T}{(e^{\|A_1\| T} - \|I\|)} \right) \mathrm{R}_1 H^*_1,$$

$$\leq \left(\frac{(e^{\|A_1\| T} - 2e^{\|A_1\| t} + \|I\|)t + (e^{\|A_1\| t} - \|I\|)T}{(e^{\|A_1\| T} - \|I\|)} \right) \mathrm{R}_1 H^*_1 = \varrho_1(t)\mathrm{R}_1 H^*_1.$$

Hence, $\|x_1(t, x_0, y_0) - x_0\| \leq \varrho_1(t)\mathrm{R}_1 H^*_1$.
Also, from (23) and lemma (4) and the same above steps we obtain that, $\|y_1(t, x_0, y_0) - y_0\| \leq \varrho_2(t)\mathrm{R}_2 H^*_2$.
For, $m \geq 1$, and by mathematical induction, we have obtained the following inequalities:

$$\|x_m(t, x_0, y_0) - x_0\| \leq \varrho_1(t)\mathrm{R}_1 H^*_1, \tag{35}$$

$$\|y_m(t, x_0, y_0) - y_0\| \leq \varrho_2(t)\mathrm{R}_2 H^*_2. \tag{36}$$

Then from (31), (32) and for all $t \in [0, T]$ where, $x_0 \in G_f$, $y_0 \in G_g$, then $x_m(t, x_0, y_0) \in G_0$ and $y_m(t, x_0, y_0) \in G_1$.

Now, we have to prove that the sequences $\{x_m(t, x_0, y_0)\}_{m=0}^{\infty}$ and $\{y_m(t, x_0, y_0)\}_{m=0}^{\infty}$ are uniformly convergent on (1). Thus, by lemma (4) and (21)-(24) when, $m = 1$, we obtain that

$$\|x_2(t, x_0, y_0) - x_1(t, x_0, y_0)\|$$

$$\leq \mathrm{R}_1\varrho_1(t) \left(\|B_1(t)\| + \Gamma_1 + \Gamma_3 \left(\big(h_1 + h_3 H_1^{\gamma}\big) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|x_1 - x_0\|$$

$$+ \mathrm{R}_1\varrho_1(t) \left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3 \left(\big(h_2 + h_3 H_1^{\gamma}\big) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|y_1 - y_0\|.$$

And

$$\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\| \le \mathrm{R}_2\varrho_2(t)\left(\|C_1+D_1(t)\|+\Sigma_1+\Sigma_3\left(\left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|x_1(t)-x_0\| + \mathrm{R}_2\varrho_2(t)\left(\|D_2(t)\|+\Sigma_2+\Sigma_3\left(\left(l_2+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|y_1(t)-y_0\|.$$

Hence for $m>1$, and by mathematical induction, we can prove the following inequalities:

$$\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \le \mathrm{R}_1\varrho_1(t)\left(\|B_1(t)\|+\Gamma_1+\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| + \mathrm{R}_1\varrho_1(t)\left(\|A_2+B_2(t)\|+\Gamma_2+\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|, \tag{37}$$

$$\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\| \le \mathrm{R}_2\varrho_2(t)\left(\|C_1+D_1(t)\|+\Sigma_1+\Sigma_3\left(\left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| + \mathrm{R}_2\varrho_2(t)\left(\|D_2(t)\|+\Sigma_2+\Sigma_3\left(\left(l_2+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|. \tag{38}$$

So, from (37) and (38) we obtain the system:

$$\begin{pmatrix}\|x_{m+1}(t)-x_m(t)\|\\ \|y_{m+1}(t)-y_m(t)\|\end{pmatrix} \le \begin{pmatrix}\varphi_1(t) & \varphi_2(t)\\ \varphi_3(t) & \varphi_4(t)\end{pmatrix}\begin{pmatrix}\|x_m(t)-x_{m-1}(t)\|\\ \|y_m(t)-y_{m-1}(t)\|\end{pmatrix}. \tag{39}$$

Getting the maximum value t of two sides, and by iterated recurrence on (39) we obtain that

$$\begin{pmatrix}\|x_{m+1}(T)-x_m(T)\|\\ \|y_{m+1}(T)-y_m(T)\|\end{pmatrix} \le \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x_m(T)-x_{m-1}(T)\|\\ \|y_m(T)-y_{m-1}(T)\|\end{pmatrix} = \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^2\begin{pmatrix}\|x_{m-1}(T)-x_{m-2}(T)\|\\ \|y_{m-1}(T)-y_{m-2}(T)\|\end{pmatrix},$$
$$= \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^3\begin{pmatrix}\|x_{m-2}(T)-x_{m-3}(T)\|\\ \|y_{m-2}(T)-y_{m-3}(T)\|\end{pmatrix}\dots = \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^m\begin{pmatrix}\|x_1(T)-x_0(T)\|\\ \|y_1(T)-y_0(T)\|\end{pmatrix}.$$

So, we get

$$\begin{pmatrix}\|x_{m+1}(T)-x_m(T)\|\\ \|y_{m+1}(T)-y_m(T)\|\end{pmatrix} \le \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^m\begin{pmatrix}\varrho_1(T)\mathrm{R}_1H^*_1\\ \varrho_2(T)\mathrm{R}_2H^*_2\end{pmatrix}. \quad \dots(40)$$

From (40) and for any $l\ge 0$ we conclude that

$$\begin{pmatrix}\|x_{m+l}(T)-x_m(T)\|\\ \|y_{m+l}(T)-y_m(T)\|\end{pmatrix} \le \begin{pmatrix}\|x_{m+l}(T)-x_{m+l-1}(T)\|\\ \|y_{m+l}(T)-y_{m+l-1}(T)\|\end{pmatrix} + \begin{pmatrix}\|x_{m+l-1}(T)-x_{m+l-2}(T)\|\\ \|y_{m+l-1}(T)-y_{m+l-2}(T)\|\end{pmatrix} + \dots + \begin{pmatrix}\|x_{m+1}(T)-x_m(T)\|\\ \|y_{m+1}(T)-y_m(T)\|\end{pmatrix},$$
$$\le \left(\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{m+l-1} + \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{m+l-2} + \dots + \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{m}\right)\begin{pmatrix}\|x_1(T)-x_0(T)\|\\ \|y_1(T)-y_0(T)\|\end{pmatrix},$$
$$\le \left(\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{l-1} + \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{l-2} + \dots + E\right)\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{m}\begin{pmatrix}\|x_1(T)-x_0(T)\|\\ \|y_1(T)-y_0(T)\|\end{pmatrix}.$$

Then we obtain that

$$\begin{pmatrix}\|x_{m+l}(T)-x_m(T)\|\\ \|y_{m+l}(T)-y_m(T)\|\end{pmatrix} \le \left(\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{l-1} + \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{l-2} + \dots + E\right)\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}^{m}\begin{pmatrix}\|x_1(T)-x_0(T)\|\\ \|y_1(T)-y_0(T)\|\end{pmatrix}.$$

Rewrite this inequality in a vector form to obtain, $\Omega_{m+l}(\mathrm{T}) \le \left(E-\varphi_\gamma(T)\right)^{-1}\varphi_\gamma^m(T)\Omega_1(T)$, where $\Omega_{\mathrm{m}+1}(t) = \begin{pmatrix}\|x_{m+l}(T)-x_m(T)\|\\ \|y_{m+l}(T)-y_m(T)\|\end{pmatrix}$, $\varphi_\gamma(T) = \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}$ and $\Omega_1(T) = \begin{pmatrix}\varrho_1(t)\mathrm{R}_1H^*_1\\ \varrho_2(t)\mathrm{R}_2H^*_2\end{pmatrix}$.

Hence, by (25) we obtain that, $\lim_{n\to\infty}\varphi_\gamma^m(t)=0$.

Then the sequence of functions $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ converge uniformly on the domains (14) and (15).

Let $\lim_{m\to\infty} x_m(t) = x(t)$ and $\lim_{m\to\infty} y_m(t) = y(t)$, then from this we can prove that the inequalities (33) and (34) are true for all $m\ge 0$ and hence $x(t)$ and $y(t)$ are periodic in t of period T.

3. Uniqueness solution of integro-differential equations of (VF) and (FV) types

The uniqueness solution of the periodic system (VF), (FV) is given by the following theorem.

Theorem 5: With all hypotheses and all conditions of theorem 4, the periodic solution of system (V F), (F V) is unique on the domains (14) and (15) in t of T period.

