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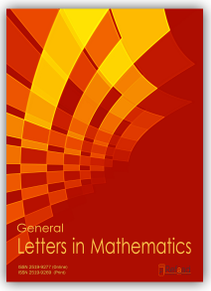

Automorphism groups of Cayley graphs of order pq^2 where $p \neq q$ are prime numbers

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Abstract. In this paper, we determine when two Cayley graphs over the cyclic group $\mathbb{Z}_{pq^2}$ are to be isomorphic, for $p \neq q$ prime numbers. and we determine the all types of automorphism groups of these graphs.

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1 Introduction

A Cayley graph over a finite group H defined by a connection set $S \subseteq H$ has H as a set of nodes and arc set $Cay(H, S) := \{(x, y) : xy^{-1} \in S\}$. A circulant graph is a Cayley graph over a cyclic group. Two Cayley graphs $Cay(H, S)$ and $Cay(K, T)$ are Cayley isomorphic if there exists a group isomorphism $f : H \to K$ which is a graph isomorphism too, that is $Cay(H, S)^f = Cay(K, T) \Leftrightarrow S^f = T$.

For a group H, let H_R be the right regular group of H consisting of the right translations $h_R : x \mapsto x_h$ for all $h \in H$. The group $\mathcal{H}(H) = N_{S_H}(H_R)$ is called the holomorph of H, and $\mathcal{H}(H) = (AutH)H_R$. Let $Aut\Gamma$ be the automorphism group of Γ, i.e the group of all permutations $f \in Sym(H)$ such that $(x, y) \in E(\Gamma) \quad \Leftrightarrow (x^f, y^f) \in E(\Gamma)$. The automorphism group of a Cayley graph $Cay(H, S)$ contains a regular subgroup $H_R \leq Sym(H)$, and any graph $\Gamma = (\Omega, E)$ is isomorphic to a Cayley graph over a group H if and only if $Aut(\Gamma)$ contains a regular subgroup isomorphic to H.

There are two main approaches to the isomorphism problem of Cayley graphs: group-theoretical and algebraic-combinatorial. The first one was developed by [11, 12, 13, 14, 15, 16]. It may be used not only for Cayley graphs but also for arbitrary Cayley combinatorial structures. The second approach, based on the ideas of algebraic combinatorics (more precisely, on the theory of Schur rings), was proposed by [1, 9, 5, 4]. This approach was recently developed and extended by [2].

In 1936, ([9]) asked the following question: When a given abstract group can be interpreted as the group of a graph and if this is the case, how the corresponding graph can be constructed. The answer to this question did not prove in a general case.

[1] studied the automorphism groups of circulant graphs of order p and $p.q$ where $p \neq q$ are prime numbers. and They first solved the case p^2, and later the solution was presented by [2] in the case p^3 . Although they have announced several times the complete solution covering all p^m , p an odd prime. (see.[3]) I. Kovacs in 2008 determined the automorphism groups of Cayley graphs on 2^m vertices see [6]. In 2012 Klin and I. Kovacs determined the automorphism groups of rational Cayley graphs of order n for $n \in \mathbb{N}$ (see [5]). and the only method to do that

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was by using Schur ring theory. In this paper we study this problem for Cayley graphs on cyclic groups of order $p.q^2$ where $p \neq q$ are prime numbers.
From now on Z_n will denote a cyclic group of order n. and S_n will denote the Symmetric group of degree n.

2 Schur- Rings:

Let H be a finite group with identity element e. We denote by $\mathbb{Q}H$ the group algebra of H over the field $\mathbb{Q}$ of rational numbers. For a subset $T \subset H$, let T denote the group algebra element $\sum_{x \in H} a_x \underline{x}$ with: $a_x = 1$ if $x \in T$, and $a_x = 0$ otherwise. such elements called simple quantities. A subalgebra $\mathcal{A}$ of $\mathbb{Q}H$ is called a Schur ring (or S-ring) over H if the following axioms are satisfied:

1. There exists a basis of $\mathcal{A}$ consisting of simple quantities $\underline{T_0}, \underline{T_1}, \dots, \underline{T_r}$;
2. $T_0 = \{0\}$, $H = \cup_{i=0}^{r} T_i$.
3. For every $i \in \{1, 2, ..., r\}$ there exists $j \in \{1, 2, ..., r\}$ such that $\underline{T_i^{-1}} = \underline{T_j}$

The sets T_i are called the basic sets, and the simple quantities $\underline{T_i}$ the basic quantities of $\mathcal{A}$. We set $Bs(\mathcal{A}) = \{T_0, T_1, \dots, T_r\}$. A subset $A \subseteq H$ is called an $\mathcal{A}$-set, and a subgroup $K \leq H$ is called an $\mathcal{A}$-subgroup if $\underline{S} \in \mathcal{A}$ and $\underline{K} \in \mathcal{A}$ respectively. An S-ring $\mathcal{A}' \subseteq \mathbb{Q}H$ is called S-subring of $\mathcal{A}$. if every element $z \in \mathcal{A}'$ equal to sum of elements from $\mathcal{A}$. If K is a $\mathcal{A}$-subgroup of H then the subalgebra $\mathcal{A}_K = \mathcal{A} \cap \mathbb{Q}H$ is an S-ring over K and $Bs(\mathcal{A}_K) = \{T \in Bs(\mathcal{A}) : \ T \subseteq K\}$. If $K \trianglelefteq H$, then we can define a quotient S-ring $\mathcal{A}/K$ over the factor-group H/E: $Bs(\mathcal{A}_{H/K}) = \{T/K : T \in Bs(\mathcal{A})\}$. Let G be an arbitrary group such that $H_R < G < S_H$ and let $T_0 = \{e\}, T_1, ...T_d$ be the set of all G_e-orbits. The vector space spanned by the simple quantities $T_0 = \{e\}, T_1, ...T_d$ is called the transitivity module of G and is denoted by $\mathcal{V}(H, G_e)$. and by (Wielandt [20,Theorem 24.1]), The transitivity module $\mathcal{V}(H, G_e)$ is an S-ring over H.

2.1 Wreath and direct product:

let $\mathcal{A}$ be S-ring over a group H, and $E, F \leq H$ be tow subgroups of H such that

1. $H = E.F$ and $E \cap F = \{1\}$
2. If $T \in Bs(\mathcal{A})$ then $T = RS$ where $R \in Bs(\mathcal{A}_E)$, $S \in Bs(\mathcal{A}_F)$.

then $\mathcal{A}$ is a direct product of $\mathcal{A}_E$ and $\mathcal{A}_F$ and write $\mathcal{A} = \mathcal{A}_E \times \mathcal{A}_F$.

