General Letters in Mathematic, Vol. 3, No.2 , Oct 2017, pp. 121 -131
e-ISSN 2519-9277, p-ISSN 2519-9269
Available online at http:\\ www.refaad.com

$(\in, \in \vee q)$-Fuzzy Hemirings and $(\overline{\in}, \overline{\in} \vee \overline{q})$ –Fuzzy Hemirings

Md. Yasin Ali[1, 2*], Kanak Ray Chowdhury[3], Abeda Sultana[2] and Nirmal Kanti Mitra[4]

[1]School of Science and Engineering, University of Information Technology & Sciences, Dhaka, Bangladesh.
ali.mdyasin56@gmail.com
[2]Department of Mathematics, Jahangirnagar University, Savar, Bangladesh.
[3]Department of Mathematics, Mohammadpur Model School and College, Mohammadpur, Dhaka, Bangladesh
[4]Department of Mathematics, Bangladesh University of Business and Technology, Dhaka, Bangladesh

Abstract In this paper, using "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points and fuzzy sets, the concept of $(\in, \in \vee q)$-fuzzy hemirings and $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy hemirings have been introduced and some of its properties have been investigated.

Keywords: Fuzzy set, Hemiring, fuzzy hemiring, $(\in, \in \vee q)$- fuzzy hemiring, $(\overline{\in}, \overline{\in} \vee \overline{q})$ - fuzzy hemiring.
2000 MSC No: 16Y60, 08A72, 03G25, 03E72.

1 Introduction

After the introduction of fuzzy set by Zadeh [1], a host of researchers applied this impression in different branches of algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. In algebra, fuzzy set was first applied by Rosenfeld [2]; the idea of fuzzy group was defined and its structures were investigated. The $(\in, \in \vee q)$ -fuzzy group which is generalization of Rosenfeld's fuzzy group was initiated Bhakat and Das [3] by using the joint notions of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points and fuzzy sets, which was instigated by Pu and Liu [4]. Later these structures were studied by some researchers [5-13]. In hemirings structures, Dudek et al. [14] introduced the concept of $(\in, \in \vee q)$ -fuzzy *h*-ideals (*k*-ideal) of a hemirings, and gave its characterizations. Afterward there have been done some works related to $(\in, \in \vee q)$ -fuzzy *h*-ideals of a hemiring [15-20]. In this paper, we have introduced $(\in, \in \vee q)$-fuzzy hemirings and $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy hemirings and some related properties have been described.

2 Preliminaries

Definition 2.1 [21]: A ***hemiring*** is a nonempty set H on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

$[H_1]$: $(H;+)$ is a commutative monoid with identity element 0;
$[H_2]$: $(H;\cdot)$ is a semigroup;
$[H_3]$: Multiplication distributes over addition from either side;
$[H_4]$: The element 0 is the absorbing element of the multiplication i.e., $0 \cdot r = 0 = r \cdot 0$.

Note: We shall omit the symbol "$\cdot$" writing ab for $a \cdot b (a, b \in H)$.

Definition 2.2 [22]: Suppose $(H;+,\cdot)$ is a hemiring and μ is a fuzzy set of H.

(i) For any $t \in [0,1]$ the set $\mu_t = \{x \in H : \mu(x) \geq t\}$ is called ***t- level set*** of μ.
(ii) $Supp(\mu) = \{x \in H : \mu(x) > 0\}$ is called ***support*** of μ.

Definition 2.3 [11]: Suppose $(H;+,\cdot)$ is a hemiring and μ is a fuzzy set of H with support $\{x\}$. Then μ is called a ***fuzzy point*** where,

$$\mu(y) = \begin{cases} t \in]0,1] & if \quad y = x \\ 0 & otherwise \end{cases}$$

We denote the fuzzy point with support $\{x\}$ and value t by x_t. For a fuzzy point x_t and a fuzzy set μ of H, **Pu and Liu [4]** introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \wedge q, \in \vee q\}$. For a fuzzy point x_t and a fuzzy set μ of H, we say that a fuzzy point x_t

(i) belongs to μ, denoted by $x_t \in \mu$ if $\mu(x) \geq t$;
(ii) quasi-coincident with μ, denoted by $x_t q \mu$ if $\mu(x) + t > 1$.

For a fuzzy point x_t and a fuzzy set μ of H, we say that

(iii) $x_t \in \vee q \mu$ if $x_t \in \mu$ or $x_t q \mu$;
(iv) $x_t \in \wedge q \mu$ if $x_t \in \mu$ and $x_t q \mu$.

The symbol $x_t \overline{\in} \mu$, $x_t \overline{q} \mu$ and $x_t \overline{\in \vee q} \mu$ mean that $x_t \in \mu$, $x_t q \mu$ and $x_t \in \vee q \mu$ do not hold respectively.

