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A [49 16 6]-Linear Code as Product Of The [7 4 2] Code Due to Aunu and the Hamming [7 4 3] Code

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***Abstract*:** The enumeration of the construction due to "Audu and Aminu"(AUNU) Permutation patterns, of a [7 4 2]- linear code which is an extended code of the [6 4 1] code and is in one-one correspondence with the known [7 4 3] - Hamming code has been reported by the Authors. The [7 4 2] linear code, so constructed was combined with the known Hamming [7 4 3] code using the (u|u+v)-construction method to obtain a new hybrid and more practical single [14 8 3] error- correcting code. In this paper, we provide an improvement by obtaining a much more practical and applicable double error correcting code whose extended version is a triple error correcting code, by combining the same codes as in [1]. Our goal is achieved through using the product code construction approach with the aid of some proven theorems.

Keywords: *Cayley tables, AUNU Scheme, Hamming codes, Generator matrix,Tensor product*
2010 MSC No: *97H60, 94A05 and 94B05*

1. Introduction

Historically, Claude Shannon's paper titled "A Mathematical theory of Communication" in the early 1940s signified the beginning of coding theory and the first error-correcting code to arise was the presently known Hamming $[7 \quad 4 \quad 3]$ code, discovered by Richard Hamming in the late 1940's (David & Joh-Lark, 2011). As it is central, the main objective in coding theory is to devise methods of encoding and decoding so as to effect the total elimination or minimization of errors that may have occurred during transmission due to disturbances in the channel. The special class of the (132) *and* $(123)-$ avoiding Patterns of AUNU permutations reported by (Ibrahim & Audu ,2005), has found applications in various areas of applied Mathematics. The application of the adjacency matrix of Eulerian graphs due to the (132) -avoiding patterns of AUNU numbers in the generation and analysis of some classes of linear and cyclic codes has been reported [2],[3]. In [4],the Authors utilized the Cayley tables for $n=5$ to derive a standard form of the generator /parity check matrix for some code. An enumeration of the construction of a [7 4 2] – linear Code from the Cayley table for $n=7$ of the generated points of W as permutations of the (132) *and* (123) -avoiding patterns of the non-commutative AUNU schemes has been reported [6]. Moreover, the [7 4 2]-linear code so generated was combined with the known Hamming [7 4 3] code using the (u|u+v) construction method to obtain a new and more Practical single error correcting code with dimensions $n=14,\ k=8$ *and* $d=3$ [1].

In this paper, we construct a larger code, the [49 16 6] linear code with improved error-correcting capabilities by combining the same codes as in [1]. Here, another approach (The Product code) method a long side some proven theorems are utilized.

2. Basic Definitions (Concepts)

2.1 Product Codes: Let C_1 be a binary (n_1, k_1) linear block code and C_2 be a binary (n_2, k_2) linear block code. A code with $n_1.n_2$ symbols can be constructed by forming a rectangular array of n_1 columns and n_2 rows in which every row is a code word in C_1 and every column is a code word in C_2 as shown in the figure q below;

$$\left[\begin{array}{cccc:ccc}
v_{0,0} & v_{0,1} & \cdots & v_{0,n_1-k_1-1} & v_{0,n_1-k_1} & \cdots & v_{0,n_1-1} \\
v_{1,0} & v_{1,1} & \cdots & v_{1,n_1-k_1-1} & v_{1,n_1-k_1} & \cdots & v_{1,n_1-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{k_2-1,0} & v_{k_2-1,1} & \cdots & v_{k_2-1,n_1-k_1-1} & v_{k_2-1,n_1-k_1} & \cdots & v_{k_2-1,n_1-1} \\
\hdashline
v_{k_2,0} & v_{k_2,1} & \cdots & v_{k_2,n_1-k_1-1} & v_{k_2,n_1-k_1} & \cdots & v_{k_2,n_1-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
v_{n_2-1,0} & u_{n_2-1,1} & \cdots & v_{n_2-1,n_1-k_1-1} & v_{n_2-1,n_1-k_1-1} & \cdots & v_{n_2-1,n_1-1}
\end{array}\right]$$

Figure 1: A code array in a two – dimensional product code

The $k_1.k_2$ symbols in the upper right quadrant of the array are information symbols. The $k_2(n_1 - k_1)$ symbols in the upper left quadrant of the array are parity-check symbols formed from the parity – check rules for code C_1 and the $k_1(n_2 - k_2)$ symbols in the lower right quadrant are parity-check symbols formed from the parity- check rules for code C_2. The $(n_1 - k_1)(n_2 - k_2)$ parity-check symbols in the lower left quadrant can be formed by using either the parity check rules for code C_2 on columns or the parity-check codes for code C_1 on rows.