Proof: Let $r(t,x_0,y_0)$ and $w(t,x_0,y_0)$ be another periodic solution of integro-differential equations (V F) and (F V), i. e.

$$r(t,x_0,y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} \Bigg(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big) - \Bigg(A_1 x_0 + \frac{A_1}{(e^{A_1 T} - I)} \int_0^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big)\Big)\, ds \Bigg)\Bigg) ds,$$

With $x(0,x_0,y_0) = x_0,\ m = 0,1,2,\ldots,$ (41)

Where, $u(t,x_0,y_0) = \int_{-\infty}^{t}\int_a^b K_1(t,s)\psi_1\big(t,s,r(s),w(s),\rho(s)\big)\,dt\,ds,\ \rho(s) = \int_{h_1(s)}^{h_2(s)}\big(r(\tau) - w(\tau)\big)\,d\tau, m = 1,2,\ldots$ and

$$w(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \Bigg(\big(C_1 + D_1(s)\big)r(s) + D_2(s)w(s) + g\big(s,r(s),w(s),v(s)\big) - \Bigg(C_2 y_0 + \frac{C_2}{(e^{C_2 T} - I)} \int_0^T e^{C_2(T-s)} \Big(\big(C_1 + D_1(s)\big)r(s) + D_2(s)w(s) + g\big(s,r(s),w(s),v(s)\big)\Big)\, ds \Bigg)\Bigg) ds,$$

With $y(0,x_0,y_0) = y_0,\ m = 0,1,2,\ldots,$ (42)

Where, $v(t) = \int_a^b \int_{-\infty}^{t} K_2(t,s)\psi_2\big(t,s,r(s),w(s),\omega(t)\big)\,ds\,dt, \omega(t) = \int_{h_3(s)}^{h_4(s)}\big(r(\tau) - w(\tau)\big)\,d\tau, m = 1,2,\ldots.$

$$
\begin{aligned}
&\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\
&\qquad = \Bigg\| x_0 e^{A_1 t} \\
&\qquad + \int_0^t e^{A_1(t-s)} \Bigg(B_1(s)x(s) + \big(A_2+B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big) \\
&\qquad - \Bigg(A_1 x_0 + \frac{A_1}{(e^{A_1T}-I)} \int_0^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2+B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big)\Big)\, ds \Bigg)\Bigg) ds \\
&\qquad - \Bigg(x_0 e^{A_1 t} \\
&\qquad + \int_0^t e^{A_1(t-s)} \Bigg(B_1(s)r(s) + \big(A_2+B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big) \\
&\qquad - \Bigg(A_1 x_0 \\
&\qquad + \frac{A_1}{(e^{A_1T}-I)} \int_0^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2+B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big)\Big)\, ds \Bigg)\Bigg) ds \Bigg)\Bigg\|, \\
&= \Bigg\| \int_0^t e^{A_1(t-s)} \Bigg(B_1(s)x(s) + \big(A_2+B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big) \\
&\qquad - \Bigg(\frac{A_1}{(e^{A_1T}-I)} \int_0^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2+B_2(s)\big)y(s) + f\big(s,x(s),y(s),u(s)\big)\Big)\, ds \Bigg)\Bigg) ds \\
&\qquad - \Bigg(\int_0^t e^{A_1(t-s)} \Bigg(B_1(s)r(s) + \big(A_2+B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big) \\
&\qquad - \Bigg(\frac{A_1}{(e^{A_1T}-I)} \int_0^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2+B_2(s)\big)w(s) + f\big(s,r(s),w(s),u_r(s)\big)\Big)\, ds \Bigg)\Bigg) ds \Bigg)\Bigg\|,
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg\| & \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \\
& \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \\
& \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \\
& - \left(\int_0^t e^{A_1(t-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \right. \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) \right. \\
& \left. + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \right) ds \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) \right. \\
& \left. \left. + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \right) ds \right) \Bigg\|,
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg\| & \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \\
& - \left[\frac{e^{A_1 t}}{A_1} - \frac{I}{A_1}\right] \frac{A_1}{(e^{A_1 T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) \right. \\
& \left. + f\big(s, x(s), y(s), u(s)\big)\Big)\, ds \right) ds \\
& - \left(\int_0^t e^{A_1(t-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \right. \\
& - \left[\frac{e^{A_1 t}}{A_1} - \frac{I}{A_1}\right] \frac{A_1}{(e^{A_1 T} - I)} \int_0^t e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \\
& - \int_0^t e^{A_1(t-s)} \left(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) \right. \\
& \left. \left. + f\big(s, r(s), w(s), u_r(s)\big)\Big)\, ds \right) ds \right) \Bigg\|,
\end{aligned}
$$

$$= \Bigg\| \left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big) \Big)\, ds$$

$$- \int_0^t e^{A_1(t-s)} \Bigg(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s)$$

$$+ f\big(s, x(s), y(s), u(s)\big) \Big)\, ds \Bigg)\, ds$$

$$- \Bigg(\left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds$$

$$- \int_0^t e^{A_1(t-s)} \Bigg(\frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s)$$

$$+ f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds \Bigg)\, ds \Bigg) \Bigg\|,$$

$$= \Bigg\| \left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big) \Big)\, ds$$

$$- \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big) \Big)\, ds$$

$$- \Bigg(\left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds$$

$$- \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s)$$

$$+ f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds \Bigg) \Bigg\|,$$

$$\le \Bigg\| \left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big) \Big)\, ds$$

$$- \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)x(s) + \big(A_2 + B_2(s)\big)y(s) + f\big(s, x(s), y(s), u(s)\big) \Big)\, ds \Bigg\|$$

$$+ \Bigg\| \left(I - \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \right) \int_0^t e^{A_1(t-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds$$

$$- \frac{e^{A_1 t} - I}{(e^{A_1 T} - I)} \frac{A_1}{(e^{A_1 T} - I)} \int_t^T e^{A_1(T-s)} \Big(B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big) \Big)\, ds \Bigg\|,$$

$$\leq \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) \int_0^t \|e^{A_1(t-s)}\| \|B_1(t)x(t) + (A_2 + B_2(t))y(t) + f(t, x(t), y(t), u(t))\| \, ds$$

$$+ \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|} \int_t^T \|e^{A_1(T-s)}\| \|B_1(t)x(t) + (A_2 + B_2(t))y(t) + f(t, x(t), y(t), u(t))\| \, ds$$

$$+ \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) \int_0^t \|e^{A_1(t-s)}\| \|B_1(t)r(t) + (A_2 + B_2(t))w(t) + f(t, r(t), w(t), u_r(t))\| \, ds$$

$$+ \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|} \int_t^T \|e^{A_1(T-s)}\| \|B_1(t)r(t) + (A_2 + B_2(t))w(t) + f(t, r(t), w(t), u_r(t))\| \, ds,$$

$$\leq \mathrm{R}_1 t \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) (\|B_1(t)\| \|x(t) - r(t)\| + \|A_2 + B_2(t)\| \|y(t) - w(t)\|$$
$$+ \|f(t, x(t), y(t), u(t)) - f(t, r(t), w(t), u_r(t))\|)$$
$$+ \mathrm{R}_1 T \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) (\|B_1(t)\| \|x(t) - r(t)\| + \|A_2 + B_2(t)\| \|y(t) - w(t)\|$$
$$+ \|f(t, x(t), y(t), u(t)) - f(t, r(t), w(t), u_r(t))\|)$$
$$- \mathrm{R}_1 t \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) (\|B_1(t)\| \|x(t) - r(t)\| + \|A_2 + B_2(t)\| \|y(t) - w(t)\|$$
$$+ \|f(t, x(t), y(t), u(t)) - f(t, r(t), w(t), u_r(t))\|),$$

$$\leq \left(\mathrm{R}_1 t \left(\|I\| - 2\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) + \mathrm{R}_1 T \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) \right) (\|B_1(t)\| \|x(t) - r(t)\| + \|A_2 + B_2(t)\| \|y(t) - w(t)\|$$
$$+ \|f(t, x(t), y(t), u(t)) - f(t, r(t), w(t), u_r(t))\|),$$

