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Bernoulli Operational Matrix of Fractional Derivative for Solution of Fractional Differential Equations

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Abstract. The aim of this paper is to present a numerical method based on Bernoulli polynomials for numerical solutions of fractional differential equations(FDEs). The Bernoulli operational matrix of fractional derivatives[31] is derived and used together with tau and collocation methods to reduce the FDEs to a system of algebraic equations. Hence, the solutions obtained using this method give good approximations. Illustrative examples are included to demonstrate the validity and applicability of the proposed method.

Keywords: *Bernoulli polynomials, operational matrix of fractional derivatives, Caputo derivative, fractional-order differential equations.*
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1 Introduction

The Fractional order calculus is a generalization of ordinary differentiation and integration to arbitrary order. The idea of fractional calculus started from some speculations of G.W. Leibniz (1695-1697) and L. Euler (1730) [8]. A history of the development of fractional differential operators can be found in [8, 24, 27]. Fractional derivatives offer more accurate models of real world problems than integer derivatives. Fractional differential equations provides an excellent mathematical description for modeling many complex phenomena in different fields such as mechanics[29], biology[2, 14], chemistry[6], economics[4], control theory[28] and others.

The analytic results on the existence and uniqueness of solutions to the FDEs has been further investigated and discussed by many researchers. For more details, we refer the reader to [21, 27, 30]. In general, most of the FDEs do not have exact solutions. Particularly, there is no known method for finding exact solutions of FDEs. In order to investigate the diffuclty in obtaining analytical solution to the FDEs, many authors have proposed an approximate method for the numerical solution of FDEs, some of these methods are based on the use of operational matrices, which reduce the solution of the fractional order differential equations to the solution of algebraic equations. In order to achieve this, there are two approaches.

One approach is based on converting the underlying FDE to fractional integral equation through integration, and using the operational matrix of fractional order integration, to eliminate the integral operations. for example, Bernoulli polynomials [15], Bernoulli wavelets [20], Legendre polynomial [1], Chebyshev polynomials [5], Chebyshev wavelets [23] and Bernstein polynomials [34] .

Another very effective approach for solving FDEs is based on using operational matrix of fractional derivatives, to eliminate the differential operators, in order to reducing the underlying problem into solving a system of algebraic equations. In this respect, several authors employed these operational matrices for obtaining numerical solutions of FDEs, such as, Legendre polynomials [35], Legendre wavelets [16], Bernstein polynomials [33] and Genocchi

polynomials [17].

In the present paper, we introduce a the operational matrix of derivative based on Bernoulli polynomials [31], for solving numerically linear and non-linear FDEs, using tau and collocation methods respectively.

The paper is organized as follows, in section 2, some necessary definitions of the fractional calculus are introduced. An overview on Bernoulli polynomials, including some of their important properties, is also given in the same section together with arbitrary function approximation. In section 3, we derive the Bernoulli operational matrix of fractional order derivative in the Caputo sense. Section 4 is devoted to the numerical method for solving the FDEs with initial and boundary value. In Section 5 the proposed methods are applied to several examples. Finally, Section 6 concludes the paper.

2 Preliminaries and notations

2.1 The fractional derivative in the Caputo sense

There are several definitions of the fractional derivative of order μ. The most commonly used definitions are Riemann-Liouville and Caputo. The Caputo fractional derivative uses initial and boundary conditions of integer order derivatives having some physical interpretations. Because of this specific reason, in this work we shall use the Caputo fractional derivative (see [25] and [30]):

Definition 2.1. *The Caputo definition of the fractional-order derivative is*

$$D^{\mu} f(x) = \frac{1}{\Gamma(n-\mu)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu+1-n}} dt, \quad n-1 < \mu \leqslant n, n \in \mathbb{N}, \tag{1}$$

where $\mu > 0$ is the order of the derivative and n is the smallest integer greater than μ . For the Caputo derivative we have

$$D^{\mu} C = 0, \quad C \text{ is constant}, \tag{2}$$

$$D^{\mu} x^{q} = \begin{cases} 0, & \text{for } q \in \mathbb{N}_0 \text{ and } q < \lceil \mu \rceil, \\ \frac{\Gamma(q+1)}{\Gamma(q+1-\mu)} x^{q-\mu}, & \text{for } q \in \mathbb{N}_0 \text{ and } q \geq \lceil \mu \rceil \text{ or } q \notin \mathbb{N} \text{ and } q > \lfloor \mu \rfloor. \end{cases} \tag{3}$$

We use the ceiling function $\lceil \mu \rceil$ to denote the smallest integer greater than or equal to μ, and the floor function $\lfloor \mu \rfloor$ to denote the largest integer less than or equal to μ. Moreover $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\mu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operation

$$D^{\mu}(\gamma f(x) + \delta g(x)) = \gamma D^{\mu} f(x) + \delta D^{\mu} g(x), \tag{4}$$

where γ and δ are constants.