Definition 2.1. *Let $\mathcal{A}$ be an S-ring over a group H and N be an $\mathcal{A}$-subgroup such that $N \trianglelefteq H$. Then A is a wreath product, notation: $\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$ if for every $T \in \mathcal{A}$, if $T \nsubseteq N$, then T is a union of N-cosets.*

Definition 2.2. *Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$ -subgroups such that $H = EF$ and $E \cap F = \{e\}$. Then $\mathcal{A}$ is a tensor product, notation: $\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_F$ if for every $T \in Bs(\mathcal{A})$, if $T \nsubseteq E \cup F$ then $T = RS$ where $R \in Bs(\mathcal{A}) \cap E$ and $S \in Bs(\mathcal{A}) \cap F$*

Consequently, if $\mathcal{A}$ is an S-ring of the direct product $H = E \times F$ such that both E and F are A-subgroups and $\mathcal{A}_E = \mathbb{Q}E$ or $\mathcal{A}_F = \mathbb{Q}F$, then $A = \mathcal{A}_E \otimes \mathcal{A}_F$.

Let $\mathcal{A}$ be an S-ring over a group H, $\mathcal{A}$ is called wreath-decomposable (or shortly decomposable), if there is a nontrivial, proper subgroup $N < H$ such that for every basic set T, $T \subset N$ or $T = \cup_{x \in T} xN$. Otherwise, $\mathcal{A}$ is called indecomposable. If an S-ring is decomposable, then it can be obtained as the wreath product of two smaller S-rings.

2.2 Automorphisms of S-rings:

Let $\mathcal{A} = \langle \underline{T_0}, \underline{T_1}, \dots, \underline{T_r} \rangle$ be an S-ring over a group H. By an automorphism of $\mathcal{A}$ we mean a permutation of H is an automorphism of $\mathcal{A}$ if it is automorphism of all Cayley graphs $\Gamma_i = Cay(H, T_i), T \in Bs(\mathcal{A})$ and The group of all automorphisms of $\mathcal{A}$ will be denoted by $Aut(\mathcal{A})$, and it is

$$Aut(\mathcal{A}) = \bigcap_{i=1}^{r} Aut(\Gamma_i)$$

so we can see that a permutation g of H is $g \in Aut(\mathcal{A})$ if $u - v \in T \Leftrightarrow u^g - v^g \in T$, $T \in Bs(\mathcal{A})$.

Now, let $Q \subseteq H$, and let $\langle\langle Q\rangle\rangle$ be the intersection of all S-ring $\mathcal{A}$ such that $\underline{Q} \in \mathcal{A}$ then $\langle\langle Q\rangle\rangle$ be an S-ring over H and the smallest S-ring contain Q.

Theorem 2.3. *$Aut\,Cay(H, Q) = Aut\langle\langle Q\rangle\rangle$.*

Proof. let $G = Aut(Cay(H, S))$ and $A = \mathcal{V}(H, G_e)$. $Q \subseteq H$ then $Q = \cup_{i=1}^{k} T_i$ for $T_i \in Bs(\langle\langle Q\rangle\rangle)$ so $Aut\langle\langle Q\rangle\rangle \leq \bigcap_{i=1}^{k} Aut\,Cay(H, T_i) \leq G$. On the other, since $G = Aut(\mathcal{A})$ and $\underline{Q} \in \mathcal{A}$ then $G = Aut(\mathcal{A}) \leq Aut\langle\langle Q\rangle\rangle$. □

3 S-rings over $\mathbb{Z}_n$

Let $H = \mathbb{Z}_n$ be cyclic group of order n and $\mathcal{P}(n)$ be the group of all $a \in \mathbb{Z}_n$ relatively prime to n. then $\mathcal{P}(n)$ can be considered as permutation group acting on the set $\mathbb{Z}_n$ by right multiplication: $\mathbb{Z}_n \to \mathbb{Z}_n : x \to ax, a \in \mathcal{P}(n), x \in \mathbb{Z}_n$ thus $\mathcal{P}(n) = Aut(\mathbb{Z}_n)$.

Definition 3.1. *Let $n \in \mathbb{N}, n \geq 2$ and let $\mathcal{A}$ be an S-ring over $\mathbb{Z}_n$, and $D_n = \{d_0, d_1, \ldots, d_{k-1}\}$ the set of all divisors (not equal to n) of n, with $d_O = 1$, and for $K \subseteq \mathbb{Z}_n$ and $d \in D_n$ define $K/d = \{x \in K : g.c.d(x, n) = d\}$ and the binary relation $\Theta(\mathcal{A})$ on D_n, defined by $(d, d') \in \Theta(\mathcal{A}) \Leftrightarrow T_{(d)}/d' \neq \phi$ is an equivalence relation called the basic equivalence af $\mathcal{A}$.*

Definition 3.2. *Let H, K be two groups and $\mathcal{A} = \langle T_0, T_1, ..., T_r\rangle \subseteq \mathbb{Q}H$, $\mathcal{B} = \langle S_0, S_1, ..., S_s\rangle \subseteq \mathbb{Q}K$ be two S-rings, then we say that $\mathcal{A}$ is Algabra-isomorphism with $\mathcal{B}$ and denote $\mathcal{A} \cong_{Alg} \mathcal{B}$ if and only if there exists a bijection $g : Bset(\mathcal{A}) \to Bset(\mathcal{B})$ such that g is an isomorphism between the algebra $\mathcal{A}$ and $\mathcal{B}$. and we say that $\mathcal{A}$ is combinatorial-isomorphism with $\mathcal{B}$ and denote $\mathcal{A} \cong_{com} \mathcal{B}$ if and only if there exists a bijection $f : H \to K$ such that $\{\Gamma_H(T_i)^f : i = 0, 1, 2, .., r\} = \{\Gamma_K(S_j) : j = 0, 1, 2, .., d\}$, so $s = d$ and f induces a bijection $f^* : Bset(\mathcal{A}) \to Bset(\mathcal{B})$ such that $\Gamma_H(T)^f = \Gamma_K(T^{f^*})$, for all $T \in Bsets(A)$. If f is group isomorphism we say that f is Cayley isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and denote $\mathcal{A} \cong_{Cay} \mathcal{B}$*

Lemma 3.3. *([7]): let $\Gamma_H(S), \Gamma_K(R)$ two Cayley graphs, and $f : H \to K$ an isomorphism between $\Gamma_H(S)$, $\Gamma_K(R)$ then:*

1. *f is combinatorial isomorphism between $\langle\langle S\rangle\rangle$ and $\langle\langle R\rangle\rangle$*
2. *If $S = \cup\{B : B \in I\}$ be the decomposition of S in to a disjoint union of basic sets of $\langle\langle S\rangle\rangle$ with $I \subseteq Bset(\langle\langle S\rangle\rangle)$ then $R = \cup\{B^{f^*} : B \in I\}$*