$$\leq \mathrm{R}_1 \left(\frac{(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|)t + (e^{\|A_1\|t} - \|I\|)T}{(e^{\|A_1\|T} - \|I\|)}\right) \Bigg(\|B_1(t)\| \|x(t) - r(t)\| + \|A_2 + B_2(t)\| \|y(t) - w(t)\|$$
$$+ \Gamma_1 \|x(t) - r(t)\| + \Gamma_2 \|y(t) - w(t)\|$$
$$+ \Gamma_3 \left((h_1 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \|x(t) - r(t)\|$$
$$+ \Gamma_3 \left((h_2 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \|y(t) - w(t)\| \Bigg),$$

$$\leq \mathrm{R}_1 \left(\frac{(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|)t + (e^{\|A_1\|t} - \|I\|)T}{(e^{\|A_1\|T} - \|I\|)}\right) \Bigg(\|B_1(t)\| + \Gamma_1$$
$$+ \Gamma_3 \left((h_1 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \Bigg) \|x(t) - r(t)\|$$
$$+ \mathrm{R}_1 \left(\frac{(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|)t + (e^{\|A_1\|t} - \|I\|)T}{(e^{\|A_1\|T} - \|I\|)}\right) \Bigg(\|A_2 + B_2(t)\| + \Gamma_2$$
$$+ \Gamma_3 \left((h_2 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \Bigg) \|y(t) - w(t)\|,$$

$$\leq \mathrm{R}_1 \varrho_1(t) \left(\|B_1(t)\| + \Gamma_1 + \Gamma_3 \left((h_1 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|x(t) - r(t)\|$$
$$+ \mathrm{R}_1 \varrho_1(t) \left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3 \left((h_2 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|y(t) - w(t)\|.$$

Therefore, we obtain that

$\|x(t, x_0, y_0) - r(t, x_0, y_0)\|$

$$\leq \mathrm{R}_1 \varrho_1(t) \left(\|B_1(t)\| + \Gamma_1 + \Gamma_3 \left((h_1 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|x(t) - r(t)\|$$
$$+ \mathrm{R}_1 \varrho_1(t) \left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3 \left((h_2 + h_3 H_1^{\gamma}) \left(\frac{\delta_1}{\gamma^2{}_1} (e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 t} \right) \right)^{\gamma} \right) \|y(t) - w(t)\|,$$

$$\|x(t, x_0, y_0) - r(t, x_0, y_0)\| \leq \varphi_1(T) \|x(t, x_0, y_0) - r(t, x_0, y_0)\| + \varphi_2(T) \|y(t, x_0, y_0) - w(t, x_0, y_0)\|. \quad (43)$$

Same steps are done for finding, $\|y(t, x_0, y_0) - w(t, x_0, y_0)\|$. Thus

$$\|y(t,x_0,y_0)-w(t,x_0,y_0)\| \le \varphi_3(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\|+\varphi_4(T)\|y(t,x_0,y_0)-w(t,x_0,y_0)\|. \tag{44}$$

Rewrite (43) and (44) in a vector form, we obtain that

$$\begin{pmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\|\end{pmatrix}.$$

Hence from (25) the greatest Eigen value of matrix $\varphi_\gamma(T)$ is less than one, so we conclude that $x(t)=r(t)$ and, $y(t)=w(t)$. This implies that the system (VF), (FV) has a unique solution.

4. Existence of periodic solution of integro-differential equations of (V F) and (F V) types

The existence solution of the system (V F), (F V) in t of period T is uniquely linked with the existence of zeros of the functions $\Delta^1(t,x_0,y_0)\in G_{f1}\times G_{g1}\to R$ and $\Delta^2(t,x_0,y_0)\in G_{f2}\times G_{g2}\to R$ and defined by (3) and (5)-(7) respectively where, $u(t)$, $v(t)$, $\rho(s)$ and $\omega(s)$ are defined by (6) and (7). From the approximate solutions (3) and (5) we obtain the sequences of functions (22) and (24).

Theorem 6: Suppose that all assumptions and conditions of theorem 4 are satisfied, thus the following inequality holds

$$\begin{pmatrix}\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|\\ \|\Delta^2(t,x_0,y_0)-\Delta^2_m(t,x_0,y_0)\|\end{pmatrix} \le \mathrm{J}\varphi_\gamma^{m+1}(T)\left(E-\varphi_\gamma(T)\right)^{-1}\Omega_1(T), \tag{45}$$

where, $\Omega_1(T)=\begin{pmatrix}\varrho_1(T)\mathrm{R}_1H^*_1\\ \varrho_2(T)\mathrm{R}_2H^*_2\end{pmatrix}$, E is an identity matrix, $J=(J_1,J_2)$, $J_1=\frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|}$ and $J_2=\frac{\|C_2\|}{e^{\|C_2\|T}-\|I\|}$ for $m\ge 0$.

Proof: From the equations (3) and (22) we have

$$\begin{aligned}
&\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|\\
&\quad=\Bigg\|A_1x_0+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x(s)+\big(A_2+B_2(s)\big)y(s)+f\big(s,x(s),y(s),u(s)\big)\Big)ds\\
&\quad-\Bigg(A_1x_0\\
&\quad+\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_m(s)+\big(A_2+B_2(s)\big)y_m(s)+f\big(s,x_m(s),y_m(s),u_m(s)\big)\Big)ds\Bigg)\Bigg\|,\\
&=\Bigg\|\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x(s)+\big(A_2+B_2(s)\big)y(s)+f\big(s,x(s),y(s),u(s)\big)\Big)ds\\
&\quad-\Bigg(\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_m(s)+\big(A_2+B_2(s)\big)y_m(s)+f\big(s,x_m(s),y_m(s),u_m(s)\big)\Big)ds\Bigg)\Bigg\|,\\
&=\Bigg\|\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(\Big(B_1(s)x(s)+\big(A_2+B_2(s)\big)y(s)+f\big(s,x(s),y(s),u(s)\big)\Big)\\
&\quad-\Big(B_1(s)x_m(s)+\big(A_2+B_2(s)\big)y_m(s)+f\big(s,x_m(s),y_m(s),u_m(s)\big)\Big)\Big)ds\Bigg\|,\\
&\le\frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|}\int_0^T\|e^{A_1(T-s)}\|\big(\|B_1(t)\|\|x(t)-x_m(t)\|+\|A_2+B_2(t)\|\|y(t)-y_m(t)\|\\
&\quad+\big\|f\big(t,x(t),y(t),u(t)\big)-f\big(t,x_m(t),y_m(t),u_m(t)\big)\big\|\big)ds,\\
&\le J_1R_1T\big(\|B_1(t)\|\|x(t)-x_m(t)\|+\|A_2+B_2(t)\|\|y(t)-y_m(t)\|\\
&\quad+\big\|f\big(t,x(t),y(t),u(t)\big)-f\big(t,x_m(t),y_m(t),u_m(t)\big)\big\|\big),\\
&\le J_1R_1T\big(\|B_1(t)\|\|x(t)-x_m(t)\|+\|A_2+B_2(t)\|\|y(t)-y_m(t)\|+\Gamma_1\|x(t)-x_m(t)\|^\alpha+\Gamma_2\|y(t)-y_m(t)\|^\beta\\
&\quad+\Gamma_3\|u(t)-u_m(t)\|^\gamma\big),
\end{aligned}$$

$$\leq J_1R_1T\Bigg(\|B_1(t)\|\|x(t)-x_m(t)\|+\|A_2+B_2(t)\|\|y(t)-y_m(t)\|+\Gamma_1\|x(t)-x_m(t)\|^{\alpha}+\Gamma_2\|y(t)-y_m(t)\|^{\beta}$$
$$+\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|x(t)-x_m(t)\|^{\gamma}$$
$$+\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|y(t)-y_m(t)\|^{\gamma}\Bigg),$$
$$\leq J_1R_1T\Bigg(\|B_1(t)\|\|x(t)-x_m(t)\|+\|A_2+B_2(t)\|\|y(t)-y_m(t)\|+\Gamma_1\|x(t)-x_m(t)\|+\Gamma_2\|y(t)-y_m(t)\|$$
$$+\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|x(t)-x_m(t)\|$$
$$+\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|y(t)-y_m(t)\|\Bigg),$$
$$\leq J_1R_1T\left(\|B_1(t)\|+\Gamma_1+\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|x(t)-x_m(t)\|$$
$$+J_1R_1T\left(\|A_2+B_2(t)\|+\Gamma_2+\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|y(t)-y_m(t)\|.$$