2.2 Properties of the Bernoulli polynomials

In this section, we recall some properties of the Bernoulli polynomials which will be of fundamental importance in the sequel. The classical Bernoulli polynomials $B_n(t)$ is usually defined by means of the exponential generating functions [3]

$$\frac{xe^{xt}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t) x^n}{n!}, \tag{5}$$

where $B_n(t)$ is the Bernoulli polynomials of degree n and is given on the interval $[0, 1]$ by [9]

$$B_n(t) = \sum_{i=0}^{n} \binom{i}{n} \alpha_{n-i} t^i, \tag{6}$$

where $\alpha_i = B_i(0)$, $i = 0, 1, ..., n$ are Bernoulli numbers. These numbers produce the following exponential generating function [3]

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i(0) \frac{t^i}{i!}.$$

The first Bernoulli numbers are

$$\alpha_0 = 1,\ \alpha_1 = -\frac{1}{2},\quad \alpha_2 = -\frac{1}{6}, \cdots,$$

with $\alpha_{2i+1} = 0$, $i = 1, 2, 3, \cdots$, and the first Bernoulli polynomials are

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t - \frac{1}{6},\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \cdots$$

The Bernoulli polynomials satisfy the following relations [12]:

$$\begin{aligned} B_n(0) = B_n(1) = \alpha_n, n \neq 1, \\ B_n^{'}(t) = nB_{n-1}(t), n \geq 1, \\ \int_0^1 B_n(t)\,\mathrm{d}t = 0, n \geq 1, \end{aligned} \tag{7}$$

Another interesting property that they satisfy is given by [3]:

$$\int_0^1 B_n(t)B_m(t)\,\mathrm{d}t = \frac{(-1)^{n+m}}{\binom{n+m}{n}}\alpha_{n+m},\quad n, m \geq 1. \tag{8}$$

According to [22], Bernoulli polynomials form a complete basis over the interval $[0, 1]$. If we introduce the Bernoulli vector $B(t)$ in the form

$$B(t)^T = [B_0(t), B_1(t), ..., B_N(t)], \tag{9}$$

then the derivative of $B(t)$, with the aid of (7), can be expressed in the matrix form by

$$\underbrace{\begin{bmatrix} B_0'(t) \\ B_1'(t) \\ B_2'(t) \\ \vdots \\ B_{N-1}'(t) \\ B_N'(t) \end{bmatrix}}_{B'(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N & 0 \end{bmatrix}}_{\mathbf{D}^{(1)}} \underbrace{\begin{bmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ \vdots \\ B_{N-1}(t) \\ B_N(t) \end{bmatrix}}_{B(t)} \tag{10}$$

where $\mathbf{D}^{(1)}$ is the $(N+1) \times (N+1)$ Bernoulli polynomials operational matrix of derivative.

2.3 Function approximation in terms of Bernoulli Polynomials

Suppose that $H = L^2([0, 1])$ and $\{B_0(t), B_1(t), ..., B_N(t)\} \subset H$ be the set of Bernoulli polynomials and $Y = Span\{B_0(t), B_1(t), ..., B_N(t)\}$. Since Y is a finite dimensional subspace, Y is a complete subspace of H [22]. Thus for an arbitrary element $g(t)$ in H, there exists a unique best approximation $(t) \in Y$ such that

$$\forall y(t) \in Y, \|g(t) - \widehat{g}(t)\| \leq \|g(t) - y(t)\|.$$

This implies that

$$\forall y(t) \in Y,\quad \langle g(t) - \widehat{g}(t), y(t)\rangle = 0, \tag{11}$$

where $\langle , \rangle$ denotes the inner product defined by

$$\langle f(t), g(t)\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t.$$

Since $\widehat{g} \in Y$, there exist unique vector $G^T = [g_0, g_1, ..., g_N]$ such that

$$g \simeq \widehat{g} = \sum_{n=0}^{N} g_n B_n(t) = G^T B(t).$$

Using Equation (11), we get

$$\langle g(t) - G^T B(t), B_i(t)\rangle = 0,\ \forall B_i(t) \in Y,$$

or equivalently

$$G^T \langle B(t), B(t)\rangle = \langle g(t), B(t)\rangle, \tag{12}$$

where $\langle B(t), B(t)\rangle = \int_0^1 B(t)B^T(t)\,\mathrm{d}t$ is an $(N+1)\times(N+1)$ matrix.

Let $M = \langle B(t), B(t)\rangle$, this matrix can be calculated using Eq (8). Therefore, any function $g(t) \in L^2[0,1]$ can be expanded by Bernoulli polynomials as $g(t) = G^T B(t)$, where

$$G = M^{-1}\langle g(t), B(t)\rangle. \tag{13}$$

2.3.1 Error bound

The error bounds of the truncated Bernoulli series expansion is discussed in [19]

Lemma 2.2. *Suppose $f(t) \in C^{m+1}[0,1]$ and $S_m = Span\{B_0(t), B_1(t), ..., B_N(t)\}$. If $C^T B(t)$ is the best approximation $f(t)$ out of S_m, then*

$$\left\|f(t) - C^T B(t)\right\|_{L^2[0,1]} \leq \frac{K}{(m+1)!\sqrt{2m+3}},$$

where $K = \max_{t\in[0,1]} \left|f^{(m+1)}(t)\right|$.

3 Bernoulli Operational Matrix of Fractional Order Derivative

The main objective of this section is to generalize the Bernoulli operational matrix of derivatives to the fractional calculus. From Eq.(10), it is clear that

$$\frac{d^n B(t)}{dt} = \left(\mathbf{D}^{(1)}\right)^n B(t), \tag{14}$$

where $n \in \mathbb{N}$ and the superscript in $\mathbf{D}^{(1)}$, denotes the matrix powers. Thus

$$\mathbf{D}^{(n)} = \left(\mathbf{D}^{(1)}\right)^n, \quad n = 1, 2, \cdots. \tag{15}$$

Lemma 3.1. *Let $B_i(t)$ be the ith Bernoulli polynomial. Then*

$$D^\mu B_i(t) = [0, 0, ..., 0]B_i(t), \qquad i = 0, 1, ..., \lceil\mu\rceil - 1,\ \mu > 0. \tag{16}$$

Proof. The lemma can easily be proved using Eqs. (2)–(6). □

In the following theorem we derive the operational matrix of fractional order derivative for the Bernoulli polynomials.