Lemma 3.4. *([7]): Let $\mathcal{A} = \langle T_0, T_1, ..., T_r\rangle$ be an S-ring over cyclic group $\mathbb{Z}_n$ and $\lambda : Bset(\mathcal{A}) \to Bset(\mathcal{A})$ be algebraic automorphism of $\mathcal{A}$ then:*

1. *For all $m \in \mathcal{P}(n)$ we have $T \in Bset(\mathcal{A}) \Leftrightarrow T^m \in Bset(\mathcal{A})$*
2. *For $T \in Bset(\mathcal{A})$ there exists $m_T \in \mathcal{P}(n)$ such that $T^\lambda = T^{m_T}$.*

Definition 3.5. *Let $S \subseteq \mathbb{Z}_n$ and $d|n$ we put $S_{(d)} = \{x \in S : g.c.d(x, n) = d\} = S \cap (\mathbb{Z}_n)_{(d)}$*

Lemma 3.6. *Let f be algebraic automorphism of $\mathcal{A}$, then for all $d|n$, there exists $m_d \in \mathcal{P}(n)$ such that $T^f = T^{m_d}$ for all $T \in Bset(\mathcal{A})$ with $T \cap (\mathbb{Z}_n)_{(d)} \neq \varnothing$*

Proof. Let $T \in Bset(\mathcal{A})$ with $T \cap (\mathbb{Z}_n)_{(d)} \neq \varnothing$ then: $T^{m_T} \in Bset(\mathcal{A})$ for some $m_T \in \mathcal{P}(n)$. If $T' \in Bset(\mathcal{A})$ with $T' \cap (\mathbb{Z}_n)_{(d)} \neq \varnothing$ then there exists $m \in \mathcal{P}(n)$ such that $T^m \cap T' \neq \varnothing$ then $T'^f = (T^{(m)})^f = ((T^{(f)})^m) = (T^{(m_T)})^m = T^{mm_T} = (T^m)^{m_T} = (T')^{m_T}$. and the proof completes with $m_T = m_d$. □

Theorem 3.7. *If $\Gamma_n(S)$, $\Gamma_n(R)$ be two isomorphic Cayley graphs, then for each $d|n$ there exists $m_d \in \mathcal{P}(n)$ such that $R_{(d)} = m_d.S_{(d)}$*

Proof. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be an isomorphism between $\Gamma_n(S)$ and $\Gamma_n(R)$ then f is combinatorial isomorphism between tow S-rings $\langle\langle S\rangle\rangle$ and $\langle\langle R\rangle\rangle$, and f induces an algebraic isomorphism $f^* : Bset(\langle\langle S\rangle\rangle) \to Baset(\langle\langle S\rangle\rangle)$ then for $d|n$ we have:

$$\begin{aligned}
R_(d) &= \cup\{T^{f^*} \cap (\mathbb{Z})_{(d)} : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S\} \\
&= \cup\{T^{m_T} \cap (\mathbb{Z})_{(d)} : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S\} \\
&= \cup\{(T \cap (\mathbb{Z})_{(d)})^{m_T} : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S\} \\
&= \cup\{(T \cap (\mathbb{Z})_{(d)})^{m_T} : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S, T \cap (\mathbb{Z})_{(d)} \neq \varnothing\} \\
&= \cup\{(T \cap (\mathbb{Z})_{(d)})^{m_d} : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S, T \cap (\mathbb{Z})_{(d)} \neq \varnothing\} \\
&= (\cup\{T \cap (\mathbb{Z}_{(d)}) : T \in Bset(\langle\langle S\rangle\rangle), T \subseteq S, T \cap (\mathbb{Z})_{(d)} \neq \varnothing\})^{m_d} \\
&= S_{(d)}^{m_d}
\end{aligned}$$

□

4 Main results

As we know that for $n = p$, If $\Gamma_p(S)$, $\Gamma_p(R)$ are two Cayley graphs then we have: $\Gamma_p(S) \cong \Gamma_p(R) \Leftrightarrow \exists m \in \mathcal{P}(p) : R = mS$. and for $n = p^2$ we have $\Gamma_{p^2}(S) \cong \Gamma_{p^2}(R)$ if and only if there exists $m_0, m_1 \in \mathcal{P}(n)$ such that $R_{(1)} = m_0 S_{(1)}$, $R_{(p)} = m_1 S_{(p)}$ and $m_0 = m_1$ when $(1+p)S_{(1)} \neq S_{(1)}$

Now let $n = pq^2$ where $p \neq q$ are odd prime. then $\mathbb{Z}_{pq^2} = \mathbb{Z}_p \oplus \mathbb{Z}_{q^2}$ and every $z \in \mathbb{Z}_n$ have single decomposition $z = [x, r + sq]$ wher $x \in \mathbb{Z}_p$, $r, s \in \mathbb{Z}_{q^2}$. then we have the following main result.

Theorem 4.1. *Let $n = pq^2$, and $\Gamma_n(S)$, $\Gamma_n(R)$ are two Cayley graphs where $R, S \subseteq \mathbb{Z}_n$. Assume that there exists $\mu \in \mathbb{Z}_p$ and γ_0, $\gamma_1 \in \mathbb{Z}_{p^2}$ such that for all $d|n$ we have*

1. *$R_d = m_d.S_d$ where $m_d = [\mu, \gamma]$.*
2. *$\gamma_0 \equiv \gamma_1 (\mod q)$*