Example 2.3 (a): Let $H = \{0,1,2\}$ be a hemiring with the operations addition $(+)$ and multiplication $(\cdot)$ as follows:

$+$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\cdot$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Now, define a fuzzy set μ of H as follows:

$$\mu(x) = \begin{cases} 0.3 & if \quad x = 0 \\ 0.6 & if \quad x = 1 \\ 0.8 & if \quad x = 2 \end{cases}$$

Where $Supp(\mu) = \{0,1,2\}$. Now consider the fuzzy point 0_t where $t \in]0, 0.3]$ then $0_t \in \mu$. Again we consider the fuzzy point 2_t where $t \in]0.2, 0.8]$ then $2_t q \mu$.

Definition 2.4 [23]: Let $(H;+,\cdot)$ be a hemiring. A fuzzy set μ of H is said to be ***fuzzy hemiring*** of H if it satisfies the following conditions:

$$[FH_1]: \mu(x+y) \geq \min\{\mu(x), \mu(y)\};$$
$$[FH_2]: \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}; \forall x, y \in H.$$

Definition 2.5: Let μ_1 and μ_2 be two fuzzy hemirings of a hemiring $(H;+,\cdot)$. We define the ***sum and product***, denoted by $\mu_1 + \mu_2$ and $\mu_1 \cdot \mu_2$ of μ_1 and μ_2 by

$$(\mu_1 + \mu_2)(x) = \max_{x=y+z}\{\min[\mu_1(y), \mu_2(z)]\}.$$

If x can be expressed as $x = y + z$ otherwise $(\mu_1 + \mu_2)(x) = 0.$

$$\text{and } (\mu_1 \cdot \mu_2)(x) = \max_{x=yz}\{\min[\mu_1(y), \mu_2(z)]\}$$

If x can be expressed as $x = yz$ otherwise $(\mu_1 \cdot \mu_2)(x) = 0.$

3 $(\in, \in \vee q)$ -Fuzzy Hemirings

Definition 3.1: A fuzzy set μ of a hemiring $(H;+,\cdot)$ is called $(\in, \in \vee q)$***-fuzzy hemiring*** of H if $\forall t, r \in]0,1]$ and $\forall x, y \in H$, the following conditions hold:

$$[FH1a]: x_t \in \mu \text{ and } y_r \in \mu \Rightarrow (x+y)_{\min\{t,r\}} \in \vee q \mu;$$

$$[FH2a]: x_t \in \mu \text{ and } y_r \in \mu \Rightarrow (xy)_{\min\{t,r\}} \in \vee q \mu.$$

Example 3.1 (a): Let $H = N \cup \{0\}$. Then H is a hemiring with usual addition and multiplication. Let μ be a fuzzy set defined by

$$\mu(x) = \begin{cases} 0.9 & if \quad x \quad is \quad even \\ 0.8 & if \quad x \quad is \quad odd \\ 0.6 & if \quad x = 0 \end{cases}$$

Then routine calculation gives that, μ is an $(\in, \in \vee q)$ - fuzzy hemiring of H.

Theorem 3.2: The conditions $[FH1a]$ and $[FH2a]$ in definition 3.1 are equivalent to the following conditions $[FH1b]$ and $[FH2b]$ respectively, where

$$[FH1b]: \mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\};$$

$$[FH2b]: \mu(x \cdot y) \geq \min\{\mu(x), \mu(y), 0.5\}.$$

Proof: $[FH1a] \Rightarrow [FH1b]$: Suppose that $x, y \in H$, we consider the following cases:

(a) $\min\{\mu(x), \mu(y)\} < 0.5,$

(b) $\min\{\mu(x), \mu(y)\} \geq 0.5.$

Case (a): Assume that there exists $x, y \in H$ such that

$$\mu(x+y) < \min\{\mu(x), \mu(y), 0.5\};$$

Then for $t \in]0,0.5]$, we have,

$$\mu(x+y) < t \leq \min\{\mu(x), \mu(y)\}.$$

It follows that $x_t \in \mu$ and $y_r \in \mu$ but $(x+y)_t \overline{\in} \mu.$

Moreover, $\mu(x+y)+t<1$ and so $(x+y)_t \overline{q}\mu$.

Hence $(x+y)_t \overline{\in \vee q}\mu$, a contradiction.

Case (b): Assume that there exists $x, y \in H$ such that

$\mu(x+y)<0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, but $(x+y)_{0.5} \overline{\in}\mu$..

Also $\mu(x+y)+0.5<1$ and so $(x+y)_{0.5} \overline{q}\mu$.

Therefore $(x+y)_{0.5} \overline{\in \vee q}\mu$, a contradiction

Hence $[FH1b]$ holds.

$[FH1b] \Rightarrow [FH1a]$: Let $x_t \in \mu$ and $y_r \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$.