Theorem 3.2. *Let $B(t)$ be the Bernoulli vector given in (9) and suppose $\mu > 0$. Then,*

$$D^\mu B(t) \simeq \mathbf{D}^{(\mu)} B(t), \tag{17}$$

where $\mathbf{D}^{(\mu)}$ is $(N+1)\times(N+1)$ operational matrix of fractional derivative of order μ in the Caputo sense and is defined as follows

$$\mathbf{D}^{(\mu)} = \begin{bmatrix} 0 & 0 & ... & 0 \\ \vdots & \vdots & ... & 0 \\ 0 & 0 & .. & 0 \\ \theta^{(\mu)}_{\lceil\mu\rceil,0,r} & \theta^{(\mu)}_{\lceil\mu\rceil,1,r} & ... & \theta^{(\mu)}_{\lceil\mu\rceil,N,r} \\ \vdots & \vdots & ... & \vdots \\ \sum\limits_{r=\lceil\mu\rceil}^{i} \theta^{(\mu)}_{i,0,r} & \sum\limits_{r=\lceil\mu\rceil}^{i} \theta^{(\mu)}_{i,1,r} & ... & \sum\limits_{r=\lceil\mu\rceil}^{i} \theta^{(\mu)}_{i,N,r} \\ \vdots & \vdots & ... & \vdots \\ \sum\limits_{r=\lceil\mu\rceil}^{N} \theta^{(\mu)}_{N,0,r} & \sum\limits_{r=\lceil\mu\rceil}^{N} \theta^{(\mu)}_{N,1,r} & ... & \sum\limits_{r=\lceil\mu\rceil}^{N} \theta^{(\mu)}_{N,N,r} \end{bmatrix}, \tag{18}$$

where $\theta_{i,j,r}^{(\mu)}$ *is given by*

$$\theta_{i,j,r}^{(\mu)} = \frac{i\,!\alpha_{i-r}}{(i-r)\,!\Gamma(r+1-\mu)} c_{r,j}, \tag{19}$$

with α_i *being the Bernoulli number and* $c_{r,j}$ *can be obtained from (13). Note that in* $\mathbf{D}^{(\mu)}$*, the first* $\lceil \mu \rceil$ *rows coefficients are all equal to zero.*

Proof. Using Equations (3), (4) and (6) we have

$$\begin{aligned} D^{\mu} B_i(t) &= \sum_{r=0}^{i} \binom{i}{r} \alpha_{i-r} D^{\mu} t^r \\ &= \sum_{r=0}^{i} \binom{i}{r} \alpha_{i-r} \frac{\Gamma(r+1)}{\Gamma(r+1-\mu)} t^{r-\mu} \\ &= \sum_{r=0}^{i} \frac{i!}{r!\,(i-r)!} \alpha_{i-r} \frac{\Gamma(r+1)}{\Gamma(r+1-\mu)} t^{r-\mu} \\ &= \sum_{r=0}^{i} \frac{i!}{(i-r)!} \frac{\alpha_{i-r}}{\Gamma(r+1-\mu)} t^{r-\mu} \end{aligned}$$

or

$$D^{\mu} B_i(t) = \sum_{r=\lceil \mu \rceil}^{i} b_{i,r}^{(\mu)} t^{r-\mu}, \qquad i = \lceil \mu \rceil, ..., N. \tag{20}$$

where

$$b_{i,r}^{(\mu)} = \frac{i! \alpha_{i-r}}{(i-r)! \Gamma(r+1-\mu)}.$$

Assuming that $t^{r-\mu}$ can be expanded in terms of Bernoulli polynomials as

$$t^{r-\mu} \simeq \sum_{j=0}^{N} c_{r,j} B_j(x), \tag{21}$$

and substituting Eq(21) in Eq (20) for $j = 0, 1, ..., N$, we get

$$D^{\mu} B_i(x) \simeq \sum_{r=\lceil \mu \rceil}^{i} b_{i,r}^{(\mu)} \sum_{j=0}^{N} c_{r,j} \beta_j(x) = \sum_{j=0}^{N} \left(\sum_{r=\lceil \mu \rceil}^{i} \theta_{i,j,r}^{(\mu)} \right) \beta_j(x), \tag{22}$$

where

$$\theta_{i,j,r}^{(\mu)} = b_{i,r}^{(\mu)} c_{r,j}.$$

Equation (22) can be rewritten as

$$D^{\mu} B_i(t) \simeq \left[\sum_{r=\lceil \mu \rceil}^{i} \theta_{i,0,r}^{(\mu)}, \sum_{r=\lceil \mu \rceil}^{i} \theta_{i,1,r}^{(\mu)}, ..., \sum_{r=\lceil \mu \rceil}^{i} \theta_{i,N,r}^{(\mu)} \right] B(t), \; i = \lceil \mu \rceil, ..., N. \tag{23}$$

Finally, combining Lemma 3.1 and relations (23), leads to the desired result. □

Remark 3.3. *If* $\mu \in \mathbb{N}$*, then Theorem* 3.2 *gives the same result as Eq.*(10)*.*

4 The numerical method

In this section, we we apply the Bernoulli operational matrix of fractional derivative for solving linear and non-linear fractional differential equations. For the existence, uniqueness and continuous dependence of the solution of the problem see [10].