then $\Gamma_n(S) \cong \Gamma_n(R)$

Proof. We define a permutation $f : \mathbb{Z}_n \to \mathbb{Z}_n$ as follows: $([x, r + sq])^f = [\mu x, \gamma_0 r + \gamma_1 sq]$. and let $z, z' \in \mathbb{Z}_n$ then $z = [x, r + sq], z' = [x', r' + s'q]$ where $x, x' \in \mathbb{Z}_p$ and $r, r', s, s' \in \mathbb{Z}_q$. and we have $z - z' = [x - x', r - r' + (s - s')q]$. If $z - z' \in S$ such that $z - z' \in S_{(d)}$ for $d|n$ then we get $d|z - z'$ so we have the follows: if $d = p = [p, 1]$ then $p|z - z' \Rightarrow p|x - x' \Rightarrow x = x'$ so $z - z' = [0, (r - r') + (s - s')q]$ thus: $z^f - z'^f = [0, \gamma_0(r - r') + \gamma_1(s - s')q]$, by $\gamma_0 \equiv \gamma_1 (\mod q)$ we have $z^f - z'^f = [1, \gamma_0].[0, (r - r') + (s - s')q] = m_p(z - z') (\mod q) \in m_p S_p = R_p \ : m_p = [1\gamma_0]$.
If $d = q$ then $d = [1, q]|z - z' \Leftrightarrow q|(r - r') + (s - s')q \Leftrightarrow q|r - r' \Leftrightarrow r = r' \Leftrightarrow z - z' = [x - x', (s - s')q]$. thus $z^f - z'^f = [\mu(x - x'), \gamma_1(s - s')q] = [\mu, \gamma_1].[x - x', (s - s')q] = m_q(z - z') \in m_q S_q = R_q \ : m_q = [\mu, \gamma_1]$.
If $d = q^2$ then $d = [1, q^2]|z - z' \Leftrightarrow q^2|r - r' + (s - s')q \Leftrightarrow r = r', s = s' \Leftrightarrow z - z' = [x - x, 0]$. thus $z^f - z'^f = [\mu(x - x'), 0] = [\mu, 1].[x - x', 0] = m_{q^2}(z - z') \in m_{q^2} S_{q^2} = R_{q^2} \ : m_{q^2} = [\mu, 1]$.
If $d = pq$ then $d = [p, q]|z - z' \Leftrightarrow p|x - x'$ and $q|r - r' + (s - s')q \Leftrightarrow x = x', r = r' \Leftrightarrow z - z' = [0, (s - s')q]$. thus $z^f - z'^f = [0, \gamma_1(s - s')q] = [1, \gamma_1].[0, (s - s')q] = m_{pq}(z - z') \in m_{pq} S_{pq} = R_{pq} \ : m_{pq} = [1, \gamma_1]$.
If $d = 1$ then $d = [1, 1]|z - z' \Leftrightarrow z - z' = [x - x', r - r' + (s - s')q]$. thus $z^f - z'^f = [\mu(x - x'), \gamma_0(r - r') + \gamma_1(s - s')q] = [\mu, \gamma_0].[x - x', r - r' + (s - s')q] = m_1(z - z') \in m_1 S_1 = R_1 \ : m_1 = [\mu, \gamma_0]$.
then for all $d|n$ we have $z - z' \in S_{(d)} \Leftrightarrow z^f - z'^f \in R_{(d)}$. consequently, f is an isomorphism between $\Gamma_n(S)$ and $\Gamma_n(R)$ □

Let $\mathcal{A}_d$ be S-ring over the cyclic group $\mathbb{Z}_r$ and put $G_r = Aut(\mathcal{A}_r)$ then we have:

Theorem 4.2. *Let $\mathcal{A}$ be a S-ring over $\mathcal{A}_n$ where $n = p.q^2$ and $p \neq q$ are prime numbers then $G = Aut(\mathcal{A})$ has one of the following forms:*

1. *S_n*
2. *$G_r \wr G_{\frac{n}{r}}$: $n \in \{p, q, q^2, pq\}$*
3. *$G_p \times G_{q^2}$ or $G_q \times G_{pq}$*

4. $W\mathbb{Z}_n$ where $W \leq \mathcal{P}(n)$ and W does not split into a direct product of subgroup in $\mathcal{P}(p)$ with subgroup in $\mathcal{P}(q^2)$.

Proof. Let $G = \mathbb{Z}_n$ which is genarated by ρ and consider the normal subgroup of G as following: $H =< \rho^{q^2} >$, $K =< \rho^p >$, $L =< \rho^{p.q} >$, $M =< \rho^q >$. Now let $\mathcal{A}$ be S-ring over G with $\Theta(\mathcal{A}) = \Theta$ then we can see that Θ has one of the following forms:

- $\Theta_1 = \{1, p, q, q^2, p.q\}$ then there is one class that contain all elements of $G\backslash\{0\}$ and the g.c.d of all these elements is 1. so $Bset(\mathcal{A}) = \{0, G\backslash\{0\}\}$. thus $Aut(\mathcal{A}) = S_n$
- $\Theta_2 = \{\{1\}, \{p\}, \{q\}, \{q^2\}, \{p.q\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_2$ then: $Bset(\mathcal{A}) = \{\{0\}, \{H\backslash\{0\}\}, \{L\backslash\{0\}\}, \{K\backslash L\}, \{M\backslash H \cup L\}, \{G\backslash K \cup M\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_2$ has the form $\mathcal{A} = \mathcal{A}_p \times (\mathcal{A}_q \wr \mathcal{A}_q)$. and generated by the following basic quantities:

$$\begin{aligned} T_{(q^2z)} &= W''q^2z : W'' \leq \mathcal{P}(p) \\ T_{(qz)} &= W''q^2z + pqW'z : W' \leq \mathcal{P}(q) \\ T_{(pqz)} &= W'pqz : W \leq \mathcal{P}(q) \\ T_{(pz)} &= W'pz + pq\mathbb{Z}_q : W' \leq \mathcal{P}(q) \\ T_{(pz)} &= W''q^2z + W'pz + pq\mathbb{Z}_q : W'' \leq \mathcal{P}(p), W' \leq \mathcal{P}(q) \end{aligned}$$

  consequenrly, $Aut(\mathcal{A}) = Aut(\mathcal{A}_p) \times (Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_q))$
- $\Theta_3 = \{\{q^2\}, \{1, p, q, pq\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_3$ then: $Bset(\mathcal{A}) = \{\{0\}, \{H\backslash\{0\}\}, \{G\backslash H\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_3$ has the form $\mathcal{A} = \mathcal{A}_{q^2} \wr \mathcal{A}_p$. where $\mathcal{A}_{q^2}$ is indecomposable S-ring. and $\mathcal{A}$ generated by the following basic quantities:

$$\begin{aligned} T_{(q^2z)} &= W''q^2z : W'' \leq \mathcal{P}(p) \\ T_{(z)} &= \mathbb{Z}_{q^2}\backslash\{0\} + q^2\mathbb{Z}_p \text{ or } T_{(z)} = Wz + q^2\mathbb{Z}_p : W \leq \mathcal{P}(q^2), 1 + q \notin W \end{aligned}$$

  consequenrly, $Aut(\mathcal{A}) = Aut(\mathcal{A}_{q^2}) \wr Aut(\mathcal{A}_p)$
- $\Theta_4 = \{\{p.q\}, \{1, p, q, q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_4$ then: $Bset(\mathcal{A}) = \{\{0\}, \{L\backslash\{0\}\}, \{G\backslash L\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_4$ has the form $\mathcal{A} = \mathcal{A}_{p.q} \wr \mathcal{A}_q$. where $\mathcal{A}_{p.q}$ is indecomposable S-ring. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(p.qz)} &= W'q^2z : W' \leq \mathcal{P}(q) \\ T_{(z)} &= \mathbb{Z}_{p.q}\backslash\{0\} + p.q\mathbb{Z}_q \text{ or } T_{(z)} = Wz + pq\mathbb{Z}_p : W \leq \mathcal{P}(pq) \end{aligned}$$

