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Lie algebra and Laguerre Matrix Polynomials of One Variable

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Abstract. The main object of this present paper is to introduce an extended family of Laguerre matrix polynomials and to derive a new class of certain summation formulas with the help of the Lie algebra method using techniques.

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1 Introduction

The analogous extension to the matrix framework for the classical case of Laguerre, modified Laguerre, and Laguerre-Konhauser matrix polynomials have been studied in [1, 4, 6, 7, 9, 11, 12, 14, 15, 16, 17, 18]. Indeed, in recent papers, Laguerre matrix polynomials have significant emergent. Some results and properties of classical Laguerre matrix functions have been extended to Laguerre matrix polynomials given in [2, 3, 8, 10]. Our main aim in this study is to construct an extended family of Laguerre matrix polynomials and to discuss a new class of certain summation formulas of these matrix functions with the help of the Lie algebra method.

To achieve the purpose of this work, we recall here definitions and properties, which will be used below. Complex space $\mathbb{C}^{N\times N}$ of all square complex matrices of common order N are considered.

Definition 1.1. *[5] If P is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix function $\Gamma(P)$ is defined by*

$$\Gamma(P)=\int_0^\infty e^{-t}t^{P-I}dt;\quad t^{P-I}=\exp\Big((P-I)\ln t\Big), \tag{1}$$

where I is an identity matrix in $\mathbb{C}^{N\times N}$.

From [5] if $A+nI$ is an invertible matrix for all integers $n\geq 0$ then the matrix version of the Pochhammer symbol is

$$(A)_n=A(A+I)\dots(A+(n-1)I)=\Gamma(A+nI)\Gamma^{-1}(A);\ n\geq 1;\ (A)_0=I, \tag{2}$$

and $\Gamma(A)$ is an invertible matrix, its inverse coincides with $\Gamma^{-1}(A)$.

Definition 1.2. *(Jódar and Sastre [3]) Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition*

$$-k\notin\sigma(A)\ \textit{for every integer}\ k>0, \tag{3}$$

where $\sigma(A)$ is the set of all eigenvalues of A and λ is a complex number whose real part is a positive number. Then the Laguerre matrix polynomials is defined by

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k\,(A+I)_n\,[(A+I)_k]^{-1}\,(\lambda x)^k}{k!\,(n-k)!};\lambda\geq 0. \tag{4}$$

We recall that the properties of Laguerre matrix polynomials are

$$\Big[nI - xDI\Big]L_n^{(A,\lambda)}(x) = (A + nI)L_{n-1}^{(A,\lambda)}(x);\ n \geq 1, \tag{5}$$

and

$$\Big[xDI + (A + (n + 1 - \lambda x)I)\Big]L_n^{(A,\lambda)}(x) = (n + 1)L_{n+1}^{(A,\lambda)}(x). \tag{6}$$

2 Group-Theoretic discussion for Laguerre matrix polynomials

By starting with the equations (5) and (6) give the raising and lowering operators $\mathbb{A}$ and $\mathbb{B}$ with respect to the index n. We consider a spurious variable y and define Laguerre matrix polynomials $\Phi_n^{(A,\lambda)}(x, y)$ in the form

$$\Phi_n^{(A,\lambda)}(x, y) = e^{ny}L_n^{(A,\lambda)}(x), n \in \mathbb{N}. \tag{7}$$

In the present investigation for local transformation of Lie groups, we have the following matrix recurrence relations:

$$\mathbb{A}\Phi_n^{(A,\lambda)}(x, y) = (A + nI)\Phi_{n-1}^{(A,\lambda)}(x, y), \tag{8}$$

$$\mathbb{B}\Phi_n^{(A,\lambda)}(x, y) = (n + 1)\Phi_{n+1}^{(A,\lambda)}(x, y), \tag{9}$$

and

$$\mathbb{C}\Phi_n^{(A,\lambda)}(x, y) = (A + (2n + 1)I)\Phi_n^{(A,\lambda)}(x, y), \tag{10}$$

where the first order linear differential operators

$$\mathbb{A} = \left[\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right]e^{-y}I, \tag{11}$$

$$\mathbb{B} = \left[A + (x\frac{\partial}{\partial x} + 1 - \lambda x + \frac{\partial}{\partial y})I\right]e^{y}, \tag{12}$$

and

$$\mathbb{C} = 2\frac{\partial}{\partial y}I + A + I. \tag{13}$$

From the above commutator relations, we can state the next theorem.