Therefore, we get

$\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|$
$$\leq J_1R_1T\left(\|B_1(t)\|+\Gamma_1+\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|x(t)-x_m(t)\|$$
$$+J_1R_1T\left(\|A_2+B_2(t)\|+\Gamma_2+\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\right)\|y(t)-y_m(t)\|,$$
$$\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|\leq J_1\varphi_1(t)\|x(t)-x_m(t)\|+J_1\varphi_2(t)\|y(t)-y_m(t)\|. \quad (46)$$

Under the same inequalities and conditions and from (5) and (24) we use the same steps to get

$$\|\Delta^2(t,x_0,y_0)-\Delta^2_m(t,x_0,y_0)\|\leq J_2\varphi_1(t)\|x(t)-x_m(t)\|+J_2\varphi_2(t)\|y(t)-y_m(t)\|. \quad (47)$$

Then from (46) and (47) and by rewriting them in a vector form we obtain (34) as follows

$$\begin{pmatrix}\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|\\ \|\Delta^2(t,x_0,y_0)-\Delta^2_m(t,x_0,y_0)\|\end{pmatrix}=\begin{pmatrix}J_1\varphi_1(t)J_1\varphi_2(t)\\ J_2\varphi_1(t)J_2\varphi_2(t)\end{pmatrix}\begin{pmatrix}\|x(t)-x_m(t)\|\\ \|y(t)-y_m(t)\|\end{pmatrix},$$
$$\begin{pmatrix}\|\Delta^1(t,x_0,y_0)-\Delta^1_m(t,x_0,y_0)\|\\ \|\Delta^2(t,x_0,y_0)-\Delta^2_m(t,x_0,y_0)\|\end{pmatrix}\leq J\varphi_\gamma^{m+1}(T)\left(E-\varphi_\gamma(T)\right)^{-1}\Omega_1(T).$$

Thus, we prove that from the inequalities (46) and (47) and the periodic functions $\Delta^1(t,x_0,y_0)$ and, $\Delta^2(t,x_0,y_0)$, there exists an isolated singular points such that $\Delta^1(t,x_0,y_0)=0$ and $\Delta^2(t,x_0,y_0)=0$, i. e. the system (V F), (F V) has periodic solutions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$.

Theorem 7: Suppose that the system (V F), (F V) of period T defined on an interval $a\leq x\leq b$ and, $c\leq y\leq d$. Then for $m\geq 1$ the sequences of vector functions $\Delta^1_m(t,x_0,y_0)$ and, $\Delta^2_m(t,x_0,y_0)$, which are defined in (22) and (24) satisfy the following inequalities:

$$\left.\begin{aligned}\min_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta^1_m(t,x_0,y_0)\leq -\omega_{1m}\\ \max_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta^1_m(t,x_0,y_0)\geq \omega_{1m}\end{aligned}\right\}, \quad (48)$$

$$\left.\begin{aligned}\min_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c1+e_2\leq y\leq d_1-e_2}}\Delta^2_m(t,x_0,y_0)\leq -\omega_{1m}\\ \max_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c1+e_2\leq y\leq d_1-e_2}}\Delta^2_m(t,x_0,y_0)\geq \omega_{1m}\end{aligned}\right\}. \quad (49)$$

Then the system (V F), (F V) has periodic solutions $x=x(t,x_0,y_0)$ and $y=y(t,x_0,y_0)$ such that $x_0\in[a_1+\varrho_1(t)R_1{H^*}_1, b_1-\varrho_1(t)R_1{H^*}_1]$ and $y_0\in[c_1+\varrho_2(t)R_2{H^*}_2, d_1-\varrho_2(t)R_2{H^*}_2]$.

Proof: Let the points, x_1, x_2 be defined in the interval $[a_1+\varrho_1(t)R_1{H^*}_1, b_1-\varrho_1(t)R_1{H^*}_1]$, and y_1, y_2 be any two points that defined in the interval $[c_1+\varrho_2(t)R_2{H^*}_2, d_1-\varrho_2(t)R_2{H^*}_2]$, such that

$$\left.\begin{array}{l}\Delta_m^1(t,x_1,y_1)=\min\limits_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}}\Delta_m^1(t,x_0,y_0)\\ \Delta_m^1(t,x_1,y_1)=\max\limits_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}}\Delta_m^1(t,x_0,y_0)\end{array}\right\}, \tag{50}$$

$$\left.\begin{array}{l}\Delta_m^2(t,x_1,y_1)=\min\limits_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}}\Delta_m^2(t,x_0,y_0)\\ \Delta_m^2(t,x_1,y_1)=\max\limits_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}}\Delta_m^2(t,x_0,y_0)\end{array}\right\}. \tag{51}$$

From the inequalities of system (45), we obtain that

$$\left.\begin{array}{l}\Delta^1(t,x_1,y_1)=\Delta_m^1(t,x_1,y_1)+\left(\Delta^1(t,x_1,y_1)-\Delta_m^1(t,x_1,y_1)\right)<0\\ \Delta^1(t,x_2,y_2)=\Delta_m^1(t,x_2,y_2)+\left(\Delta^1(x_2,y_2)-\Delta_m^1(t,x_2,y_2)\right)>0\end{array}\right\}, \tag{52}$$

$$\left.\begin{array}{l}\Delta^2(t,x_1,y_1)=\Delta_m^2(t,x_1,y_1)+\left(\Delta^2(t,x_1,y_1)-\Delta_m^2(t,x_1,y_1)\right)<0\\ \Delta^2(t,x_2,y_2)=\Delta_m^2(t,x_2,y_2)+\left(\Delta^2(x_2,y_2)-\Delta_m^2(t,x_2,y_2)\right)>0\end{array}\right\}, \tag{53}$$

and from the continuity of the functions $\Delta^1(t,x_0,y_0)$ and $\Delta^2(t,x_0,y_0)$ also the inequalities (52) and (53) there exists an isolated singular point $(x^o,y^o)=(x_0,y_0)$ and $x^o\in[x_1,x_2]$, $y^o\in[y_1,y_2]$ where, $\Delta^1(t,x_0,y_0)$ and $\Delta^2(t,x_0,y_0)$ are equal to zero, this means that the system (V F), (F V) has periodic solutions $x=x(t,x_0,y_0)$ and $y=y(t,x_0,y_0)$ for $x_0\in[a_1+\varrho_1(t)\mathrm{R}_1H_1^*,b_1-\varrho_1(t)\mathrm{R}_1H_1^*]$ and $y_0\in[c_1+\varrho_2(t)\mathrm{R}_2H_2^*,d_1-\varrho_2(t)\mathrm{R}_2H_2^*]$.

Remark 1: Theorem 7 is proved when x_0 are y_0 are scalar singular points which should be isolated (For this remark, see [20]).

Theorem 8: Suppose that the vector functions, $f(t,x,y,u)$ and $g(t,x,y,v)$ are defined and continuous on the domain (1). Then they are periodic in t of T period, bounded and satisfy all assumptions and conditions of theorem 4. Then the functions $f(t,x,y,u)$ and $g(t,x,y,v)$ on domain (1) are odd functions, i. e.

$$\left.\begin{array}{l}f(-t,x,y,u)=-f(t,x,y,u)\\ g(-t,x,y,u)=-g(t,x,y,u)\end{array}\right\}. \tag{54}$$

Then the solutions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ of (V F), (F V) for which $x_0\in G_f$ and $y_0\in G_g$ are periodic in t of T period.