The rectangular array shown on figure 1 is called a code array that consist of $k_1.k_2$ information symbols and $n_1.n_2 - k_1.k_2$ parity- check symbols. There are $2^{k_1.k_2}$ such code arrays. The sum of two code arrays is an array obtained by adding either their corresponding rows or their corresponding columns. Since the rows or columns are code words in C_1 or C_2, the sum of two corresponding rows (or columns) in two code arrays is a code word in C_1 (or C_2). Consequently, the sum of two code arrays is another code array. Hence the $2^{k_1.k_2}$ code arrays form a two dimensional $(n_1.n_2, k_1.k_2)$ linear block code denoted by $C_1 \otimes C_2$, which is called the direct product or (simply the product) of C_1 and C_2 [7].

2.2 Definition Let C_1 and C_2 be two $[n_1, k_1\ d_1]$ and $[n_2, k_2\ d_2]$ codes respectively and $n = n_1.n_2$. Then the product code denoted by $C_1 \otimes C_2$ is defined as

$$C_1 \otimes C_2 = \left\{ (C_{ij})1 \le i \le n_1, 1 \le j \le n_2 \left| \begin{array}{l} (C_{ij})_{1 \le i \le n_1} \text{ for all } j \\ (C_{ij})_{1 \le j \le n_2} \text{ for all } i \end{array} \right. \right\}.$$

Remark

From the above definition, the product code $C_1 \otimes C_2$ is exactly the set of all $n_1 \times n_2$ arrays whose columns belongs to C_1 and rows to C_2. In literature, the product code is called the direct product or Kronecker product or Tensor product

Theorem1: Let $x_1, x_2, \ldots x_{k_1}$ in $F_q^{n_2}$ be a basis for C_1 and $y_1, y_2, \ldots y_{n_2}$ in $F_q^{n_2}$ be a basis for C_2, then $\{x_i \otimes y_j \,|1 \le i \le k_1, 1 \le j \le k_2\}$ is a basis for $C_1 \otimes C_2$.

Proof

The given set is an independent set by proven results. This set is a subset of $C_1 \otimes C_2$. So the dimension of $C_1 \otimes C_2$ is at least $k_1.k_2$. Now we will show that they form in fact a basis for $C_1 \otimes C_2$. Without lost of generality, we may assume that C_1 is systematic at the first k_1 coordinates with generator matrix $(I_{k_1} \mid A)$ and C_2 is systematic at the first k_2 coordinates with generator matrix $(I_{k2} \mid B)$. Then U is an $L \times n_2$ matrix with rows in C_2 *iff* $U = (M \,|MB)$ where M is an $L \times n_2$ matrix and V is an $n_1 \times M$ matrix, with columns in C_1 *ff* $V^T = (N \,|NA)$, where N is an $M \times k_1$ matrix. Now, let M be an $k_1 \times k_2$ matrix, then $(M \,|MB)$ is a $k_1 \times n_2$ matrix with rows in C_2 and $\begin{pmatrix} M \\ A^T M \end{pmatrix}$ is an $n_1 \times k_2$ matrix with columns in C_1. Therefore $\left(\begin{array}{c|c} M & MB \\ \hline A^T M & A^T MB \end{array}\right)$ is an $n_1 \times n_2$ matrix with columns in C_1 and rows in C_2 for every $k_1 \times k_2$ matrix M. And conversely, every code word of $C_1 \otimes C_2$ is of this form. Hence the dimension of $C_1 \otimes C_2$ is equal to $k_1.k_2$ and the given set is a basis of $C_1 \otimes C_2$.□

Theorem 2:

Let C_1 and C_2 be respectively (n_1, k_1, d_1) and (n_2, k_2, d_2) codes. Then, the product code $C_1 \otimes C_2$ is an $(n_1.n_2, k_1.k_2, d_1.d_2)$ code.

Proof

By definition, $n = n_1.n_2$ is the length of the product code. It has been remarked that $C_1 \otimes C_2$ is a linear subspace of $F_q^{n_1.n_2}$. The dimension of the product code is $k_1.k_2$ by theorem 1. Next, we proof that the minimum distance of $C_1 \otimes C_2$ is $d_1.d_2$.

Now, for any code word of $C_1 \otimes C_2$, which is a $n_1 \times n_2$ array, every non-zero column has weight $\ge d_1$, and every non-zero row has weight $\ge d_2$. So the weight of a non-zero code word of the product code is at least $d_1.d_2$. This implies that the minimum distance of $C_1 \otimes C_2$ is at least $d_1.d_2$. Suppose $x \in C_1$ has weight d_1 and $y \in C_2$ has weight d_2, then $x \otimes y$ is a code word of $C_1 \otimes C_2$ and has weight $d_1.d_2$.□ [8]

2.3 Remark

The generator matrix $G_1 \otimes G_2$ of the product code $C_1 \otimes C_2$ is given by the tensor product of two matrices which in block form is given by $A \otimes B = \begin{bmatrix} b_{1,1}C & \ldots\ldots & b_{1,m}C \\ \vdots & \ddots & \vdots \\ b_{n,1}C & \ldots\ldots & b_{n,m}C \end{bmatrix}$ [9].

Theorem 3:
Let G_1 be a generator matrix of C_1 and G_2 be a generator matrix of C_2, then $G_1 \otimes G_2$ is a generator matrix of $C_1 \otimes C_2$.