  And W does not split into a direct products $W'e_p + W''e_q$ with $W' \in \mathcal{P}(q)$, and $W'' \in \mathcal{P}(p)$. Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_{pq}) \wr Aut(\mathcal{A}_q)$
- $\Theta_5 = \{\{p\}, \{p.q\}, \{1, q, q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_5$ then: $Bset(\mathcal{A}) = \{\{0\}, \{L\backslash\{0\}\}, \{K\backslash L\}, \{G\backslash K\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_5$ has the form $\mathcal{A} = \mathcal{A}_p \wr \mathcal{A}_q \wr \mathcal{A}_q$. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(p.qz)} &= W'p.qz : W' \leq \mathcal{P}(q) \\ T_{(pz)} &= W'pz + p.q\mathbb{Z}_q : W' \leq \mathcal{P}(q) \\ T_{(z)} &= W''z + p\mathbb{Z}_{q^2} : W'' \leq \mathcal{P}(p) \end{aligned}$$

  Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_p) \wr Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_q)$
- $\Theta_6 = \{\{1\}, \{p\}, \{p.q\}, \{q, q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_6$ then: $Bset(\mathcal{A}) = \{\{0\}, \{L\backslash\{0\}\}, \{K\backslash L\}, \{M\backslash L\}, \{G\backslash M \cup K\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_6$ has the form $\mathcal{A} = (\mathcal{A}_p \times \mathcal{A}_q) \wr \mathcal{A}_q$. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(p.qz)} &= W'p.qz : W' \leq \mathcal{P}(q) \\ T_{(pz)} &= W'pz + p.q\mathbb{Z}_q : W' \leq \mathcal{P}(q) \\ T_{(qz)} &= W''.qz + p.q\mathbb{Z}_q : W'' \leq \mathcal{P}(p) \\ T_{(z)} &= W''.qz + W'pz + p.q\mathbb{Z}_q : W'' \leq \mathcal{P}(p), W' \leq \mathbb{P}(q) \end{aligned}$$

  Consequently, $Aut(\mathcal{A}) = (Aut(\mathcal{A}_p) \times Aut(\mathcal{A}_q)) \wr Aut(\mathcal{A}_q)$

- $\Theta_7 = \{\{1,p\},\{p.q\},\{q\},\{q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_7$ then: $Bset(\mathcal{A}) = \{\{0\},\{H\backslash\{0\}\},\{L\backslash\{0\}\},\{M\backslash H\cup L\},\{G\backslash M\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_7$ has the form $\mathcal{A} = \mathcal{A}_q \wr (\mathcal{A}_q \times \mathcal{A}_q)$. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(q^2z)} &= W''q^2z : W'' \leq \mathcal{P}(p) \\ T_{(p.qz)} &= W'p.qz : W' \leq \mathcal{P}(q) \\ T_{(qz)} &= W''.q^2z + W'p.qz : W'' \leq \mathcal{P}(p), W' \leq \mathcal{P}(q) \\ T_{(z)} &= W''z + q\mathbb{Z}_{pq} : \end{aligned}$$

Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_q) \wr (Aut(\mathcal{A}_q) \times Aut(\mathcal{A}_q))$

- $\Theta_8 = \{\{q^2\},\{1,q\},\{p,p.q\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_8$ then: $Bset(\mathcal{A}) = \{\{0\},\{H\backslash\{0\}\},\{K\backslash\{0\}\},\{G\backslash H\cup K\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_8$ has the form $\mathcal{A} = \mathcal{A}_p \times \mathcal{A}_{q^2}$. where $\mathcal{A}_{q^2}$ is indecomposable S-ring, and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(q^2z)} &= W''q^2z : W'' \leq \mathcal{P}(p) \\ T_{(pz)} &= T_{(p.qz)} = p(\mathbb{Z}_{q^2}\backslash\{0\}) \\ \text{or } T_{(pz)} &= T_{(p.qz)} = Wpzq : W \leq \mathcal{P}(q^2), 1+q \notin W \\ T_{(z)} &= W''q^2z + Wpz : W \leq \mathcal{P}(q^2), 1+q \notin W \\ \text{or } T_{(z)} &= W''q^2z + p(\mathbb{Z}_{q^2}\backslash\{0\}) : W \leq \mathcal{P}(q^2), 1+q \notin W \end{aligned}$$

Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_p) \wr Aut(\mathcal{A}_{q^2})$

- $\Theta_9 = \{\{p.q\},\{1,p\},\{q,q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_9$ then: $Bset(\mathcal{A}) = \{\{0\},\{L\backslash\{0\}\},\{M\backslash L\},\{G\backslash M\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_9$ has the form $\mathcal{A} = \mathcal{A}_q \wr \mathcal{A}_q \wr \mathcal{A}_q$. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(p.qz)} &= W'pqz : W' \leq \mathcal{P}(q) \\ T_{(qz)} &= T_{(q^2z)} = W''qz + pq\mathcal{Z}_q :: W' \leq \mathcal{P}(q) \\ T_{(z)} &= W'z + q\mathcal{Z}_{p.q} : W' \leq \mathcal{P}(q) \end{aligned}$$

Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_p) \wr Aut(\mathcal{A}_q)$

- $\Theta_{10} = \{\{1,p\},\{q,q^2,p.q\}\}$ then there are one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{10}$ then: $Bset(\mathcal{A}) = \{\{0\},\{M\backslash\{0\}\},\{G\backslash M\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{10}$ has the form $\mathcal{A} = \mathcal{A}_q \wr \mathcal{A}_{p.q}$. where $\mathcal{A}_{p.q}$ is indecomposable S-ring, and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(z)} &= W'z + p\mathbb{Z}_{p.q} : W' \leq \mathcal{P}(q) \\ T_{(qz)} &= q(\mathbb{Z}_{p.q}\backslash\{0\}) \\ \text{or } &= Wqz : W \leq \mathcal{P}(p.q) \end{aligned}$$

And W does not split into a direct product $W'e_p + W''e_q$ with $W' \in \mathcal{P}(q)$, and $W'' \in \mathcal{P}(p)$. Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_{p.q})$