4.1 Linear fractional differential equation

Consider the linear Caputo fractional differential equations of the form

$$D^{\mu}y(t) = a_k D^{\alpha_k}y(t) + a_{k-1}D^{\alpha_{k-1}}y(t) + \cdots + a_1 D^{\alpha_1}y(t) + a_0 y(t) + g(t), t \in (0, L) \tag{24}$$

subject to the initial conditions

$$y^{(i)}(0) = y_i, \qquad i = 0, 1, ..., m-1, \tag{25}$$

where $a_0, a_1, ..., a_k$ are real constant coefficients and also $\mu \in]m-1, m[$, $0 < \alpha_1 < \alpha_2 < ... < \alpha_k < \mu$. Here D^{μ} is the derivative of y of order μ in the sense of Caputo fractional differential operator. Moreover, $y(t)$ is an unknown function of the independent variable t. The values of y_i, $i = 0, ..., m-1$ describe the initial state of $y(t)$, and $g(t)$ is a given source function.

Now, we approximate $y(t)$ and $g(t)$ with the Bernoulli polynomial as

$$y(t) \simeq \sum_{i=0}^{N} c_i B_i(t) = C^T B(t). \tag{26}$$

$$g(t) \simeq \sum_{i=0}^{N} G_i B_i(t) = G^T B(t). \tag{27}$$

where the vector G is known (can be obtained from (13)) and is described in subsection (2.3), but $C^T = [c_0, c_1, ..., c_N]$ is an unknown vector. In virtue of Theorem 3.2 and Eq.(26), the fractional state rates $D^{\mu}y(t)$ and $D^{\alpha_j}y(t)$ can be approximated as

$$D^{\mu}y(t) \simeq C^T \mathbf{D}^{(\mu)} B(t), \tag{28}$$

and

$$D^{\alpha_j}y(t) \simeq C^T \mathbf{D}^{(\alpha_j)} B(t), \qquad j = 1, ..., k. \tag{29}$$

From Eqs.(26)-(29), the residual $R(t)$ for Eq.(24) can be written as

$$R(t) = \left[C^T (\mathbf{D}^{(\mu)} - \sum_{j=1}^{k} a_j \mathbf{D}^{(\alpha_j)} - a_0 I_{N+1}) - G^T \right] B(t) = 0 \tag{30}$$

where I_N denotes the identity matrix of dimension $N+1$. As in a typical tau method [7] , we can generate $N-m+1$ linear equations by applying

$$\langle R(t), B_i(t) \rangle = \int_0^L R(t) B_i(t)\, \mathrm{d}t = 0, \qquad i = 0, ..., N-m. \tag{31}$$

Moreover, substituting Eqs.(14) and (26) into Eq. (25), we obtain

$$y^{(i)}(0) = C^T \mathbf{D}^{(i)} B(0) = y_i, \quad i = 0, 1, ..., m-1. \tag{32}$$

Eqs. (31) and (32) generate $(N-m+1)$ and m linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $y(t)$ given in Eq. (26) can be calculated and gives a solution of Eq.(24) with the initial conditions (25).

4.2 Non-linear fractional differential equation

Consider the following non-linear fractional-order differential equation (NFDE):

$$D^{\mu}y(t) = F(t, y(t), D^{q_1}y(t), ..., D^{q_k}y(t)), \tag{33}$$

subject to the initial conditions (32). Where $m-1 < \mu \leq m$, $0 < q_1 < q_2 < ... < q_k < \mu$. It is to be noted here that F can be non-linear in general.

We approximate $y(t)$, $D^{\mu}y(t)$ and $D^{q_j}y(t)$ for $j = 1, ..., k$ as in the previous section, hence the residual $\overline{R}_N(t)$ of Eq.(33) is given by:

$$\overline{R}(t) = C^T \mathbf{D}^{(\mu)} B(t) - F(t, C^T B(t), C^T \mathbf{D}^{(q_1)} B(t), ..., C^T \mathbf{D}^{(q_k)} B(t)), \tag{34}$$

where $C^T = [c_0, c_1, ..., c_N]$ is an unknown vector. To find the solution of (33), we first calculate Eq. (34) at $N-m+1$ points. In terms of order the optimal choice is to let the collocation points be the $N-m+1$ roots of shifted Legendre polynomial $P_{N-m+1}(t)$ [13]. We obtain

$$\overline{R}(t_i) = 0, \quad i = 1, ..., N-m+1. \tag{35}$$

Eqs. (35) with (32) generate $N-1$ non-linear system of equations in the unknown expansion coefficients C of dimension $(N+1)$. Any standard iteration technique, like Newton's iterative technique can be employed for solving this system and hence the approximate solution $y(t)$ can be obtained.

Remark 4.1. *It is worthy to mention here that the method described above in section* 4.2*, can be also applied to linear fractional differential equations.*

5 Illustrative examples

Example 5.1. *We consider the following non-linear initial value problem (see [18])*

$$D^3 y(t) + D^{\frac{5}{2}} y(t) + y^2(t) = t^4, t \in [0,1] \tag{36}$$

$$y(0) = y'(0) = 0, \ y''(0) = 2. \tag{37}$$

The exact solution of this problem is $y(x) = t^2$.