Proof: Consider the sequences of functions $\{x_m(t,x_0,y_0)\}_{m=1}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=1}^{\infty}$ are defined in (21) and (23). Since $f(t,x,y,u)$ and $g(t,x,y,v)$ on domain (1) are odd functions, then $\overline{f(t,x,y,u)}=0$ and $\overline{g(t,x,y,v)}=0$. Hence for $m=0$ we obtain that

$$x_1(t,x_0,y_0)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)-\Delta_0^1(t,x_0,y_0)\Big)ds=x_1(t+T,x_0,y_0),$$

$$y_1(t,x_0,y_0)=y_0e^{C_2t}+\int_0^t e^{C_2(t-s)}\Big(\big(C_1+D_1(s)\big)x_0(s)+D_2(s)y_0(s)+g\big(s,x_0(s),y_0(s),v_0(s)\big)-\Delta_0^2(t,x_0,y_0)\Big)ds=y_1(t+T,x_0,y_0).$$

Where,

$$\Delta_0^1(t,x_0,y_0)=A_1x_0+\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0(s)+\big(A_2+B_2(s)\big)y_0(s)+f\big(s,x_0(s),y_0(s),u_0(s)\big)\Big)ds,$$

$$\Delta_0^2(t,x_0,y_0)=C_2y_0+\frac{C_2}{(e^{C_2T}-I)}\int_0^T e^{C_2(T-s)}\Big(\big(C_1+D_1(s)\big)x_0(s)+D_2(s)y_0(s)+g\big(s,x_0(s),y_0(s),v_0(s)\big)\Big)ds.$$

That is, $x_1(t,x_0,y_0)$ and $y_1(t,x_0,y_0)$ are periodic of period T in t. Moreover, $\|x_1(t,x_0,y_0)-x_0\|\le\varrho_1(t)\mathrm{R}_1H_1^*$ and $\|y_1(t,x_0,y_0)-y_0\|\le\varrho_2(t)\mathrm{R}_2H_2^*$, i. e. the functions $x_1(t,x_0,y_0)\in G_f$ and $y_1(t,x_0,y_0)\in G_g$.
Finally, we obtain that $x_1(t,x_0,y_0)=x_1(-t,x_0,y_0)$ and $y_1(t,x_0,y_0)=y_1(-t,x_0,y_0)$ as the integral of an odd function. Beside that, it is clear by induction and for $m\ge1$, the functions $x_m(t,x_0,y_0)$ and $y_m(t,x_0,y_0)$ are defined and periodic of period T in t. So, from (54) we obtain that $x_m(t,x_0,y_0)=x_m(-t,x_0,y_0)$ and $y_m(t,x_0,y_0)=y_m(-t,x_0,y_0)$ and also, we have obtained the inequalities (35) and (36).

Theorem 9: Suppose that the periodic system (V F), (F V) be given and continuous on (1) and also $G_f\in G_0$ and $G_g\in G_1$. Then for G_f and G_g to obtain points at which the constants $\Delta^1(0,x_0,y_0)$ and $\Delta^2(0,x_0,y_0)$ be zero, it is required that for all m's and for any $x_1\in G_0$ and $y_1\in G_1$, the following vector inequality holds:

$$\begin{pmatrix}\|\Delta_m^1(t,x_0,y_0)\|\\ \|\Delta_m^2(t,x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}x_0\|A_1\|+J_1R_1T(\|B_1(t)\|\|x_m(t)\|+\|A_2+B_2(t)\|\|y_m(t)\|+\vartheta_1)\\ y_0\|C_2\|+J_2R_2T(\|C_1+D_1(t)\|\|x_m(t)\|+\|A_2+B_2(t)\|\|y_m(t)\|+\vartheta_2)\end{pmatrix}\varphi_\gamma^{m+1}(T)\big(E-\varphi_\gamma(T)\big)^{-1}J\Omega_1(T). \tag{55}$$

Proof: From equations (3), (22) and (34), we have

$$\|\Delta_m^1(0,x_0,y_0)\| = \|\Delta_m^1(0,x_0,y_0) - \Delta^1(0,x_0,y_0) + \Delta^1(0,x_0,y_0)\| \le \|\Delta^1(0,x_0,y_0)\| + \|\Delta_m^1(0,x_0,y_0) - \Delta^1(0,x_0,y_0)\|,$$

$$\le \left\| A_1x_0 + \frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_m(s) + \big(A_2+B_2(s)\big)y_m(s) + f\big(s,x_m(s),y_m(s),u_m(s)\big)\Big)ds \right\|$$

$$+ J_1R_1T\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\big(h_1 + h_3H_1^\gamma\big)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1a} - e^{-\gamma_1b})e^{\gamma_1t}\right)\right)^\gamma\right)\|x(t) - x_m(t)\|$$

$$+ J_1R_1T\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\big(h_2 + h_3H_1^\gamma\big)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1a} - e^{-\gamma_1b})e^{\gamma_1t}\right)\right)^\gamma\right)\|y(t) - y_m(t)\|,$$

$$\le x_0\|A_1\| + \frac{\|A_1\|}{e^{\|A_1\|T} - \|I\|}\int_0^T \|e^{A_1(T-s)}\|\big(\|B_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \|f(t,x_m(t),y_m(t),u_m(t))\|\big)ds$$

$$+ J_1\varphi_1(t)\|x(t) - x_m(t)\| + J_1\varphi_2(t)\|y(t) - y_m(t)\|,$$

$$\le x_0\|A_1\| + J_1R_1T\big(\|B_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \|f(t,x_m(t),y_m(t),u_m(t))\|\big)$$

$$+ J_1\varphi_1(t)\|x(t) - x_m(t)\| + J_1\varphi_2(t)\|y(t) - y_m(t)\|,$$

$$\|\Delta_m^1(0,x_0,y_0)\| \le x_0\|A_1\| + J_1R_1T(\|B_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \vartheta_1) + J_1\varphi_1(t)\|x(t) - x_m(t)\| + J_1\varphi_2(t)\|y(t) - y_m(t)\|. \tag{56}$$

Under the same inequalities, conditions and the equations (5), (24) and (34), we obtain that

$$\|\Delta_m^2(0,x_0,y_0)\| = \|\Delta_m^2(0,x_0,y_0) - \Delta^2(0,x_0,y_0) + \Delta^2(0,x_0,y_0)\| \le \|\Delta^2(0,x_0,y_0)\| + \|\Delta_m^2(0,x_0,y_0) - \Delta^2(0,x_0,y_0)\|,$$

$$\le \left\| C_2y_0 + \frac{C_2}{(e^{C_2T}-I)}\int_0^T e^{C_2(T-s)}\Big(\big(C_1 + D_1(s)\big)x_m(s) + D_2(s)y_m(s) + g\big(s,x_m(s),y_m(s),v_m(s)\big)\Big)ds \right\|$$

$$+ J_2R_2T\left(\|C_1 + D_1(t)\| + \Sigma_1 + \Sigma_3\left(\big(l_1 + l_3H_2^\gamma\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma\right)\|x(t) - x_m(t)\|$$

$$+ J_2R_2T\left(\|D_2(t)\| + \Sigma_2 + \Sigma_3\left(\big(l_2 + l_3H_2^\gamma\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma\right)\|y(t) - y_m(t)\|,$$

$$\le y_0\|C_2\| + \frac{\|C_2\|}{e^{\|C_2\|T} - \|I\|}\int_0^T \|e^{C_2(T-s)}\|\big(\|C_1 + D_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\|$$

$$+ \|g(t,x_m(t),y_m(t),u_m(t))\|\big)ds + J_2\varphi_1(t)\|x(t) - x_m(t)\| + J_2\varphi_2(t)\|y(t) - y_m(t)\|,$$

$$\|\Delta_m^2(0,x_0,y_0)\| \le y_0\|C_2\| + J_2R_2T(\|C_1 + D_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \vartheta_2) + J_2\varphi_1(t)\|x(t) - x_m(t)\| + J_2\varphi_2(t)\|y(t) - y_m(t)\|. \tag{57}$$

Then from (56) and (57) we obtain the vector form (55) as follows:

$$\begin{pmatrix}\|\Delta_m^1(t,x_0,y_0)\|\\ \|\Delta_m^2(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix} x_0\|A_1\| + J_1R_1T(\|B_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \vartheta_1)\\ y_0\|C_2\| + J_2R_2T(\|C_1 + D_1(t)\|\|x_m(t)\| + \|A_2+B_2(t)\|\|y_m(t)\| + \vartheta_2)\end{pmatrix}\varphi_\gamma^{m+1}(T)\big(E - \varphi_\gamma(T)\big)^{-1}J\Omega_1(T).$$

5. Stability solutions of integro-differential equations of (VF) and (FV) types

In this section, we study the stability of periodic solution for the integral equations (V F) and (F V).