Now we have,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, r, 0.5\}.$$

If $\min\{t, r\} > 0.5$ then $\mu(x+y) \geq 0.5$, this implies that

$$\mu(x+y)+\min\{t.r\}>1.$$

If $\min\{t, r\} \leq 0.5$ then $\mu(x+y) \geq \min\{t, r\}$.

Therefore $(x+y)_{\min\{t,r\}} \in \vee q\mu.$

Similarly, we can prove that $[FH2a] \Rightarrow [FH2b]$.

Theorem 3.3: If μ is a fuzzy set of a hemiring $(H;+,\cdot)$ satisfying $\mu(x) \geq 0; \forall x \in H$, then μ is an $(\in, \in \vee q)$ -fuzzy hemiring of H.

Proof: Let $x, y \in H$ and $t, r \in]0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now $\forall x, y \in H$, we have $x+y \in H$ and $xy \in H$, then

$$\mu(x+y) \geq 0.5 \text{ and } \mu(xy) \geq 0.5 .$$

If $\min\{t, r\} \leq 0.5$, then $\mu(x+y) \geq 0.5 \geq \min\{t, r\}$

and $\mu(xy) \geq 0.5 \geq \min\{t, r\}$.

This implies $(x+y)_{\min\{t,r\}} \in \mu$ and $(xy)_{\min\{t,r\}} \in \mu.$

Again if $\min\{t, r\} > 0.5$, then $\mu(x+y)+\min\{t.r\}>1.$

and $\mu(xy)+\min\{t.r\}>1.$

This implies that $(x+y)_{\min\{t,r\}} q\mu$ and $(xy)_{\min\{t,r\}} q\mu.$

Therefore, $(x+y)_{\min\{t,r\}} \in \vee q\mu$ and $(xy)_{\min\{t,r\}} \in \vee q\mu.$

Thus μ is an $(\in, \in \vee q)$ -fuzzy hemiring of H.

Theorem 3.4: Every fuzzy hemiring is an $(\in, \in \vee q)$ -fuzzy hemiring.

Proof: Let $x, y \in H$ and $t, r \in]0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$. Since μ is a fuzzy hemiring so we have,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \text{ and } \mu(xy) \geq \min\{\mu(x), \mu(y)\}.$$

Now $\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, r\}$.

And $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, r\}$. .

If $\min\{t, r\} \leq 0.5$ then $(x+y)_{\min\{t,r\}} \in \mu$ and $(xy)_{\min\{t,r\}} \in \mu$.

Or if $\min\{t, r\} > 0.5$ then $(x+y)_{\min\{t,r\}} q\mu$ and $(xy)_{\min\{t,r\}} q\mu$.

Hence consequently $(x+y)_{\min\{t,r\}} \in \vee q\mu$ and $(xy)_{\min\{t,r\}} \in \vee q\mu$.

This implies that μ is an $(\in, \in \vee q)$-fuzzy hemiring of H.

Theorem 3.5: Any $(\in, \in \vee q)$- fuzzy hemiring μ of a hemiring $(H;+,\cdot)$ satisfying $\mu(x) < 0.5$, is an ordinary fuzzy hemiring.

Proof: Let $x, y \in H$ and $t, r \in]0,1]$ be such that $x_t \in \mu$ and $y_r \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$. Since μ is an $(\in, \in \vee q)$ fuzzy hemiring of H, so we have,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$$

and $$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}.$$

Now $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} = \min\{t, r, 0.5\} = \min\{t, r\}$.

And $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\} = \min\{t, r, 0.5\} = \min\{t, r\}$. .

This implies that $(x+y)_{\min\{t,r\}} \in \mu$ and $(xy)_{\min\{t,r\}} \in \mu$.

Thus μ is an ordinary fuzzy hemiring.

Theorem 3.6: A fuzzy set μ of a hemiring $(H;+,\cdot)$ is an $(\in, \in \vee q)$-fuzzy hemiring of H iff for any $t \in]0,0.5], \mu_t (\neq \phi)$ is a sub-hemiring of H.

Proof: Let $x, y \in \mu_t$ then $\mu(x) \geq t$ and $\mu(y) \geq t$.

We consider μ is an $(\in, \in \vee q)$-fuzzy hemiring of H and $t \in]0,0.5]$. Then we have,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$$

$$\Rightarrow \mu(x+y) \geq \min\{t, t, 0.5\}$$

$$\Rightarrow \mu(x+y) \geq t$$

$$\therefore x+y \in \mu_t.$$

And $$\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$$

$$\Rightarrow \mu(xy) \geq \min\{t, r, 0.5\}$$

$$\Rightarrow \mu(xy) \geq t$$

$$\therefore xy \in \mu_t.$$

Hence μ_t is a sub-hemiring of H.