We solve the above Eq. (36) *with initial condition* (37)*, by applying the technique described in Section* 4.2 *with* $N = 3$. *We approximate the solution as*

$$y(t) = \sum_{i=0}^{3} c_i B_i(t) = C^T B(t).$$

thus,

$$D^3 y(t) = C^T \mathbf{D}^{(3)} B(t), \ D^{\frac{5}{2}} y(t) = C^T \mathbf{D}^{\left(\frac{5}{2}\right)} B(t).$$

where

$$\mathbf{D}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \ \mathbf{D}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}, \ \mathbf{D}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}^{\left(\frac{5}{2}\right)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{8}{\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} & -\frac{48}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{bmatrix} \text{ and } C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Therefore Eqs. (36) *and* (37) *become*

$$C^T \mathbf{D}^{(3)} B(t) + C^T \mathbf{D}^{\left(\frac{5}{2}\right)} B(t) + \left[C^T B(t)\right]^2 - t^4 = 0 \tag{38}$$

$$C^T B(0) = C^T \mathbf{D}^{(1)} B(0) = 0, \ C^T \mathbf{D}^{(2)} B(0) = 2. \tag{39}$$

Using Eq. (39)*, we get*

$$\begin{cases} c_0 - \frac{1}{2}c_1 + \frac{1}{6}c_2 = 0 \\ c_1 - c_2 + \frac{1}{2}c_3 = 0 \\ 2c_2 - 3c_3 = 2 \end{cases} \tag{40}$$

Now collocate Eq.(38) *at the the first root of shifted Legendre polynomials* $t_0 = \frac{1}{2}$*, to obtain*

$$C^T D^{(3)} B(t_0) + C^T D^{\left(\frac{5}{2}\right)} B(t_0) + \left[C^T B(t_0)\right]^2 - t_0^4 = 0 \tag{41}$$

Solving Eqs. (41) *and* (40) *using Newton's iterative method we obtain*

$$c_0 = \frac{1}{3}, c_1 = 1, c_2 = 1, c_3 = 0$$

Therefore

$$y(t) = \left[\frac{1}{3}, 1, 1, 0\right] \begin{bmatrix} 1 \\ t - \frac{1}{2} \\ t^2 - t + \frac{1}{6} \\ t^3 - \frac{3}{2}t^2 + \frac{1}{2}t \end{bmatrix} = t^2,$$

which is the exact solution of this problem.

Example 5.2. *Consider the following linear fractional initial value problem (see [11])*

$$D^2 y(t) + D^{\frac{1}{2}} y(t) + y(t) = g(t), \quad g(t) = t^2 + 2 + \frac{8}{3\sqrt{\pi}} t^{1.5}, \quad t \in [0, 1] \tag{42}$$

$$y(0) = y'(0) = 0. \tag{43}$$

By applying the technique described in Section 4.1 *for the case* $N = 2$, *the residual of Eq.*(42) *can be calculated by the formula:*

$$R(t) = C^T \mathbf{D}^{(2)} B(t) + C^T \mathbf{D}^{\left(\frac{1}{2}\right)} B(t) + C^T B(t) - G^T B(t) \tag{44}$$

where

$$\begin{aligned} \mathbf{D}^{(1)} &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \ \mathbf{D}^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \ \mathbf{D}^{\left(\frac{1}{2}\right)} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{3\sqrt{\pi}} & \frac{8}{5\sqrt{\pi}} & \frac{-8}{7\sqrt{\pi}} \\ -\frac{4}{15\sqrt{\pi}} & \frac{8}{7\sqrt{\pi}} & \frac{8}{3\sqrt{\pi}} \end{bmatrix}, \\ G^T &= \left[\frac{7}{3} + \frac{16}{15\sqrt{\pi}}, 1 + \frac{96}{35\sqrt{\pi}}, 1 + \frac{32}{21\sqrt{\pi}}\right] \ \textit{and} \ C^T = [c_0, c_1, c_2]. \end{aligned}$$

If we apply the method described in Section 4.1 *to Eq.* (43), *we find*

$$c_0 + \frac{4}{3\sqrt{\pi}} c_1 + (2 - \frac{4}{15\sqrt{\pi}}) c_2 - (\frac{7}{3} + \frac{16}{15\sqrt{\pi}}) = 0 \tag{45}$$

The initial conditions in Eq. (43) *give*

$$c_0 - \frac{1}{2} c_1 + \frac{1}{6} c_2 = 0 \qquad \textit{and} \qquad c_1 - c_2 = 0 \tag{46}$$

Solving Eqs. (45) *and* (46) *yield* $c_0 = \frac{1}{3}, c_1 = c_2 = 1$, *and consequently* $y(t) = t^2$ *which is the exact solution of this problem.*

Example 5.3. *Consider the following linear fractional equation*

$$D^\alpha y(t) + y(t) - t^2 - \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} = 0, \quad 0 < \alpha < 1 \tag{47}$$

with the initial condition

$$y(0) = 0. \tag{48}$$

Using the technique described in Section 4.1 *we solve Eq.* (47) *with initial condition* (48) *for* $\alpha = 0.5$ *and* $N = 2$,

$$c_0 + \frac{4}{3\sqrt{\pi}} c_1 - \frac{4}{15\sqrt{\pi}} c_2 - (\frac{1}{3} + \frac{16}{15\sqrt{\pi}}) = 0 \tag{49}$$

$$(\frac{2}{15\sqrt{\pi}} + \frac{1}{12}) c_1 + \frac{2}{21\sqrt{\pi}} c_2 - (\frac{1}{12} + \frac{8}{35\sqrt{\pi}}) = 0 \tag{50}$$

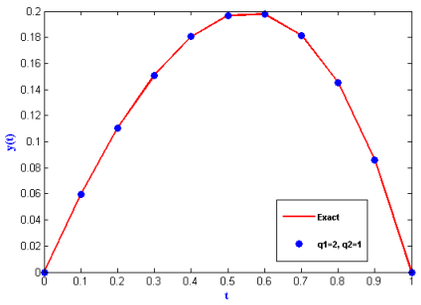


Fig. 1: Comparison of our method with $N = 5$ and the exactsolution for $q_1 = 2$ and $q_2 = 1$.

