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# Homotopy Sumudu Transformation Method for Solving Fractional Delay Differential Equations

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## Abstract

In this article, A new accurate approximate solution for a nonlinear fractional delay differential equations are obtained using an effective algorithm so called homotopy analysis sumudu transformation method (HASTM). Some examples have been solved to demonstrate the methodology of this procedure. The accuracy of the obtained numerical results has been achieved via rapid rate of convergence for the obtained approximate solutions. Numerical comparison which was displayed and presented in tables and also shown graphically in figures prove and confirm the high accuracy, capability and efficiency of this procedure.


Keywords: Fractional Calclus, Sumudu Transform, Homotopy Analysis Method.
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## 1. Introduction

In the last years, there has been much great attention in the field of study fractional differential equations, since the modern research in applied science and engineering have demonstrate that the dynamic of many system are modeled using differential equations of non-integer order due to their broad applications in complex physical system and viscoelastic materials, fluid mechanics and electromagnetism [26, 13, 25, 4, 14, 1, 5]. Delay differential equations (DDEs) are considered as a kind of functional differential equations that has many applications in several areas of studies such as biology, economy, electrodynamics, control and quantum mechanics. DDEs are more general than classical differential equations because the derivative of the unknown function at certain time depend on the value of the function at previous time or equations which have delayed argument. In many field of applied science and engineering DDEs play an important role and numerous approximated and numerical methods have been employed to solve different classes of DDEs [28, 6, 11, 19, 24, 8, 9]. Recently, fractional delay differential equations (FDDEs) have attracted the attentions of researchers from all over the world in a wide range area of science and engineering such as ecology, physical science and groundwater flow. For instance, the following equation indicate to the food-limited model that relates the nourishment rage of area the effect of food on the population of that area in surveyed.

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$$ {}_0D_x^{\alpha}u(x) = ru(x)(\frac{k-u(x-\tau)}{k+mu(x-\tau)}), \quad u(x) = 0.5(-\tau \leqslant x \leqslant 0) \tag{1.1} $$

Also, the number of blood cells in human body can be represented by the following FDDE

$$ {}_0D_x^{\alpha}u(x) = \frac{ca^r y(x-\tau)}{a^r + y^r(x-\tau)} - gy(x), \quad u(x) = 0.02(-\tau \leqslant x \leqslant 0) \tag{1.2} $$

Many authors studied FDDEs to obtain approximate solutions using various approximated analytical method such as HAM, OHAM, ADM and so on besides other type of differential equations of integer or non integer order [30, 18, 15, 27, 29, 23, 20, 31, 7, 16, 21, 2, 22, 3, 10], but the most important challenge that faces the researchers and should be taken under consideration is to find efficient and more accurate procedure which gives exact or accurate results with a very low remarkable error. In this paper, an effective procedure which gives solutions in form of convergent series solutions with easily computations and without linearization or restrictive assumptions is presented.

This paper is summarized as follows, Section two, include the fundamental properties of fractional calculus and sumudu transform method. Section 3 is devoted for description of the basic idea of the HASTM. Section 4 is devoted to prove the validity and capability of this procedure by presenting and discussing some numerical experiments, Finally, we end by including the conclusion of this study in last Section.

## 2. Fundamental Properties oF Fractional Calculus and Sumudu Transform Method

This Section deals with some notation and basic definitions of the fractional calculus theory and Sumudu transform that will be used further in this paper([12]).

**Definition 2.1.** The real function $f(x), x > 0$ said to be in the space $C_\mu, \mu \in \Re$ if there exists a real number $(p > \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in c[0, \infty)$ and it is said to be in the space $C_\mu^m$ iff $f^(m) \in C_\mu$ where $m \in N$

**Definition 2.2.** The left sided Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$ is defined as:

$$ J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, x > 0 $$

and $\Gamma(\alpha)$ is the gamma function.