Theorem 10: If the function $\Delta^1(0,x_0,y_0) \in D_f \times D_g \to R$ and $\Delta^2(0,x_0,y_0) \in D_f \times D_g \to R$ from (3) and (5) are defined respectively, where functions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are the limit functions of the sequence functions (21) and (23). Then from the functions $\Delta^{1*}(0,x_0^1,y_0^1)$, $\Delta^{1*}(0,x_0^2,y_0^2)$, $\Delta^{2*}(0,x_0^1,y_0^1)$ and $\Delta^{2*}(0,x_0^2,y_0^2)$, also $x(t,x_0,y_0)$, $r(t,x_0,y_0)$, $y(t,x_0,y_0)$ and $w(t,x_0,y_0)$ we have to verify that the following vector inequalities hold:

$$\begin{pmatrix}\|\Delta^{1*}(0,x_0^1,y_0^1) - \Delta^{1*}(0,x_0^2,y_0^2)\|\\ \|\Delta^{2*}(0,x_0^1,y_0^1) - \Delta^{2*}(0,x_0^2,y_0^2)\|\end{pmatrix} \le \begin{pmatrix}\|A_1\|\|x_0^1 - x_0^2\|\\ \|C_2\|\|y_0^1 - y_0^2\|\end{pmatrix}\begin{pmatrix}J_1\varphi_1(T)\|x_0^1(t) - x_0^2(t)\| + J_1\varphi_2(T)\|y_0^1(t) - y_0^2(t)\|\\ J_2\varphi_3(T)\|x_0^1(t) - x_0^2(t)\| + J_2\varphi_4(T)\|y_0^1(t) - y_0^2(t)\|\end{pmatrix}. \tag{58}$$

$$\begin{pmatrix}\|x(t,x_0,y_0) - r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0) - w(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\|x_0^1 - x_0^2\|\|e^{A_1t}\|\\ \|y_0^1 - y_0^2\|\|e^{C_2t}\|\end{pmatrix} + \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0) - r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0) - w(t,x_0,y_0)\|\end{pmatrix}. \tag{59}$$

Proof: From (3) and the assumptions of theorem 4 we have

$$\|\Delta^{1*}(0,x_0^1,y_0^1)-\Delta^{1*}(0,x_0^2,y_0^2)\|$$

$$=\Bigg\|A_1x_0^1+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^1(s)+\big(A_2+B_2(s)\big)y_0^1(s)+f\big(s,x_0^1(s),y_0^1(s),u_0^1(s)\big)\Big)ds$$

$$-\Bigg(A_1x_0^2+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^2(s)+\big(A_2+B_2(s)\big)y_0^2(s)+f\big(s,x_0^2(s),y_0^2(s),u_0^2(s)\big)\Big)ds\Bigg)\Bigg\|,$$

$$\leq\|A_1\|\|x_0^1-x_0^2\|+\Bigg\|\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^1(s)+\big(A_2+B_2(s)\big)y_0^1(s)+f\big(s,x_0^1(s),y_0^1(s),u_0^1(s)\big)\Big)ds$$

$$-\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^2(s)+\big(A_2+B_2(s)\big)y_0^2(s)+f\big(s,x_0^2(s),y_0^2(s),u_0^2(s)\big)\Big)ds\Bigg\|,$$

$$\leq\|A_1\|\|x_0^1-x_0^2\|+\frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|}\Bigg\|\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^1(s)+\big(A_2+B_2(s)\big)y_0^1(s)+f\big(s,x_0^1(s),y_0^1(s),u_0^1(s)\big)\Big)ds$$

$$-\int_0^T e^{A_1(T-s)}\Big(B_1(s)x_0^2(s)+\big(A_2+B_2(s)\big)y_0^2(s)+f\big(s,x_0^2(s),y_0^2(s),u_0^2(s)\big)\Big)ds\Bigg\|,$$

$$\leq\|A_1\|\|x_0^1-x_0^2\|+\frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|}\int_0^T e^{A_1(T-s)}\big(\|B_1(t)\|\|x_0^1(t)-x_0^2(t)\|+\|A_2+B_2(t)\|\|y_0^1(t)-y_0^2(t)\|$$
$$+\big\|f\big(t,x_0^1(t),y_0^1(t),u_0^1(t)\big)-f\big(t,x_0^2(t),y_0^2(t),u_0^2(t)\big)\big\|\big)\,ds,$$

$$\leq\|A_1\|\|x_0^1-x_0^2\|+J_1R_1T\big(\|B_1(t)\|\|x_0^1(t)-x_0^2(t)\|+\|A_2+B_2(t)\|\|y_0^1(t)-y_0^2(t)\|$$
$$+\big\|f\big(t,x_0^1(t),y_0^1(t),u_0^1(t)\big)-f\big(t,x_0^2(t),y_0^2(t),u_0^2(t)\big)\big\|\big),$$

$$\leq\|A_1\|\|x_0^1-x_0^2\|+J_1R_1T\big(\|B_1(t)\|\|x_0^1(t)-x_0^2(t)\|+\|A_2+B_2(t)\|\|y_0^1(t)-y_0^2(t)\|+\Gamma_1\|x_0^1(t)-x_0^2(t)\|^{\alpha}$$
$$+\Gamma_2\|y_0^1(t)-y_0^2(t)\|^{\beta}+\Gamma_3\|u_0^1(t)-u_0^2(t)\|^{\gamma}\big),$$

where,

$$\|u_0^1(t)-u_0^2(t)\|^{\gamma}=\left(\big(h_1+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1t}\right)\right)^{\gamma}\|x_0^1(t)-x_0^2(t)\|^{\gamma}$$
$$+\left(\big(h_2+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1t}\right)\right)^{\gamma}\|y_0^1(t)-y_0^2(t)\|^{\gamma},$$

$\leq\big(h_1+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1T}\right)\|x_0^1(t)-x_0^2(t)\|+\big(h_2+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1T}\right)\|y_0^1(t)-y_0^2(t)\|,$

$\leq\|A_1\|\|x_0^1-x_0^2\|+J_1R_1T\Big(\|B_1(t)\|\|x_0^1(t)-x_0^2(t)\|+\|A_2+B_2(t)\|\|y_0^1(t)-y_0^2(t)\|+\Gamma_1\|x_0^1(t)-x_0^2(t)\|+\Gamma_2\|y_0^1(t)-y_0^2(t)\|+\Gamma_3\big(h_1+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1T}\right)\|x_0^1(t)-x_0^2(t)\|+\Gamma_3\big(h_2+h_3H_1^{\gamma}\big)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1a}-e^{-\gamma_1b})e^{\gamma_1T}\right)\|y_0^1(t)-y_0^2(t)\|\Big),$

$$\leq \|A_1\|\|x_0^1 - x_0^2\| + J_1R_1T\left(\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|x_0^1(t) - x_0^2(t)\| + \left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|y_0^1(t) - y_0^2(t)\|\right).$$

Therefore, we obtain that

$$\|\Delta^{1*}(0, x_0^1, y_0^1) - \Delta^{1*}(0, x_0^2, y_0^2)\| \leq \|A_1\|\|x_0^1 - x_0^2\| + J_1R_1T\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|x_0^1(t) - x_0^2(t)\| + J_1R_1T\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|y_0^1(t) - y_0^2(t)\|,$$

$$\|\Delta^{1*}(0, x_0^1, y_0^1) - \Delta^{1*}(0, x_0^2, y_0^2)\| \leq \|A_1\|\|x_0^1 - x_0^2\| + J_1\varphi_1(T)\|x_0^1(t) - x_0^2(t)\| + J_1\varphi_2(T)\|y_0^1(t) - y_0^2(t)\|. \quad (60)$$

Same iterations are done by (5) for finding, $\|\Delta^{2*}(0, x_0^1, y_0^1) - \Delta^{2*}(0, x_0^2, y_0^2)\|$. Thus

$$\|\Delta^{2*}(0, x_0^1, y_0^1) - \Delta^{2*}(0, x_0^2, y_0^2)\| \leq \|C_2\|\|y_0^1 - y_0^2\| + J_2\varphi_3(T)\|x_0^1(t) - x_0^2(t)\| + J_2\varphi_4(T)\|y_0^1(t) - y_0^2(t)\|. \quad (61)$$

Rewrite (60) and (61) in a vector form, we obtain that

$$\begin{pmatrix}\|\Delta^{1*}(0, x_0^1, y_0^1) - \Delta^{1*}(0, x_0^2, y_0^2)\| \\ \|\Delta^{2*}(0, x_0^1, y_0^1) - \Delta^{2*}(0, x_0^2, y_0^2)\|\end{pmatrix} \leq \begin{pmatrix}\|A_1\|\|x_0^1 - x_0^2\| \\ \|C_2\|\|y_0^1 - y_0^2\|\end{pmatrix}\begin{pmatrix}J_1\varphi_1(T)\|x_0^1(t) - x_0^2(t)\| + J_1\varphi_2(T)\|y_0^1(t) - y_0^2(t)\| \\ J_2\varphi_3(T)\|x_0^1(t) - x_0^2(t)\| + J_2\varphi_4(T)\|y_0^1(t) - y_0^2(t)\|\end{pmatrix}.$$