- $\Theta_{11} = \{\{p,p.q\},\{1,q,q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{11}$ then: $Bset(\mathcal{A}) = \{\{0\},\{K\backslash\{0\}\},\{G\backslash K\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{11}$ has the form $\mathcal{A} = \mathcal{A}_p \wr \mathcal{A}_{q^2}$. where $\mathcal{A}_{q^2}$ is indecomposable S-ring, and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(z)} &= W''z + p\mathbb{Z}_{q^2} : W'' \leq \mathcal{P}(p) \\ T_{(pz)} = T_{(p.qz)} &= p(\mathbb{Z}_{q^2}\backslash\{0\}) \\ \text{or } &= Wpz : W \leq \mathcal{P}(q^2), 1+q \notin W \end{aligned}$$

consequenrly, $Aut(\mathcal{A}) = Aut(\mathcal{A}_p) \wr Aut(\mathcal{A}_{q^2})$

- $\Theta_{12} = \{\{q, p.q\}, \{1, p\}, \{q^2\}\}$ then there is one rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{12}$ then:
  $Bset(\mathcal{A}) = \{\{0\}, \{H\backslash\{0\}\}, \{M\backslash H\}, \{G\backslash M\}\}$ so every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{12}$ has the form $\mathcal{A} = \mathcal{A}_q \wr \mathcal{A}_q \wr \mathcal{A}_p$. and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(q^2z)} &= W''q^2z : W'' \leq \mathcal{P}(p) \\ T_{(qz)} = T_{(p.qz)} &= W'qz + q^2\mathbb{Z}_p : W' \leq \mathcal{P}(q) \\ T_{(z)} &= W'z + q\mathbb{Z}_{p.q} : W' \leq \mathbb{P}(q) \end{aligned}$$

  Consequently, $Aut(\mathcal{A}) = Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_q) \wr Aut(\mathcal{A}_q)$

- $\Theta_{13} = \{\{1\}, \{p\}, \{p.q\}, \{q, q^2\}\}$ then there isn't any rational S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{13}$ and every S-ring $\mathcal{A}$ with $\Theta(\mathcal{A}) = \Theta_{13}$ has the form $\mathcal{A} = (\mathcal{A}_p \wr \mathcal{A}_q) \times \mathcal{A}_p$, and $\mathcal{A}$ is generated by the following basic quantities:

$$\begin{aligned} T_{(pz)} &= W'pz : W' \leq \mathcal{P}(q) \\ T_{(p.qz)} &= W'pqz : W' \leq \mathcal{P}(q) \\ T_{(q.z)} &= W''qz + p.q\mathbb{Z}_q : W'' \leq \mathcal{P}(p) \\ T_{q^2z} &= W'pqz + W'pz : W' \leq \mathcal{P}(q) \\ T_{(z)} &= W'pz + W''qz + p.q\mathbb{Z}_q : W'' \leq \mathcal{P}(p), W' \leq \mathcal{P}(q) \end{aligned}$$

  consequenrly, $Aut(\mathcal{A}) = Aut(\mathcal{A}_p) \wr Aut(\mathcal{A}_q) \times Aut(\mathcal{A}_q)$

- Let $\mathcal{A} = \langle \underline{0}, \underline{W}, \underline{l_1W}, \ldots \underline{l_sW} \rangle$, where $W \leq \mathcal{P}(n)$. If W split into a direct product of subgroup in $\mathcal{P}(p)$ with subgroup in $\mathcal{P}(q^2)$, then we return to one of previous cases. Let now W does not split into a direct product of subgroup in $\mathcal{P}(p)$ with subgroup in $\mathcal{P}(q^2)$. We define $A' \equiv W(\mod q^2)$, $A'' \equiv W(\mod p)$, then $B' = A'\mathbb{Z}_{q^2} \leq S_{q^2}$ and $B'' = A''\mathbb{Z}_p \leq S_p$. so $W\mathbb{Z}_{pq^2}$ is a subgroup in direct product $B' \times B''$. thus $\mathcal{A}$ is a s-subring in direct product of $\mathcal{A}_{q^2}$ with $\mathcal{A}_p$, and so $G = Aut(\mathcal{A}) \leq G_{q^2} \times G_p$ is a set of pairs $g = (g', g'')$ acting on $\mathbb{Z}_{pq^2}$ with $g' \in G_{q^2}$ and $g'' \in G_p$. Thus we have two cases:
  If $G \leq \mathcal{H}(\mathbb{Z}_{pq^2})$ then $G = \overline{W}\ \mathbb{Z}_{pq^2}\ : \overline{W} \leq \mathcal{P}(p.q^2)$. Thus:

$$W = T_{(1)}^{\mathcal{V}(\mathbb{Z}_{pq^2}, W)} = T_{(1)}^{\mathcal{V}(\mathbb{Z}_{pq^2}, \overline{W})} = \overline{W}$$

  and then $G = W\mathbb{Z}_{pq^2}$.
  Assume now $G \nleq \mathcal{H}(\mathbb{Z}_{pq^2})$ then there is at least $g_0 = (g_0', g_0'') \in G$ such that $g_0 = (g_0', g_0'') \notin \mathcal{H}(\mathbb{Z}_{pq^2})$. Thus, at least one of the components, for instance, $g_0'' \notin \mathcal{H}(\mathbb{Z}_p)$. But $\mathcal{H}(\mathbb{Z}_p) = N_{S_p}(\mathbb{Z}_p)$, so there exists an element $h_1 \in \mathbb{Z}_p$ such that $g_1'' = g_0''h_1g_0''^{-1} \notin \mathcal{H}(\mathbb{Z}_p)$.
  Define $G' = \{g' \in S_{q^2} : (g', id_p) \in G\}$, $G' = \{g' \in S_p : (id_{q^2}, g'') \in G\}$, (where id_r=identity on the group $\mathbb{Z}_r : r = p, q^2$). Clearly, $\mathbb{Z}_{q^2} \leq G'$, $\mathbb{Z}_p \leq G''$. But $h = (id_{q^2}, h_1) = h_1e_{q^2} \in \mathbb{Z}_n \leq G$ and $g_0 \in G$, so $(id_{q^2}, g_1'') = g_0hg_0^{-1} \in G$ Thus, $g_1'' \in G''$ but $g_1'' \nleq \mathcal{H}(\mathbb{Z}_p)$ so $G'' = S_p$, and G splits into direct product $G' \times G''$. Thus:

$$\begin{aligned} W =& (W\mathbb{Z}_{pq^2})_0 \cap \mathcal{P}(pq^2) = T_{(1)}^{\mathcal{V}(\mathbb{Z}_{pq^2}, (W\mathbb{Z}_{pq^2})_0)} \\ =& T_{(1)}^{\mathcal{V}(\mathbb{Z}_{pq^2}, (G)_0)} = (G)_0 \cap \mathcal{P}(pq^2) \\ =& (G_0' \times G_0'') \cap (\mathcal{P}(q^2) \times \mathcal{P}(p)) \\ =& (G_0' \cap \mathcal{P}(q^2)) \times (G_0'' \cap \times \mathcal{P}(p)) \end{aligned}$$