Some properties of the operator $J^{\alpha}$, where $f \in C_\mu, \mu \geqslant -1, \alpha, \beta \geqslant 0$ and $\gamma > -1$

- $J_a^{\alpha} J_a^{\beta} f(x) = J_a^{\alpha+\beta} f(x) = J_a^{\beta} J_a^{\alpha} f(x)$
- $J_a^{\alpha} C = \frac{c}{\Gamma(\alpha+1)} t^{\alpha}$, where C is constant
- $J_a^{\alpha} t^{\gamma} = \frac{\gamma+1}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

**Definition 2.3.** The caputo definition of fractional derivative operator is given by $D_*^{\alpha} af(x) = J^{n-\alpha_a D^n f(x)} = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^(n)(\tau) d\tau, t > 0, \forall \quad n-1 < \alpha \leqslant n, n \in N$

**Definition 2.4.** The Sumudu transformation is defined as follows,
Consider | vjhgk |

Properties of the Sumudu transform are given as:

- $S[1] = 1$
- $S[e^{at}] = \frac{1}{1-au}$
- $S[\frac{t^n}{\Gamma(n+1)}] = u^n, \quad n > 0$
- $S[\alpha f(x) + \beta g(x)] = \alpha S[f(x) + \beta S[g(x)]$

**Definition 2.5.** If $G^n(u)$ is Summudu transformation of $n$'th order derivative of $f6n(t), \forall n \geqslant 1$then we have:
$G^n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}$
where $-1 \leqslant n-1 \leqslant \alpha < n$

**Definition 2.6.** The Sumudu transform $S[f(x)]$ of the fractional derivative introduced by Caputo is given by $S[D^\alpha f(x)] = \frac{S[f(x)]}{u^\alpha} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}, \quad -1 < \alpha \leqslant n$

## 3. Homotopy Analysis Sumudu transform Method

To explain the basic idea of the homotopy analysis Sumudu transform method (HASTM) for the fractional delay differential equations [**?** ], we consider

$$D^\alpha[u'(x)] = N[u(x)]. \tag{3.1}$$

Applying Sumudu transform on both sides of equation (3.1), we have:

$$\frac{S[u(x)]}{v^\alpha} - \sum_{n=0}^{m-1} v^{-\alpha+k} u^k(0) = S[N[u(x)]] \tag{3.2}$$

and this yields

$$S[u(x)] = \sum_{n=0}^{m-1} v^{+k} u^k(0) + v^\alpha [N[u(x)]] \tag{3.3}$$

$$S[u(x)] = g(v, f_i) + v^\alpha [N[u(x)]] \tag{3.4}$$

where $g(v, f_i) = f_0(x) + vf_1(x) + v^2 f_2(x)v + ... + v^{m-1} f_{m-1}(x)$. Some time it's difficult to find $u(x;t)$ by taking Sumudu inverse for both side of Eq. (3.4) particularly for nonlinear $N[u(x)]$. So we can utilized the HAM by defining the map:

$$(1-q)S[\phi(x;q)] - u_0(x)] = hqH(x)N[u(x)] \tag{3.5}$$

where $q \in [0,1]$ is the embedding parameter and $H(x;q)$ is real function of $x$ and $q, N$ nonlinear operator defined by:

$$N[\phi(x;q)] = S[\phi, q;] - v^\alpha [N[u(x)]] - g[v, f_i] \tag{3.6}$$

Obeviosly, when $q \in [0,1]$ it's hold $\phi(x;q) = u_0$ and also at$q = 1$ it's hold $\phi(x;q) = u(x)$. Thus as $q$ increase from 0 to 1 the solution various from the initial guess to the exact solution. We can expand $\phi(x;q)$ in Taylor series with respect to $q$ as

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \tag{3.7}$$

where $u_m(x) = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}$
the convergent of the series solution depends on $h$, if we probably chose $h, u_0(x)$ and $H(x;q)$, the series (3.7) convergent at $q = 1$, hence, we obtain

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{3.8}$$

which is a solution of original nonlinear equation, this equation represents the relationship between the initial guess $u_0(x)$ and the exact solution $u(x)$ by means of the term $u_m(x)$, which are still to be calculated. Defined the vector $\vec{u} = (u_0, u_1, ..., u_m)$. Differentiating the zeroth-order deformation equation (5.23) m-times with respect to $q$ and then dividing them by $m!$ and finally setting $q = 0$, we get the following $m'$th-order deformation equation:

$$S[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)R_m(\vec{u_{m-1}}) \tag{3.9}$$