N	2	3	5	7	9
l^2 *Error*	4.22×10^{-02}	1.47×10^{-02}	2.17×10^{-05}	2.66×10^{-08}	4.42×10^{-11}
l^∞ *Error*	2.02×10^{-02}	7.28×10^{-03}	1.21×10^{-05}	1.52×10^{-08}	1.09×10^{-11}

Table 1: The l^2 and l^∞ errors for Example 4.

$$c_0 - \frac{1}{2}c_1 + \frac{1}{6}c_2 = 0 \tag{51}$$

Solving the system of Eqs. (49), (50)*and*(51) *we obtain*

$$y(t) = \left[\frac{1}{3}, 1, 1\right] \begin{bmatrix} 1 \\ t - \frac{1}{2} \\ t^2 - t + \frac{1}{6} \end{bmatrix} = t^2,$$

which is the exact solution.

Example 5.4. *Consider the following linear boundary value problem [32]*

$$D^{q_1}y(t) - D^{q_2}y(t) = -1 - e^{t-1},\ 0 < t < 1,\ 1 < q_2 \leq 2,\ 0 < q_1 \leq 1. \tag{52}$$

$$u(0) = u(1) = 0. \tag{53}$$

The exact solution of Eq.(52) *with boundary condition* (54) *corresponding to* $q_1 = 2$ *and* $q_2 = 1$ *is* $y(t) = t(1 - e^{t-1})$. *In Table* 1, *we compare numerical results of* $y(t)$ *using our method with* $N = 6$ *and* $N = 7$ *with the exact solution and those obtained in* [*20*, *36*]. *Table* 1 *shows the* l^2 *and* l^∞ *errors between the exact and approximate solutions for* $q_1 = 2$ *and* $q_2 = 1$ *with different values of* N. *In Figure 1, the exact solution* $y(t)$ *and the approximate solution for* $q_1 = 2$ *and* $q_2 = 1$ *are plotted. Figure 2 illustrates the approximate solutions corresponding to* $q_1 = 2$ *and different values of* q_2 *near the value 1 (*$q_2 = 0.7, 0.8, 0.9, 1$*). Furthermore, figure* 3 *displays the numerical results obtained for* $q_2 = 1$ *and* $q_1 = 1.7, 1.8, 1.9, 2$ *together with the exact solution. These Figures illustrate that series expansion of* $y(t)$ *has a good convergence rate.*

Example 5.5. *Consider the following non-linear fractional Riccati differential equation [23, 26]*

$$D^q y(t) = 2y(t) - y(t)^2 + 1, \qquad 0 < q \leq 1, \tag{54}$$

with initial condition

$$y(0) = 0. \tag{55}$$

The exact solution of this problem for the case $q = 1$ *is*

$$y(t) = 1 + \sqrt{2}\tanh\left[\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right].$$

This example is solved in [*20*] *using Bernoulli wavelets method, with* $k = 2$ *and* $M = 10$. *It was also studied in* [*23*] *and* [*26*] *using the Chebyshev wavelet method and the modified homotopy perturbation method, respectively. Here we*

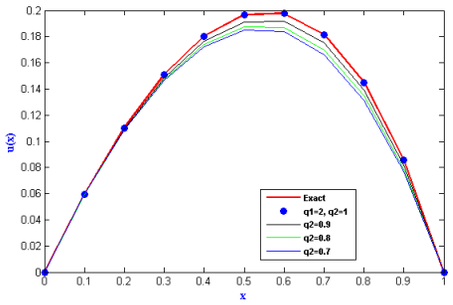


Fig. 2: Comparison of $y(t)$ for $N = 6$ in the case corresponding to $q_1 = 2$ and $q_2 = 0.7, 0.8, 0.9, 1$ with exact solution for Example 4

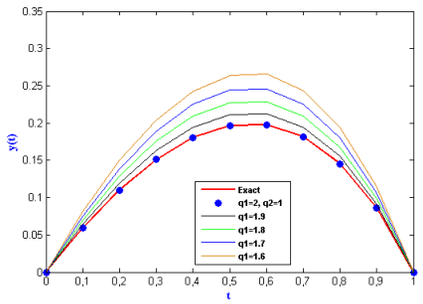


Fig. 3: Comparison of $y(t)$ for $N = 6$ in the case corresponding to $q_2 = 1$ and $q_1 = 1.7, 1.8, 1.9, 2$ with exact solution for Example 4

t	*[26]*	*[23]*	*[20]*	*Our Method*	*Exact solution for q = 1*
0	0	-	0.000001	0	0
0.1	0.1102940	0.110311	0.110295	0.110295	0.110295
0.2	0.2419650	0.241995	0.241977	0.241976	0.241976
0.3	0.395106	0.395123	0.395105	0.395104	0.395104
0.4	0.568115	0.567829	0.567812	0.567812	0.567812
0.5	0.757564	0.756029	0.756015	0.756014	0.756014
0.6	0.958259	0.953576	0.953566	0.953566	0.953566
0.7	1.163459	1.152955	1.152949	1.152949	1.152948
0.8	1.365240	1.346365	1.346364	1.346363	1.346363
0.9	1.554960	1.526909	1.526911	1.526911	1.526911
1	1.723810	1.689494	1.689497	1.6894983	1.6894983