  Thus, W splits into direct product of $G_0' \cap \mathcal{P}(q^2) \leq \mathcal{P}(q^2)$ and $G_0'' \cap \mathcal{P}(p) \leq \mathcal{P}(p)$. this contradicts to our assumption that W does not split into direct product. and so our assumption $G \nleq \mathcal{H}(\mathbb{Z}_{pq^2})$ is false. Thus $G = W\mathbb{Z}_{pq^2}$

□

4.1 Example

Let $n = 12 = 2^2 \times 3$, From theorem we get the following list of all S-rings over $\mathbb{Z}_n$ and the corresponding automorphism groups:

heightnumber	S-ring	type	$Aut(\mathcal{A})$
1	$\langle \underline{0}, \underline{1,2,3,4,5,6,7,8,9,10,11}, \rangle$	$\langle \underline{0}, \underline{\mathbb{Z}_{12}\backslash 0} \rangle$	S_{12}
2	$\langle \underline{0}, \underline{1,5,7,11}, \underline{2,4}, \underline{8,10}, \underline{3,9}, \underline{6} \rangle$	$\mathcal{A}_p \times (\mathcal{A}_q \wr \mathcal{A}_q)$	$S_3 \times (\mathbb{Z}_2 \wr \mathbb{Z}_2)$
3	$\langle \underline{0}, \underline{1,7}, \underline{2,8}, \underline{4,10}, \underline{5,11}, \underline{3}, \underline{6}, \underline{9} \rangle$	$\mathcal{A}_p \times (\mathcal{A}_q \wr \mathcal{A}_q)$	$\mathbb{Z}_3 \times (\mathbb{Z}_2 \wr \mathbb{Z}_2)$
4	$\langle \underline{0}, \underline{1,2,3,5,6,7,9,10,11}, \underline{4,8} \rangle$	$\mathcal{A}_{q^2} \wr \mathcal{A}_p$	$S_4 \wr S_3$
5	$\langle \underline{0}, \underline{1,2,3,5,6,7,9,10,11}, \underline{4}, \underline{8} \rangle$	$\mathcal{A}_{q^2} \wr \mathcal{A}_p$	$S_4 \wr \mathbb{Z}_3$
6	$\langle \underline{0}, \underline{1,5,9}, \underline{2,6,10}, \underline{3,7,11}, \underline{4,8} \rangle$	$\mathcal{A}_{q^2} \wr \mathcal{A}_p$	$\mathbb{Z}_4 \wr S_3$
7	$\langle \underline{0}, \underline{1,5,9}, \underline{2,6,10}, \underline{3,7,11}, \underline{4}, \underline{8} \rangle$	$\mathcal{A}_{q^2} \wr \mathcal{A}_p$	$\mathbb{Z}_4 \wr \mathbb{Z}_3$
8	$\langle \underline{0}, \underline{1,2,3,4,5,7,8,9,10,11}, \underline{6} \rangle$	$\mathcal{A}_{p.q} \wr \mathcal{A}_q$	$S_6 \wr \mathbb{Z}_2$
9	$\langle \underline{0}, \underline{1,2,4,5,7,8,10,11}, \underline{3,9}, \underline{6} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_q \wr \mathcal{A}_q$	$S_3 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$
10	$\langle \underline{0}, \underline{1,4,7,10}, \underline{2,5,8,11}, \underline{3,9}, \underline{6} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_q \wr \mathcal{A}_q$	$\mathbb{Z}_3 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$
11	$\langle \underline{0}, \underline{1,5,7,11}, \underline{2,4,8,10}, \underline{3,9}, \underline{6} \rangle$	$(\mathcal{A}_p \times \mathcal{A}_p) \wr \mathcal{A}_q$	$(S_3 \times \mathbb{Z}_2) \wr \mathbb{Z}_2$
12	$\langle \underline{0}, \underline{1,7}, \underline{2,8}, \underline{3,9}, \underline{4,10}, \underline{5,11}, \underline{6} \rangle$	$(\mathcal{A}_p \times \mathcal{A}_p) \wr \mathcal{A}_q$	$(\mathbb{Z}_3 \times \mathbb{Z}_2) \wr \mathbb{Z}_2$
13	$\langle \underline{0}, \underline{1,3,5,7,9,10}, \underline{2,10}, \underline{4,8}, \underline{6} \rangle$	$\mathcal{A}_q \wr (\mathcal{A}_p \times \mathcal{A}_p)$	$\mathbb{Z}_2 \wr (S_3 \times \mathbb{Z}_2)$
14	$\langle \underline{0}, \underline{1,3,5,7,9,10}, \underline{2}, \underline{4}, \underline{6}, \underline{8}, \underline{10} \rangle$	$\mathcal{A}_q \wr (\mathcal{A}_p \times \mathcal{A}_p)$	$\mathbb{Z}_2 \wr (\mathbb{Z}_3 \times \mathbb{Z}_2)$
15	$\langle \underline{0}, \underline{1,2,5,7,10,11}, \underline{4,8}, \underline{3,6,9} \rangle$	$\mathcal{A}_p \times \mathcal{A}_{q^2}$	$S_3 \times S_4$
16	$\langle \underline{0}, \underline{1,7,10}, \underline{2,5,11}, \underline{3,6,9}, \underline{4}, \underline{8} \rangle$	$\mathcal{A}_p \times \mathcal{A}_{q^2}$	$\mathbb{Z}_3 \times S_4$
17	$\langle \underline{0}, \underline{1,5,7,11}, \underline{2,10}, \underline{4,8}, \underline{3}, \underline{6}, \underline{9} \rangle$	$\mathcal{A}_p \times \mathcal{A}_{q^2}$	$S_3 \times \mathbb{Z}_4$
18	$\langle \underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, \underline{11} \rangle$	$\mathcal{A}_p \times \mathcal{A}_{q^2}$	$\mathbb{Z}_3 \times \mathbb{Z}_4$
19	$\langle \underline{0}, \underline{1,3,5,7,9,10}, \underline{2,4,8,10}, \underline{6} \rangle$	$\mathcal{A}_q \wr \mathcal{A}_p \wr \mathcal{A}_q$	$\mathbb{Z}_2 \wr S_3 \wr \mathbb{Z}_2$
20	$\langle \underline{0}, \underline{1,3,5,7,9,10}, \underline{2,8}, \underline{4,10}, \underline{6} \rangle$	$\mathcal{A}_q \wr \mathcal{A}_p \wr \mathcal{A}_q$	$\mathbb{Z}_2 \wr \mathbb{Z}_3 \wr \mathbb{Z}_2$
21	$\langle \underline{0}, \underline{1,3,5,7,9,10}, \underline{2,4,6,8,10} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_{p.q}$	$\mathbb{Z}_2 \wr S_6$
22	$\langle \underline{0}, \underline{1,2,4,5,7,8,10,11}, \underline{3,6,9} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_{q^2}$	$S_3 \wr S_4$
23	$\langle \underline{0}, \underline{1,4,7,10}, \underline{2,5,8,11}, \underline{3,6,9} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_{q^2}$	$\mathbb{Z}_3 \wr S_4$
24	$\langle \underline{0}, \underline{1,2,4,5,7,8,10,11}, \underline{3}, \underline{6}, \underline{9} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_{q^2}$	$\mathbb{Z}_4 \wr S_3$
25	$\langle \underline{0}, \underline{1,4,7,10}, \underline{2,5,8,11}, \underline{3}, \underline{6}, \underline{9} \rangle$	$\mathcal{A}_p \wr \mathcal{A}_{q^2}$	$\mathbb{Z}_4 \wr \mathbb{Z}_3$
26	$\langle \underline{0}, \underline{1,3,5,7,9,11}, \underline{2,6,10}, \underline{4,8} \rangle$	$\mathcal{A}_q \wr \mathcal{A}_q \wr \mathcal{A}_p$	$\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr S_3$
27	$\langle \underline{0}, \underline{1,3,5,7,9,11}, \underline{2,6,10}, \underline{4}, \underline{8} \rangle$	$\mathcal{A}_q \wr \mathcal{A}_q \wr \mathcal{A}_p$	$\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_3$
28	$\langle \underline{0}, \underline{1,5,7,11}, \underline{2,4,8,10}, \underline{3}, \underline{6}, \underline{9} \rangle$	$(\mathcal{A}_p \times \mathcal{A}_q) \wr \mathcal{A}_q$	$(S_3 \wr \mathbb{Z}_2) \times \mathbb{Z}_2$
29	$\langle \underline{0}, \underline{1,7}, \underline{2,8}, \underline{4,10}, \underline{5,11}, \underline{3}, \underline{6}, \underline{9} \rangle$	$(\mathcal{A}_p \times \mathcal{A}_q) \wr \mathcal{A}_q$	$(\mathbb{Z}_3 \wr \mathbb{Z}_2) \times \mathbb{Z}_2$
30	$\langle \underline{0}, \underline{1,5}, \underline{2,10}, \underline{4,8}, \underline{7,11}, \underline{3}, \underline{6}, \underline{9} \rangle$	$\langle \underline{0}, \underline{W}, \underline{l_1 W}, ..., \underline{l_r W} \rangle$	$\{1,5\}\mathbb{Z}_{12}$
31	$\langle \underline{0}, \underline{1,7}, \underline{5,11}, \underline{3,9}, \underline{2}, \underline{4}, \underline{6}, \underline{8}, \underline{10} \rangle$	$\langle \underline{0}, \underline{W}, \underline{l_1 W}, ..., \underline{l_r W} \rangle$	$\{1,7\}\mathbb{Z}_{12}$
32	$\langle \underline{0}, \underline{1,11}, \underline{2,10}, \underline{3,9}, \underline{4,8}, \underline{5,7}, \underline{6} \rangle$	$\langle \underline{0}, \underline{W}, \underline{l_1 W}, ..., \underline{l_r W} \rangle$	$\{1,11\}\mathbb{Z}_{12}$