Applying the inverse sumudu transform, we have

$$u_m(x) = \chi_m u_{m-1}(x)] + hS^{-1}[H(x)R_m(\vec{u}_{m-1})] \tag{3.10}$$

where

$$R_m(x)\vec{u}_{m-1} = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|q = 0 \tag{3.11}$$

and

$$\chi_m = \begin{cases} 0 & m \leqslant 1 \\ 1, & m > 1 \end{cases}$$

## 4. NUMERICAL RESULTS

In this section, we investigate the effectiveness of the proposed procedure by solving several examples of fractional delay differential equations.

### *4.1. Example 1*

Consider the following fractional delay differential equation [12]

$$D^{\alpha}u'(x) = \frac{3}{4}u(x) + u(\frac{x}{2}) - x^2 + 2, \quad 0 \leqslant x \leqslant 1, \quad 1 < \alpha \leqslant 2, \tag{4.1}$$

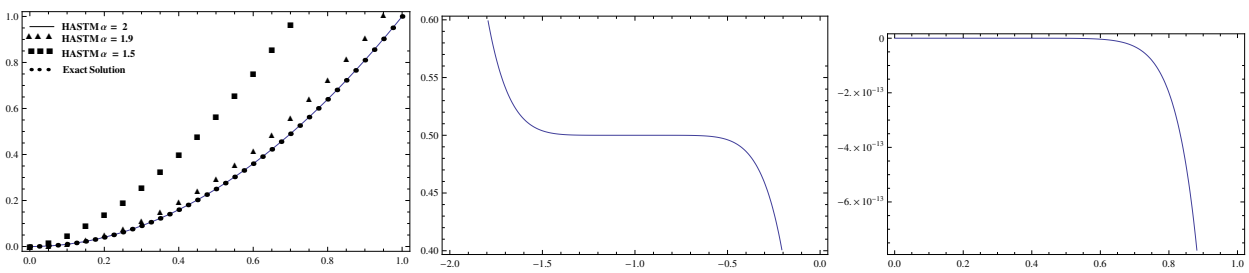
subjects to the initial conditions $u(0) = 0, \quad u'(0) = 0$.
The exact solution for this problem in case of $\alpha = 2$ is

$$u(x) = x^2. \tag{4.2}$$

Table 1 present a comparison between the absolute error obtained by HASTM approximate solution, Sumudu transform method and the exact solutions given by Eq. (4.2). It is easily observed from the results displayed in Table 1 that the approximate solution obtained by HASTM is more accurate than the approximate solution obtained by Sumudu transform method. Fig. 1 displays the plot of the exact solution and the approximate solutions for some values of $\alpha$, and also represents the plot of the $h-curve$ and the absolute error.

Table 1: The approximate solution by HASTM at different values of $\alpha$, and comparison with exact solution and Sumudu transformation method for Example 1.

| $x$ | Exact Solution | HASTM Solution when $\alpha = 1.5$ | HASTM Solution when $\alpha = 2$ | HASTM Absolute error when $\alpha = 2$ | Absolute error by when $\alpha = 2$ [12] |
|---|---|---|---|---|---|
| 0.1 | 0.01 | 0.0478911 | 0.01 | 0.000 | 0.00 |
| 0.2 | 0.04 | 0.136917 | 0.04 | 0.0000 | 0.000 |
| 0.3 | 0.09 | 0.254736 | 0.09 | 0.000 | 0.00 |
| 0.4 | 0.16 | 0.397657 | 0.16 | 0.000 | 0.00 |
| 0.5 | 0.25 | 0.563926 | 0.25 | 0.000 | 0.000001 |
| 0.6 | 0.36 | 0.752611 | 0.36 | 0.0.000 | 0.000006 |
| 0.7 | 0.49 | 0.963211 | 0.48999 | $5.6 \times 10^{-17}$ | 0.000022 |
| 0.8 | 0.64 | 1.19548 | 0.63999 | $5.6 \times 10^{-16}$ | 0.000064 |
| 0.9 | 0.81 | 1.44934 | 0.08099 | $3.6 \times 10^{-15}$ | 0.000165 |
| 1.0 | 1.0 | 1.72483 | 0.99999 | $1.9 \times 10^{-14}$ | 0.000383 |