Theorem 2.1. *Differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy the commutation relations*

$$(i)[\mathbb{A}, \mathbb{B}] = \mathbb{C}, \quad (ii)[\mathbb{A}, \mathbb{C}] = 2\mathbb{A}, \quad (iii)[\mathbb{B}, \mathbb{C}] = -2\mathbb{B}. \tag{14}$$

Proof. Now, we find that

$$\mathbb{A}\mathbb{B}\Phi_n^{(A,\lambda)}(x, y) = \left[\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right]e^{-y}\left[\left[A + (x\frac{\partial}{\partial x} + 1 - \lambda x + \frac{\partial}{\partial y})I\right]e^{y}\right]\Phi_n^{(A,\lambda)}(x, y).$$

Hence, on simplification, we get

$$\begin{aligned}\mathbb{A}\mathbb{B}\Phi_n^{(A,\lambda)}(x, y) =& \lambda x\Phi_n^{(A,\lambda)}(x, y) + (A + (2 - \lambda x)I)\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x, y) + \frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x, y)\\ &- (A + (3 - \lambda x)I)\frac{\partial}{\partial x}x\Phi_n^{(A,\lambda)}(x, y) - x^2\frac{\partial^2}{\partial x^2}\Phi_n^{(A,\lambda)}(x, y).\end{aligned} \tag{15}$$

On the other hand, we get

$$\mathbb{B}\mathbb{A}\Phi_n^{(A,\lambda)}(x, y) = \left[A + (x\frac{\partial}{\partial x} + 1 - \lambda x + \frac{\partial}{\partial y})I\right]e^{y}\left[\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right]e^{-y}\Phi_n^{(A,\lambda)}(x, y).$$

This is to be simplified as

$$\begin{aligned}\mathbb{B}\mathbb{A}\Phi_n^{(A,\lambda)}(x,y) &= -(A+(1-\lambda x)I)\Phi_n^{(A,\lambda)}(x,y) + (A-\lambda x I)\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ \frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x,y) - (A+(3-\lambda x)I)\frac{\partial}{\partial x}x\Phi_n^{(A,\lambda)}(x,y) - x^2\frac{\partial^2}{\partial x^2}\Phi_n^{(A,\lambda)}(x,y).\end{aligned} \tag{16}$$

Then for $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$, if we subtract (15) from (16), we have

$$[\mathbb{A},\mathbb{B}]\Phi_n^{(A,\lambda)}(x,y) = (A+I)\Phi_n^{(A,\lambda)}(x,y) + 2\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y). \tag{17}$$

Thus, we have the required result $[\mathbb{A},\mathbb{B}] = \mathbb{C}$.

Secondly in order to prove (ii). In a similar manner, we find

$$\mathbb{A}\mathbb{C}\Phi_n^{(A,\lambda)}(x,y) = \left[\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right]e^{-y}\left[A+I+2\frac{\partial}{\partial y}I\right]\Phi_n^{(A,\lambda)}(x,y).$$

which can be simplified as

$$\begin{aligned}\mathbb{A}\mathbb{C}\Phi_n^{(A,\lambda)}(x,y) &= (A+I)\frac{\partial}{\partial y}\left(e^{-y}\Phi_n^{(A,\lambda)}(x,y)\right) - 2e^{-y}\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ 2e^{-y}\frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x,y) - (A+I)xe^{-y}\frac{\partial}{\partial x}\Phi_n^{(A,\lambda)}(x,y) - 2xe^{-y}\frac{\partial^2}{\partial x\partial y}\Phi_n^{(A,\lambda)}(x,y),\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathbb{C}\mathbb{A}\Phi_n^{(A,\lambda)}(x,y) &= \left[A+I+2\frac{\partial}{\partial y}I\right]\left[\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right]e^{-y}\Phi_n^{(A,\lambda)}(x,y)\\ &= (A-I)\frac{\partial}{\partial y}\left(e^{-y}\Phi_n^{(A,\lambda)}(x,y)\right) - 2e^{-y}\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ 2e^{-y}\frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x,y) - (A-I)xe^{-y}\frac{\partial}{\partial x}\Phi_n^{(A,\lambda)}(x,y) - 2xe^{-y}\frac{\partial^2}{\partial x\partial y}\Phi_n^{(A,\lambda)}(x,y).\end{aligned} \tag{19}$$

Combining (18) and (19), we get the result

$$[\mathbb{A},\mathbb{C}]\Phi_n^{(A,\lambda)}(x,y) = 2\frac{\partial}{\partial y}\left(e^{-y}\Phi_n^{(A,\lambda)}(x,y)\right) - 2x\frac{\partial}{\partial x}\left(e^{-y}\Phi_n^{(A,\lambda)}(x,y)\right). \tag{20}$$