Where, $x(0, x_0^1, y_0^1)$, $x(0, x_0^2, y_0^2)$, $y(0, x_0^1, y_0^1)$ and $y(0, x_0^2, y_0^2)$ are the solutions of the equations (V F) and (F V) as follows:

$$x(0, x_0^a, y_0^a) = x_0^a e^{A_1 t} + \int_0^t e^{A_1(t-s)}\left(B_1(s)x_0^a(s) + \left(A_2 + B_2(s)\right)y_0^a(s) + f\left(s, x_0^a(s), y_0^a(s), u_0^a(s)\right) - \Delta^1(0, x_0^a, y_0^a)\right)ds,$$

$$y(0, x_0^a, y_0^a) = y_0^a e^{C_2 t} + \int_0^t e^{C_2(t-s)}\left(\left(C_1 + D_1(s)\right)x_0^a(s) + D_2(s)y_0^a(s) + g\left(s, x_0^a(s), y_0^a(s), v_0^a(s)\right) - \Delta^2(0, x_0^a, y_0^a)\right)ds,$$

Where, $a = 1,2,\dots$ and

$$\Delta^1(0, x_0^a, y_0^a) = A_1x_0^1 + \frac{A_1}{e^{A_1T} - I}\int_0^T e^{A_1(T-s)}\left(B_1(s)x_0^1(s) + \left(A_2 + B_2(s)\right)y_0^1(s) + f\left(s, x_0^1(s), y_0^1(s), u_0^1(s)\right)\right)ds,$$

$$\Delta^2(0, x_0^a, y_0^a) = C_2y_0^a + \frac{C_2}{e^{C_2T} - I}\int_0^T e^{C_2(T-s)}\left(\left(C_1 + D_1(s)\right)x_0^a(s) + D_2(s)y_0^a(s) + g\left(s, x_0^a(s), y_0^a(s), v_0^a(s)\right)\right)ds.$$

Next, by the above solutions we get

$$\|x(0, x_0^1, y_0^1), -x(0, x_0^2, y_0^2)\| = \left\| x_0^1 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\left(B_1(s)x_0^1(s) + \left(A_2 + B_2(s)\right)y_0^1(s) + f\left(s, x_0^1(s), y_0^1(s), u_0^1(s)\right) - \Delta^1(0, x_0^1, y_0^1)\right)ds - \left(x_0^2 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\left(B_1(s)x_0^2(s) + \left(A_2 + B_2(s)\right)y_0^2(s) + f\left(s, x_0^2(s), y_0^2(s), u_0^2(s)\right) - \Delta^1(0, x_0^2, y_0^2)\right)ds\right)\right\|,$$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \left\|\int_0^t e^{A_1(t-s)}\left(B_1(s)x_0^1(s) + \left(A_2 + B_2(s)\right)y_0^1(s) + f\left(s, x_0^1(s), y_0^1(s), u_0^1(s)\right) - \Delta^1(0, x_0^1, y_0^1)\right)ds - \int_0^t e^{A_1(t-s)}\left(B_1(s)x_0^2(s) + \left(A_2 + B_2(s)\right)y_0^2(s) + f\left(s, x_0^2(s), y_0^2(s), u_0^2(s)\right) - \Delta^1(0, x_0^2, y_0^2)\right)ds\right\|,$$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \left\| \int_0^t e^{A_1(t-s)} \left(B_1(s)x_0^1(s) + \left(A_2 + B_2(s)\right)y_0^1(s) + f\left(s, x_0^1(s), y_0^1(s), u_0^1(s)\right) - \Delta^1(0, x_0^1, y_0^1)\right) ds - \int_0^t e^{A_1(t-s)} \left(B_1(s)x_0^2(s) + \left(A_2 + B_2(s)\right)y_0^2(s) + f\left(s, x_0^2(s), y_0^2(s), u_0^2(s)\right) - \Delta^1(0, x_0^2, y_0^2)\right) ds \right\|,$$

$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \mathrm{R}_1 t \left(\|I\| - \frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right)\left(\|B_1(t)\|\|x_0^1(t) - x_0^1(t)\| + \|A_2 + B_2(t)\|\|y_0^1(t) - y_0^1(t)\| + \left\|f\left(t, x_0^1(t), y_0^1(t), u_0^1(t)\right) - f\left(t, x_0^2(t), y_0^2(t), u_0^2(t)\right)\right\|\right) + \mathrm{R}_1 T \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right)\left(\|B_1(t)\|\|x_0^1(t) - x_0^1(t)\| + \|A_2 + B_2(t)\|\|y_0^1(t) - y_0^1(t)\| + \left\|f\left(t, x_0^1(t), y_0^1(t), u_0^1(t)\right) - f\left(t, x_0^2(t), y_0^2(t), u_0^2(t)\right)\right\|\right) - \mathrm{R}_1 t \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right)\left(\|B_1(t)\|\|x_0^1(t) - x_0^1(t)\| + \|A_2 + B_2(t)\|\|y_0^1(t) - y_0^1(t)\| + \left\|f\left(t, x_0^1(t), y_0^1(t), u_0^1(t)\right) - f\left(t, x_0^2(t), y_0^2(t), u_0^2(t)\right)\right\|\right),$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \left(\mathrm{R}_1 t \left(\|I\| - 2\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right) + \mathrm{R}_1 T \left(\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|}\right)\right)\left(\|B_1(t)\|\|x(t) - r(t)\| + \|A_2 + B_2(t)\|\|y(t) - w(t)\| + \left\|f\left(t, x(t), y(t), u(t)\right) - f\left(t, r(t), w(t), u_r(t)\right)\right\|\right),$$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \mathrm{R}_1 \left(\frac{\left(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|\right)t + \left(e^{\|A_1\|t} - \|I\|\right)T}{\left(e^{\|A_1\|T} - \|I\|\right)}\right)\Bigg(\|B_1(t)\|\|x(t) - r(t)\| + \|A_2 + B_2(t)\|\|y(t) - w(t)\| + \Gamma_1\|x(t) - r(t)\| + \Gamma_2\|y(t) - w(t)\| + \Gamma_3\left(\left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma \|x(t) - r(t)\| + \Gamma_3\left(\left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma \|y(t) - w(t)\|\Bigg),$$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \mathrm{R}_1 \left(\frac{\left(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|\right)t + \left(e^{\|A_1\|t} - \|I\|\right)T}{\left(e^{\|A_1\|T} - \|I\|\right)}\right)\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|x(t) - r(t)\| + \mathrm{R}_1 \left(\frac{\left(e^{\|A_1\|T} - 2e^{\|A_1\|t} + \|I\|\right)t + \left(e^{\|A_1\|t} - \|I\|\right)T}{\left(e^{\|A_1\|T} - \|I\|\right)}\right)\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|y(t) - w(t)\|,$$

$$\leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \mathrm{R}_1 \varrho_1(t)\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|x(t) - r(t)\| + \mathrm{R}_1 \varrho_1(t)\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|y(t) - w(t)\|,$$

Therefore, we receive that

$$\|x(t, x_0, y_0) - r(t, x_0, y_0)\| \leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \mathrm{R}_1 \varrho_1(t)\left(\|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|x(t) - r(t)\| + \mathrm{R}_1 \varrho_1(t)\left(\|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^\gamma\right)\|y(t) - w(t)\|,$$

$$\|x(t, x_0, y_0) - r(t, x_0, y_0)\| \leq \|x_0^1 - x_0^2\|\|e^{A_1 t}\| + \varphi_1(T)\|x(t, x_0, y_0) - r(t, x_0, y_0)\| + \varphi_2(T)\|y(t, x_0, y_0) - w(t, x_0, y_0)\|. \ldots \quad (62)$$

Same steps are done for finding, $\|y(t,x_0,y_0)-w(t,x_0,y_0)\|$. Thus

$$\|y(t,x_0,y_0)-w(t,x_0,y_0)\| \le \|y_0^1-y_0^2\|\|e^{C_2t}\|+\varphi_3(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\|+\varphi_4(T)\|y(t,x_0,y_0)-w(t,x_0,y_0)\|. \quad (63)$$

Rewrite (62) and (63) in a vector form, we obtain that

$$\begin{pmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\|x_0^1-x_0^2\|\|e^{A_1t}\|\\ \|y_0^1-y_0^2\|\|e^{C_2t}\|\end{pmatrix}+\begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\|\\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\|\end{pmatrix}.$$

Hence from the condition (25), it implies that the system (VF), (FV) has a stable solution.