Proof
In this theorem, the code words are considered as the elements of F_q^n and no longer as matrices. Now, Let x_i be the i-th row of G_1 and denote by y_j the j^{th} row of G_2.
So $x_1, x_2, \dots x_{k_1} \in F_q^{n_2}$ is a basis for C_1 and $y_1, y_2, \dots y_{n_2} \in F_q^{n_2}$ is a basis for C_2. Hence the set $\{x_i \otimes y_j | 1 \le i \le k_1, 1 \le j \le k_2\}$ is a basis for $C_1 \otimes C_2$ by theorem 1. Furthermore, if $l = (i-1)k_2 + j$, then $x_i \otimes y_j$ is the l^{th} row of $G_1 \otimes G_2$. Hence $G_1 \otimes G_2$ is a generator matrix of $C_1 \otimes C_2$.□

Example 1
Let C_1 and C_2 be two ternary codes with generator matrices G_1 and G_2 respectively be as given ;
$G_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Find, i. $G_1 \otimes G_2$ ii. $X_2 \otimes Y_2$ and hence evaluate the 5^{th} row of $G_1 \otimes G_2$.

Solution
i. From remark 1, we have,

$$G_1 \otimes G_2 = \left[\begin{array}{cccc|cccc|cccc} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 \end{array}\right]$$

ii. Now, the second row of G_1 is $X_2 = (\ 0\ 1\ 2\ )$ and $Y_2 = (\ 0\ 1\ 2\ 0\ )$ is the second row of G_2.
Therefore $X_2 \otimes Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, considered as a matrix. Hence equals to $(0\ 0\ 0\ 0\ 0\ 1\ 2\ 0\ 0\ 2\ 1\ 0\ )$ written as a vector, which is in did equal to the $(2-1)3+2=5^{th}$ $-$row of $G_1 \otimes G_2$.

3. Finding/Main Results

Product Code Construction Of The [49 16 6]

In what follows, we combined the [7 4 2]- linear code constructed in (3) with the known Hamming [7 4 3] code to obtain a Hybrid double error correcting linear code.
Now, the [7 4 2] –linear code has the following parameters; $n = 7$, $k = 4$, $d = 2$ and the known [7 4 3]- Hamming code has parameters; $n = 7$, $k = 4$, $d = 3$.
On combining the two codes using the product code construction approach, we shall obtain a
$[\ n_1.n_2,\ k_1.k_2,\ d_1.d_2\ ]$
$= [\ 7 \times 7\ ,\ 4 \times 4\ ,\ 2 \times 3\ ]$
$= [\ 49, 16, 6\ ]$ - linear code.

Firstly, the [7 4 2]- linear code has generator and parity check matrices G_1 *and* H_1 as follows;

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad and \quad H_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

While for the generalize Hamming binary [7 4 3]-linear code, its generator and parity check matrices G_2 *and* H_2 are ; $G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ *and* $H_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

respectively.

Now, by the product code construction, our new code which is as a result of the combination of the above codes shall be a [49 16 6] – linear code whose generator matrix $G_* = G_1 \otimes G_2$ is;

$G_* = G_1 \otimes G_2 =$

$$\begin{bmatrix}
1000011000000000000000000000 & 100001110000111000011 \\
0100101000000000000000000000 & 010010101001010100101 \\
0010110000000000000000000000 & 001011000101100010110 \\
0001111000000000000000000000 & 000111100011110001111 \\
0000000100001100000000000000 & 000000000000001000011 \\
0000000010000100000000000000 & 000000000000000100101 \\
0000000001001000000000000000 & 000000000000000010110 \\
0000000000111100000000000000 & 000000000000000001111 \\
0000000000000010000110000000 & 000000010000110000000 \\
0000000000000001001010000000 & 000000001001010000000 \\
0000000000000000101100000000 & 000000000101100000000 \\
0000000000000000011110000000 & 000000000011110000000 \\
0000000000000000000001000011 & 100001100000000000000 \\
0000000000000000000000100101 & 010010100000000000000 \\
0000000000000000000000001110 & 001011000000000000000 \\
0000000000000000000000001111 & 0001`11100000000000000
\end{bmatrix}$$

Furthermore, the parity check matrix for the [49 16 6] linear code $H_1 \otimes H_2$ is obtained using the same approach as for $G_1 \otimes G_2$ above or using the fact that $G = [I \mid A] \rightarrow H = [A^T \mid I]$. As such, a double error correcting code of length $n = 49$, dimension $k = 16$ and Hamming distance $d = 6$ is constructed which is of increased error- correcting capability than the [14 8 3] linear code constructed in [1].

4. Conclusion/ Recommendation

This paper, provides an improvement on the [14 8 3] linear code by obtaining a much more practical and applicable double error correcting code whose extended version is a triple error correcting code, by combining the same codes using the product code construction approach. Thus, this paper has further pointed the applicability of the AUNU schemes in the direction of coding theory, i.e Codes, generation and analysis by using the single error-detection [7 4 2] code in the construction of a triple error-detection and double error-correction code. Further research on other code properties such as perfectness or otherwise and code extension of the generated code is recommended.

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