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References

[1] M. H. Klin R. Pschel,The Knig problem, the isomorphism problem for cyclic graphs and the method of Schur rings. Algebraic methods in graph theory, Vols. I, II (Szeged 1978)

[2] M. Ch. Klin R. Pschel. Circulant graphs via Schur ring theory. Automorphism groups of circulant graphs on p^m vertices, p an odd prime. Manuscript.

[3] M.H. Klin R. Pschel, Circulant graphs via Schur ring theory, II. Automorphism groups of circulant graphs on pm vertices, p an odd prime. *(Manuscript).*

[4] M. H. Klin, Automorphism groups of circulant graphs, agungsbericht of the conference Applicable Algebra, Oberwolfach, 1420 Februar, 1993, pp. 12.

[5] Mikhail Klin Istvn Kovacs, Automorphism groups of rational circulant graphs, the electronic journal of combinatorics, Vol.19 (2012), pp.35

[6] I. Kovacs M. Klin, Automorphism groups of circulant graphs on 2^m vertices. (In preparation.)

[7] M. Muzychuk R. Poschel, Isomorphism criterion for circulant graphs, Preprint.Math-AL-9-1999, Technische University a at Dresden, 1999.

[8] A. Joseph, The isomorphism problem for Cayley digraphs on groups of prime squared order, Discrete Math, 141(1-3) (1995), 173-183 https://doi.org/10.1016/0012-365x(93)e0215-p

[9] D. Konig, Theorie der Endlichen und Unendlichen Graphen, Akademische Verlagsgesellschaft, 1936.

[10] Shiu, Wai-chee, The algebraic structure and computation of Schur rings, University of Hong Kong, PH Thesis 1992.

[11] B. A LSPACH and T.D. Pareon, Isomorphism of circulant graphs and digraphs, Discrete Math, 25 (1979),97-108. https://doi.org/10.1016/0012-365x(93)e0215-p

[12] L. Babai, Isomorphism problem for a class of point-symmetric structures, Acta Math.Acad. Sci. Hungar, 29 (1977), 329-336.https://doi.org/10.1007/bf01895854

[13] L. Babai P. Frankl, Isomorphism of Cayley graphs I, Combinatorics, Keszthely,1976, Colloquia Mathematica Societatis J; a anos Bolyai 18 ,North-Holland, Amsterdam ,(1978) p35-52.

[14] L. Babai P. Frankl, Isomorphism of Cayley graphs II, Acta Math. Acad. Sci. Hungar. 34 (1979),177-183.

[15] C. D. Godsil, On Cayley graph isomorphisms, Ars Combin, 15(1983), 231-246.

[16] P. P. Palfy, Isomorphism problem for relational structures with a cyclic automorphism, European J. Combin, l8 (1987), 35-43.https://doi.org/10.1016/s0195-6698(87)80018-5

[17] K. H. Leung S.L. Ma, The structure of Schur rings over cyclic groups, J. Pure Appl.Math, 66(1990), 287-302. https://doi.org/10.1016/0022-4049(90)90032-d

[18] M. E. Muzychuk, On the structure of basic sets of Schur rings over cyclic groups, J. Algebra, 169(1994), 655-678. https://doi.org/10.1006/jabr.1994.1302

[19] M. E. Muzychuk, The structure of rational Schur rings over cyclic groups, European. J. Combin, 14 (1993), 479-490. https://doi.org/10.1006/eujc.1993.1050

[20] H. Wielandt, Finite Permutation Groups, Academic Press, Berlin, 1964. https://doi.org/10.1016/c2013-0-11702-3