6. Method of Banach Fixed Point Theorem of integro-differential equations of (VF) and (FV) types

In this section we have to study the existence and uniqueness theorem of periodic system of (V F), (F V) type that introduced by the following:

Theorem 11: Suppose that $f(t,x,y,u)$ and $g(t,x,y,v)$ be vector functions that are defined by (V F) and (F V) and continuous on domains (10) and (11) and periodic in t of period T that satisfy the inequalities and conditions in theorem 4. Then the system (V F), (F V) has a unique periodic continuous solution that satisfies the Banach fixed point theorem.

Proof: Suppose that $(S,\|.\|)$ be a Banach space on $C[0,T]$ and T^* be a mapping on S by the following vector functions

$$\begin{aligned}T^*x(t,x_0,y_0) &= x_0e^{A_1t}\\ &+\int_0^t e^{A_1(t-s)}\Bigg(B_1(s)x(s)+\big(A_2+B_2(s)\big)y(s)+f\big(s,x(s),y(s),u(s)\big)\\ &-\Bigg(A_1x_0\\ &+\frac{A_1}{(e^{A_1T}-I)}\int_0^T e^{A_1(T-s)}\Big(B_1(s)x(s)+\big(A_2+B_2(s)\big)y(s)+f\big(s,x(s),y(s),u(s)\big)\Big)\,ds\Bigg)\Bigg)ds, \quad (64)\end{aligned}$$

$$\begin{aligned}T^*y(t,x_0,y_0) &= y_0e^{C_2t}\\ &+\int_0^t e^{C_2(t-s)}\Bigg(\big(C_1+D_1(s)\big)x(s)+D_2(s)y(s)+g\big(s,x(s),y(s),v(s)\big)\\ &-\Bigg(C_2y_0\\ &+\frac{C_2}{(e^{C_2T}-I)}\int_0^T e^{C_2(T-s)}\Big(\big(C_1+D_1(s)\big)x(s)+D_2(s)y(s)+g\big(s,x(s),y(s),v(s)\big)\Big)\,ds\Bigg)\Bigg)ds. \quad (65)\end{aligned}$$

Easily to prove, $T^*: C[0,T]\to C[0,T]$. Now, we shall to prove that T^* is a contraction mapping on $[0,T]$.

Let $x(t), r(t), y(t)$ and $w(t)$ be vector functions on $[0,T]$, then
$\|T^*x(t)-T^*r(t)\| = \max_{t\in[0,T]}\{|T^*x(t)-T^*r(t)|\}$,
and
$\|T^*y(t)-T^*w(t)\| = \max_{t\in[0,T]}\{|T^*y(t)-T^*w(t)|\}$.
Thus

$$\|T^*x(t)-T^*z(t)\| \le \varphi_1(T)\|x(t)-z(t)\|+\varphi_2(T)\|y(t)-w(t)\|, \quad (66)$$

And

$$\|T^*y(t)-T^*w(t)\| \le \varphi_3(T)\|x_1(t)-x_2(t)\|+\varphi_4(T)\|y_1(t)-y_2(t)\|. \quad (67)$$

Rewrite (66) and (67) in a vector form to get

$$\begin{pmatrix}\|T^*x(t)-T^*z(t)\|\\ \|T^*y(t)-T^*w(t)\|\end{pmatrix} \le \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x(t)-z(t)\|\\ \|y(t)-w(t)\|\end{pmatrix}.$$

Then from condition (25), T^* is a contraction mapping, so from Banach fixed point theorem there exists fixed points $x(t)$ and $y(t)$ such that $T^*x(t) = x(t)$ and $T^*y(t) = y(t)$. Therefore (2) and (4) are unique solutions of (VF) and (FV).

7. Conclusion

In this work, we study the periodic solutions of nonlinear systems of Integro-differential equations that contain multiple integrals of (Volterra-Fredholm) and (Fredholm-Volterra) types with isolated singular kernels. We prove some theorems in the existence and uniqueness in closed and bounded domains. The stability of the periodic solutions was studied based on the following remark:

Remark 2: The confirm of the stability periodic solutions for the system (V F), (F V) that is when a slight change happens in the points x_0 and y_0. Therefore, a slight change will happen in the functions $\Delta_0^1(0, x_0, y_0)$ and $\Delta_0^2(0, x_0, y_0)$. (For this remark see [19]).

References:

[1] M. U. Akmetov, *Periodic solutions of some systems of differential equations*, Vestn. Kiev. Univ., Ser. Mat. Mekh., (24) (1982).

[2] A. T. Alymbaev, *Periodic solutions of systems of nonlinear integro-differential equations*, in: Studies in Integro - Differential Equations, (21) (1988).

[3] M. T. Apostol, *Mathematical Analysis, 2nd edition, Institute of Technology*, Addison-Wesley, California, (1973).

[4] M. A. Aziz, *Periodic Solutions for Some Systems of Nonlinear Ordinary Differential Equations*, thesis, University of Mosul, (2006).

[5] R. N. Butris, *Periodic solution of non-linear system of Integro - differential equations depending on the Gamma distribution*, India, Gen. Math. Notes, 15(1) (2013).

[6] R. N. Butris, and H. S. Faris, *Solutions for nonlinear systems of integro-differential equations that contain multiple integrals of (VF) and (FV) types with isolated singular kernels*, Journal of Xi'an University of Architecture & Technology, XII(V) (2020), https://doi.org/10.37896/jxat12.05/1553

[7] R. N. Butris, and H.M. Haji, *Existence, uniqueness and stability solution of Voltera-Friedholm of integro-differential equations*, International Journal of Mechanical Engineering and Technology, 10(07) (2019).

[8] R. N. Butris, and A. Sh. Rafeq, *Existence and uniqueness solution for nonlinear volterra integral equation*, J. Pure and Eng. Sciences, Dohok University, Iraq. 1(14) (2011).

[9] R. N. Butris, and R.F. Taher, *Periodic Solution of Integro-Differential Equations Depended on Special Functions with Singular Kernels and Boundary Integral Conditions*, International Journal of Advanced Trends in Computer Science and Engineering, 8(4) (2019), https://doi.org/10.30534/ijatcse/2019/73842019

[10] T. Jankowski, *Application of The Numerical-Analytic Method to Systems of Differential Equations with Parameter.*, Ukrainian Mathematical Journal, 54(4) (2002).

[11] I. I. Korol, *Numerical-analytic method for the investigation of boundary-value problems for semi-linear systems of differential equations*, nonlinear oscillations, 13(1) (2010), 45-56, https://doi.org/10.1007/s11072-010-0100-6

[12] I. I. Korol, *On Periodic Solutions of One Class of Systems of Differential Equations*, Ukrainian Mathematical Journal, 57(4) (2005).

[13] I. Korol, and M. O. Perestyuk, *Numerical-Analytic Method of Successive Periodic Approximations*, Ukrainian Mathematical Journal, 58(4) (2006), https://doi.org/10.1007/s11253-006-0083-8

[14] A. Yu. Luchka, *New Approach to the Investigation of the Existence of Periodic Solutions of Systems of Differential Equations and their Construction*, Nonlinear Oscillations, 11(1) (2008), 55-69, https://doi.org/10.1007/s11072-008-0014-8

[15] N. L. Mishra, and M. Sen, *On the concept of existence and local attractive of solutions for some quadratic Volterra integral equation of fractional order*, Applied Mathematics and Computation, 285 (2016), 174-183, https://doi.org/10.1016/j.amc.2016.03.002

[16] N. L. Mishra, M. Sen, and N. R. Mohapotra, *On Existence Theorems for Some Generalized Nonlinear Functional-Integral Equations with Applications*, Filomat, 31(7) (2017), 2081-2091, https://doi.org/10.2298/fil1707081n

[17] Yu. A. Mitropolsky, and D. I. Martynyuk, *Periodic Solutions for the Oscillations System with Retarded Argument*, Kiev, Ukraine, (1979).

[18] A. Sh. Rafeq, *Periodic solutions for some classes of non-linear systems of integro- Education*, University of Duhok, differential equations, M. Sc. Thesis., (2009).

[19] M. M. Rama, *Ordinary Differential Equations Theory and Applications*, Britain, (1981).

[20] A. M. Samoilenko, and N. I. Ronto, *Numerical-Analytic Methods for Investigations of Periodic Solutions*, Kiev, Ukraine, (1979).