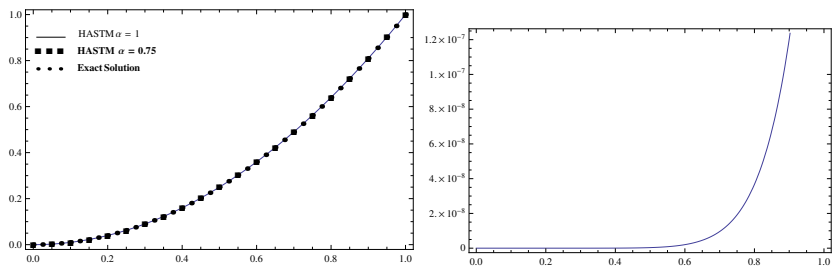



(a) (b) (c)

Figure 1: (a) Comparison between the approximate solution obtained by HASTM at different values of $\alpha$ and the exact one (b)Reprsents the $\hbar$ curve (c) HASTM absolute error

Table 2: The approximate solution by HASTM at different values of $\alpha$, and comparison with exact solution and Sumudu transformation method for Example 2.

| $x$ | Exact Solution | HASTM $\alpha = 0.75$ | HASTM $\alpha = 1$ | HASTM Absolute $\alpha = 1$ | Absolute error $\alpha = 1$ [12] |
|---|---|---|---|---|---|
| 0.1 | 0.01 | 0.01 | 0.01 | 0.00 | 0.000000 |
| 0.2 | 0.04 | 0.04 | 0.04 | $5.9 \times 10^{-16}$ | 0.000000 |
| 0.3 | 0.09 | 0.09 | 0.09 | $5.13 \times 10^{-14}$ | 0.000001 |
| 0.4 | 0.16 | 0.16 | 0.16 | $1.21 \times 10^{-12}$ | 0.000007 |
| 0.5 | 0.25 | 0.25 | 0.25 | $1.41 \times 10^{-11}$ | 0.000027 |
| 0.6 | 0.36 | 0.36 | 0.36 | $1.05 \times 10^{-10}$ | 0.00008 |
| 0.7 | 0.49 | 0.49 | 0.49 | $5.73 \times 10^{-10}$ | 0.0002 |
| 0.8 | 0.64 | 0.64 | 0.64 | $2.49 \times 10^{-9}$ | 0.000444 |
| 0.9 | 0.81 | 0.81 | 0.81 | $9.09 \times 10^{-9}$ | 0.000898 |
| 1.0 | 1.0 | 1.0 | 1.0 | $2.9 \times 10^{-8}$ | 0.001686 |




(a) (b)

Figure 2: (a) Comparison between the approximate solution obtained by HASTM at different values of $\alpha$ and the exact one (b) HASTM absolute error

### 4.2. Example 2

Consider the following fractional delay equation [12]

$$D^{\alpha}u'(x) = -u(x) + u(\frac{x}{2}) + \frac{3}{4}x^2 + \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}, \quad 0 \leqslant x \leqslant 1, \quad 0 < \alpha \leqslant 1, \tag{4.3}$$

subjects to the initial conditions $u(0) = 0$, and exact solution $u(x) = x^2$ in case of $\alpha = 1$

Table 2 shows the comparison of the absolute error between approximate solution obtained by HASTM and Sumudu transformation method while Figs. 2 $(a)$ and $(b)$ are displayed for the plot of the approximate solution, exact solution and the absolute error, respectively. The accuracy of the presented procedure is very clear compared the other method in literature, this leads to conclude and prove that the presented procedure is very powerful and suitable for this kind of differential equations.