Table 2: Numerical results for Example 5.

solve this example using the method described in section 4.2. *Our results of the approximate solution with* $N = 5$, *when* $q = 1$ *are compared with the exact solution in Figure* 4. *In Table* 2, *we compare numerical results of* $y(t)$ *using our method for* $N = 10$ *with the exact solution and those obtained in [26],[23] and [20]. The absolute error for the solution* $y(t)$ *obtained with different values of* $N = 3, 5, 7, 9, 13$ *of the case* $q = 1$ *are shown in Table 2. In Figure 5 we compare the results obtained by the present approach with* $N = 10$, *for* $q = 0.9, 0.75, 0.5$ *with the exact solution, in which the figures affirm that when* q *approaches 1, our results approach the exact solution.*

Example 5.6. *Consider another fractional order Riccati differential equation [18, 20]:*

$$D^q y(t) = 1 - y(t)^2, \qquad 0 < q \leq 1, \tag{56}$$

subject to the initial condition

$$y(0) = 0. \tag{57}$$

N	2	4	6	8	10	12
l^2 *Error*	9.60×10^{-02}	3.62×10^{-03}	1.32×10^{-04}	5.37×10^{-06}	1.64×10^{-07}	3.69×10^{-8}
l^∞ *Error*	4.75×10^{-02}	1.85×10^{-03}	7.47×10^{-05}	3.03×10^{-06}	1.09×10^{-07}	2.75×10^{-8}

Table 3: The l^2 and l^∞ errors for Example 5.

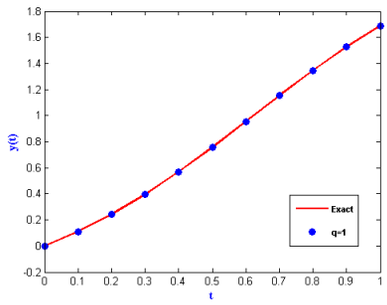


Fig. 4: Comparison of $y(t)$ for $N = 5$ in the case corresponding to $q = 1$ with exact solution for Example 5

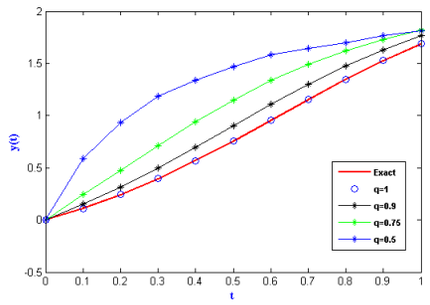


Fig. 5: Comparison of $y(t)$ for $N = 10$ in the case corresponding to $q = 1.7, 1.8, 1.9, 1$ with exact solution for Example 5

t	*in* [20]	*Our Method*		*Exaxt Solution*
		N=5	N=6	*for q=1*
0.1	0.099684	0.099658	0.099668	0.099667
0.2	0.197417	0.197407	0.197377	0.197375
0.3	0.291316	0.291333	0.291310	0.291312
0.4	0.379912	0.379931	0.379945	0.379948
0.5	0.462083	0.462083	0.462117	0.462117
0.6	0.537057	0.537036	0.537053	0.537049
0.7	0.604405	0.604387	0.604369	0.604367
0.8	0.664052	0.664061	0.664032	0.664036
0.9	0.716264	0.716289	0.716296	0.716297

Table 4: Numerical results for Example 6.

The exact solution of this system when $q = 1$ is known to be [18]

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

This example is solved in [18] using Legendre wavelets method, with $k = 1$ and $M = 25$. It was also considered in [20] by applying the Bernoulli wavelets method, with $k = 2$ and $M = 5$. We solve this example using the technique described in section 4.2. Our results of the approximate solution with $N = 5$, when $q = 1$ are compared with the exact solution in Figure 6. In Table 4, we compare numerical results of $y(t)$ using our method for $N = 10$ with the exact solution and that obtained in [20]. The absolute error for the solution $y(t)$ obtained with different values of $N = 2, 4, 6, 8, 12$ of the case $q = 1$ are shown in Table 5. In Figure 7 we compare the results obtained by our approach for $N = 10$, when $q = 0.9, 0.75, 0.5$ with the exact solution. From Figure 7 we see that as q approaches 1 the numerical solutions converge to that of integer order differential equation.

Example 5.7. *Consider the following nonlinear fractional differential equations [37]*

$$\begin{cases} D^q y_1(t) &= -1002 y_1(t) + 1000 y_2^2(t) \\ D^q y_2(t) &= y_1(t) - y_2(t) - y_2^2(t) \end{cases} \tag{58}$$

N	2	4	6	8	10	12
$l_2 Error$	1.34×10^{-02}	1.42×10^{-04}	7.62×10^{-06}	2.50×10^{-07}	4.26×10^{-09}	7.58×10^{-11}
l_∞ $Error$	8.05×10^{-02}	9.10×10^{-05}	4.24×10^{-06}	1.21×10^{-07}	2.37×10^{-09}	4.11×10^{-11}

Table 5: The l^2 and l^∞ errors for Example 6.