Hence, we have

$$[\mathbb{A},\mathbb{C}] = 2\mathbb{A}.$$

Lastly, we consider

$$\mathbb{B}\mathbb{C}\Phi_n^{(A,\lambda)}(x,y) = \left[A+(x\frac{\partial}{\partial x}+1-\lambda x+\frac{\partial}{\partial y})I\right]e^{y}\left[A+I+2\frac{\partial}{\partial y}I\right]\Phi_n^{(A,\lambda)}(x,y),$$

which can be simplified as

$$\begin{aligned}\mathbb{B}\mathbb{C}\Phi_n^{(A,\lambda)}(x,y) &= (A+I)(A+(1-\lambda x)I)e^{y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ (A+I)\frac{\partial}{\partial y}\left(e^{y}\Phi_n^{(A,\lambda)}(x,y)\right) + 2(A+(2-\lambda x)I)e^{y}\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ 2e^{y}\frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x,y) + (A+I)xe^{y}\frac{\partial}{\partial x}\Phi_n^{(A,\lambda)}(x,y) + 2xe^{y}\frac{\partial^2}{\partial x\partial y}\Phi_n^{(A,\lambda)}(x,y).\end{aligned} \tag{21}$$

Similarly, we have

$$\begin{aligned}\mathbb{C}\mathbb{B}\Phi_n^{(A,\lambda)}(x,y) &= (A+I)(A+(2-\lambda x)I)e^{y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ 2(A+(2-\lambda x)I)\frac{\partial}{\partial y}\left(e^{y}\Phi_n^{(A,\lambda)}(x,y)\right) + (A+3I)e^{y}\frac{\partial}{\partial y}\Phi_n^{(A,\lambda)}(x,y)\\ &+ 2e^{y}\frac{\partial^2}{\partial y^2}\Phi_n^{(A,\lambda)}(x,y) + (A+3I)xe^{y}\frac{\partial}{\partial x}\Phi_n^{(A,\lambda)}(x,y) + 2xe^{y}\frac{\partial^2}{\partial x\partial y}\Phi_n^{(A,\lambda)}(x,y).\end{aligned} \tag{22}$$

Subtracting (21) from (22), we get

$$[\mathbb{B},\mathbb{C}]\Phi_n^{(A,\lambda)}(x,y) = -2\Big[A + (x\frac{\partial}{\partial x} + 1 - \lambda x + \frac{\partial}{\partial y})I\Big]\Big(e^y\Phi_n^{(A,\lambda)}(x,y)\Big). \tag{23}$$

Hence, we have the commutation relation

$$[\mathbb{B},\mathbb{C}] = -2\mathbb{B},$$

which completes the proof of the theorem. □

3 Applications

Here the summation formulas of Laguerre matrix polynomials (7) have been derived from the finite differential operator $e^{-\alpha A}$. It is interesting to notice that while constructing the Lie-algebra for Laguerre matrix polynomials, we have

$$e^{-\alpha\mathbb{A}}x = \frac{x}{1-\alpha e^{-y}}, |\alpha e^{-y}| < 1 \tag{24}$$

and

$$e^{-\alpha\mathbb{A}}y = y + \log(1-\alpha e^{-y}). \tag{25}$$

Thus

$$\begin{aligned} e^{-\alpha\mathbb{A}}\Phi_n^{(A,\lambda)}(x,y) =& \Phi_n^{(A,\lambda)}\Big(\frac{x}{1-\alpha e^{-y}}, y + \log(1-\alpha e^{-y})\Big) \\ =& e^{ny}(1-\alpha e^{-y})^n L_n^{(A,\lambda)}\Big(\frac{x}{1-\alpha e^{-y}}\Big). \end{aligned} \tag{26}$$

On the other hand, from eq. (8), we have

$$\begin{aligned} e^{-\alpha\mathbb{A}}\Phi_n^{(A,\lambda)}(x,y) = e^{ny}\sum_{k=0}^{n}\frac{(-\alpha)^k}{k!}e^{-ky} \\ \times (A+nI)(A+(n-1)I)\dots(A+(n-k+1)I)L_{n-k}^{(A,\lambda)}(x). \end{aligned} \tag{27}$$

Writing $t = -\alpha e^{-y}$ and $|t| < 1$, we obtain a well-known finite sum for the Laguerre matrix polynomials

$$\begin{aligned} &(1+t)^n L_n^{(A,\lambda)}\Big(\frac{x}{1+t}\Big) \\ &= \sum_{k=0}^{n}\frac{t^k}{k!}(A+nI)(A+(n-1)I)\dots(A+(n-k+1)I)L_{n-k}^{(A,\lambda)}(x). \end{aligned} \tag{28}$$

4 Conclusion

These are the main results of investigating the family of Laguerre matrix polynomials using Lie algebra method. Starting from the modified forms of the matrix differential recurrence relations of Laguerre matrix polynomials are one of these direct methods and clearly some directions to develop more researchers and studies in that area. Also, some interested summation formulas and consequences of our results have been discussed. The results of this paper are original, variant, significant and so it is interesting and capable to develop its study in the future which plays the vital role in Mathematical Physics.

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