### 4.3. Example 3

Consider the following fractional delay equation [17]

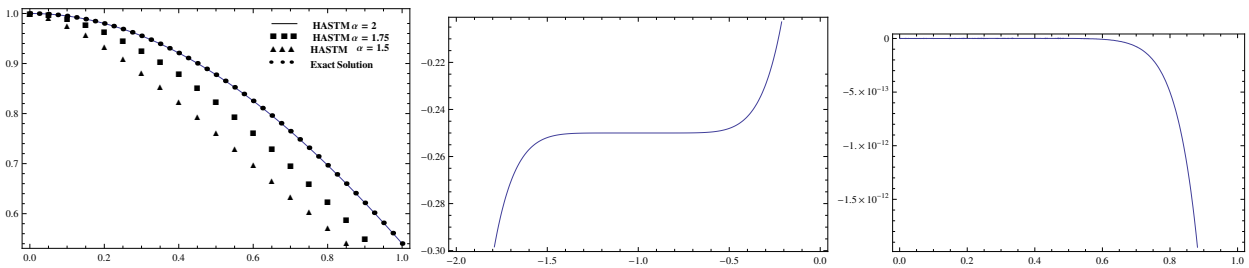
$$D^{\alpha}u(x) = 1 - 2u^2(\frac{x}{2}), \quad 0 \leqslant x \leqslant 1, \quad 0 < \alpha \leqslant 2, \tag{4.4}$$

subjects to the initial conditions $u(0) = 1, u'(0) = 0$, and exact solution $u(x) = \cos x$ in case of $\alpha = 1$

Table 3 present a comparison between the absolute error obtained by HASTM approximate solution, modified laguerre wavelets method and the exact solutions given by Eq. (4.2). It is easily observed from the results displayed in this Table that the approximate solution obtained by HASTM is more accurate

Table 3: The approximate solution by HASTM at different values of $\alpha$, and comparison with exact solution and modified Laguerre wavelets method for Example 3.

| $x$ | Exact Solution | HASTM $\alpha = 0.75$ | HASTM $\alpha = 2$ | HASTM Absolute $\alpha = 1$ | Absolute error $\alpha = 1$ [17] |
|---|---|---|---|---|---|
| 0.1 | 0.995004165 | 0.988975813 | 0.995004165 | 0.00000 | $2.11000 \times 10^{-8}$ |
| 0.2 | 0.980066578 | 0.963174957 | 0.980066578 | 0.000 | $2.09000 \times 10^{-8}$ |
| 0.3 | 0.955336489 | 0.925890213 | 0.955336489 | 0.000 | $2.09000 \times 10^{-8}$ |
| 0.4 | 0.921060994 | 0.878995420 | 0.921060994 | $1.11022 \times 10^{-16}$ | $2.08000 \times 10^{-8}$ |
| 0.5 | 0.877582562 | 0.824018155 | 0.877582562 | $1.11022 \times 10^{-16}$ | $2.06000 \times 10^{-8}$ |
| 0.6 | 0.825335615 | 0.762342705 | 0.825335615 | 0.000 | $2.04000 \times 10^{-8}$ |
| 0.7 | 0.764842187 | 0.695278761 | 0.764842187 | $2.22045 \times 10^{-16}$ | $2.03000 \times 10^{-8}$ |
| 0.8 | 0.696706709 | 0.624087729 | 0.696706709 | $1.33227 \times 10^{-15}$ | $2.00000 \times 10^{-8}$ |
| 0.9 | 0.621609968 | 0.549991883 | 0.621609968 | $8.77076 \times 10^{-15}$ | $1.99000 \times 10^{-8}$ |
| 1.0 | 0.540302306 | 0.474175618 | 0.540302306 | $4.76286 \times 10^{-14}$ | $1.97000 \times 10^{-8}$ |




(a) (b) (c)

Figure 3: (a) Comparison between the approximate solution obtained by HASTM at different values of $\alpha$ and the exact one (b)Reprsents the $h$ curve (c) HASTM absolute error

than the approximate solution obtained by modified laguerre wavelets method. Fig. 3 displays the plot of the exact solution and the approximate solutions for some values of $\alpha$, and also represents the plot of the absolute error and the $h-curve$.

## 5. conclusions

In this research article, HASTM has been employed successfully for the first time to obtain a new accurate approximate solution for a classes fractional delay differential equation. It has been found that the construction of this procedure has a very rapid convergent series solution due to the general formula of its coefficients which can be determined after a few successive iterations. This procedure has been tested upon several examples of linear and nonlinear FDDes and gives a good approximation in few terms which is converged to the exact solution.

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