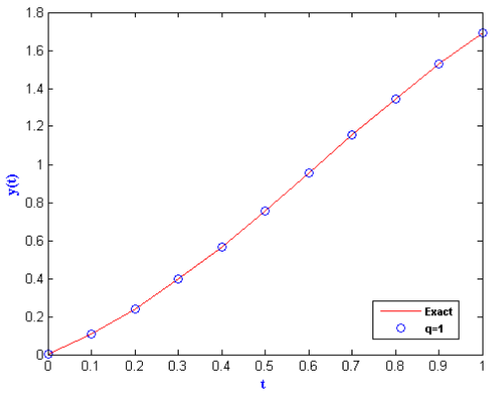


Fig. 6: Comparison of $y(t)$ for $N = 5$ in the case corresponding to $q = 1$ with exact solution for Example 6

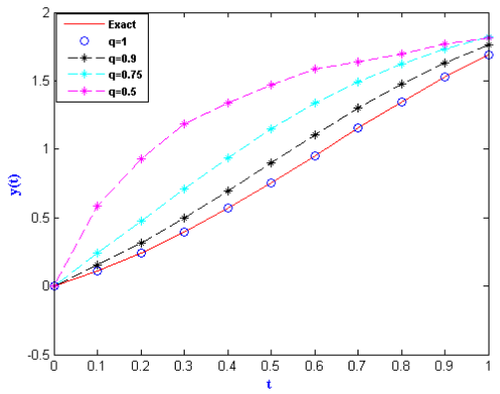
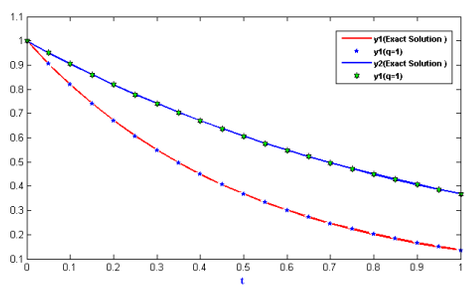


Fig. 7: Comparison of $y(t)$ for $N = 10$ in the case corresponding to $q = 0.5, 0.75, 0.9$ and 1 with exact solution for Example 6

Fig. 8: Comparison of $y_1(t)$ and $y_2(t)$ by our method with $N = 10$ in the case corresponds to $q = 0.25, 0.5, 0.75, 1$ with exact solution for Example 7



subject to the initial conditions $y_1(0) = 1$ *and* $y_2(0) = 1$. *The exact solution of this system for* $q = 1$ *is known to be* $y_1(t) = e^{-2t}$ *and* $y_1(t) = e^{-t}$. *This example is solved by our method for* $N = 10$ *and* $q = 1$. *The results are compared with the exact solution* ($q = 1$) *in Figure* 8. *The results obtained when* $q = 0.9, 0.75, 0.5$ *and* 0.25 *for* $y_1(t)$ *and* $y_1(t)$ *are plotted in Figure* 9. *The figures confirm that when* q *approaches 1 our results approach the exact solution. We also compare the absolute error obtained by our method and those obtained in [37] and [17] at* $t = 1$ *in Table* 6.

t	$y(t)$	Error in [37] with $h = 0.002$	Error in [17] with $N = 10$	Our Error with $N = 10$
1	$y_1(t)$	2.5606×10^{-7}	3.70730×10^{-8}	1.98881×10^{-9}
	$y_2(t)$	8.0150×10^{-8}	2.09476×10^{-11}	1.99036×10^{-12}

Table 6: The absolute errors obtained by the present method and that in [37] and [17] at $t = 1$ for Example 7

6 Conclusion

In this work, a general formulation for the Bernoulli operational matrix of fractional derivative has been derived. This matrix is used to approximate numerical solution of linear and nonlinear FDEs. Our approach was based on the truncated Bernoulli polynomials expansion, Tau and collocation methods. The advantage of the present operational matrix method is that it has less computational complexity because every operational matrix of differentiation involves mostly zeros entries and thus reduces the run time and provides solutions with high accuracy. The solution obtained for these examples show that this approach can effectively solve these problems and is very simple and easy in implementation.

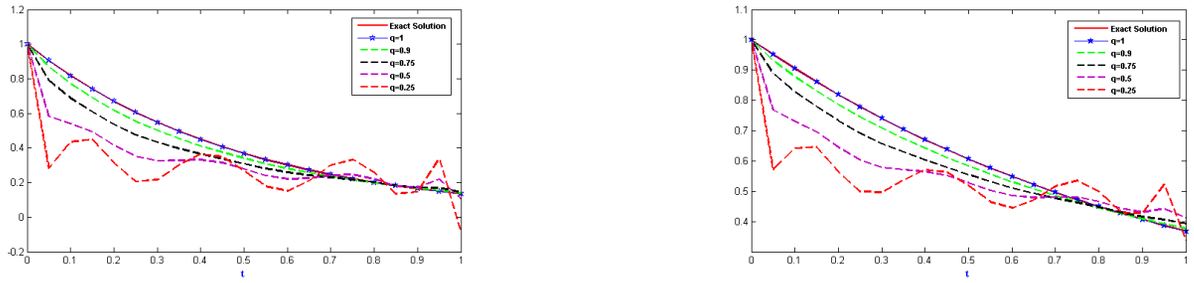


Fig. 9: Comparison of $y_1(t)$ and $y_2(t)$ for $N = 10$ in the case corresponds to $q = 0.25, 0.5, 0.75, 1$ with exact solution for Example 7

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