Conversely, suppose μ_t is a sub-hemiring of H. Then $\forall x, y \in \mu_t$, we have

$$x+y \in \mu_t, \; xy \in \mu_t \text{ and } t \in]0,0.5].$$

Now, $$x+y \in \mu_t$$

$$\Rightarrow \mu(x+y) \geq t$$

$\Rightarrow \mu(x+y) \geq \min\{t, t, 0.5\}$

$= \min\{\mu(x), \mu(y), 0.5\}$

And $xy \in \mu_t$

$\Rightarrow \mu(xy) \geq t$

$\Rightarrow \mu(xy) \geq \min\{t, t, 0.5\}$

$= \min\{\mu(x), \mu(y), 0.5\}$.

Hence μ is a $(\in, \in \vee q)$ fuzzy hemiring of H.

Definition 3.7: Let μ be any fuzzy hemiring of a hemiring $(H;+,\cdot)$.

The set $[\mu]_t = \{x \in H : x_t \in \vee q\mu\}$, where $t \in]0,1]$ is called an $(\in, \in \vee q)$ - ***level set*** of μ.

Theorem 3.8: A fuzzy set μ of a hemiring $(H;+,\cdot)$ is an $(\in, \in \vee q)$ -fuzzy hemiring of H iff $[\mu]_t$ is a sub-hemiring of $\forall t \in]0,1]$.

Proof: Let μ be an $(\in, \in \vee q)$ fuzzy hemiring of a hemiring H.

Let $x, y \in [\mu]_t; \forall t \in]0,1]$. Then $\mu(x) \geq t$ or $\mu(x)+t > 1$ and $\mu(y) \geq t$ or $\mu(y)+t > 1$. Since μ is an $(\in, \in \vee q)$ -fuzzy hemiring, we have,

Case-1: When $\mu(x) \geq t$ and $\mu(y) \geq t$.

Then $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$

$\Rightarrow \mu(x+y) \geq \min\{t, t, 0.5\}$.

If $t > 0.5$, then $\mu(x+y) \geq 0.5$, and so $\mu(x+y)+t > 1 \therefore (x+y)_t q\mu$.

Or if $t \leq 0.5$, then $\mu(x+y) \geq t$, and so $(x+y)_t \in \mu$.

Thus for $t \in]0,1]$, we have $x+y \in [\mu]_t$.

Case-2: When $\mu(x) \geq t$ and $\mu(y)+t > 1$.

Then $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$

$\Rightarrow \mu(x+y) \geq \min\{t, 1-t, 0.5\}$.

If $t > 0.5$, then $1-t < 0.5$.

This implies $\mu(x+y) > 1-t \Rightarrow \mu(x+y)+t > 1$, therefore $(x+y)_t q\mu$..

Or if $t \leq 0.5$, then $1-t \geq 0.5$, then we have $\mu(x+y) \geq t$.

Therefore $(x+y)_t \in \mu$.

Thus $x+y \in [\mu]_t; \forall t \in]0,1]$.

Case-3: Similar to case-2.

Case-4: When $\mu(x)+t > 1$ and $\mu(y)+t > 1$.

Then $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$

$\Rightarrow \mu(x+y) \geq \min\{1-t, 1-t, 0.5\}$.

If $t > 0.5$ then $1-t < 0.5$ and so $\mu(x+y) > 1-t \Rightarrow \mu(x+y)+t > 1$.

This implies $(x+y)_t q\mu$. .

If $t \le 0.5$, then $1-t \ge 0.5 \ge t$. In this case we have,

$$\mu(x+y) \ge t \Rightarrow (x+y)_t \in \mu.$$

Thus $x+y \in [\mu]_t; \forall t \in]0,1]$.

In similar way for each case we can prove that $xy \in [\mu]_t; \forall t \in]0,1]$.

Therefore, we have $x+y \in [\mu]_t$ and $xy \in [\mu]_t$.

Thus $[\mu]_t$ is a sub-hemiring of H.

Conversely, let μ be a fuzzy set and $[\mu]_t$ is a sub-hemiring of $H; \forall t \in]0,0.5]$.

If $\mu(x+y) < t < \min\{\mu(x), \mu(y), 0.5\}$ for some $t \in]0,0.5]$, then $x, y \in [\mu]_t$ and $x+y \in [\mu]_t$. Hence $\mu(x+y) \ge t$ or $\mu(x+y)+t > 1$ contradiction, similar result holds for $\mu(xy) \ge \min\{\mu(x), \mu(y), 0.5\}$.

Therefore $\mu(x+y) \ge \min\{\mu(x), \mu(y), 0.5\}$ and $\mu(xy) \ge \min\{\mu(x), \mu(y), 0.5\}$.

Again $\forall t \in]0.5,1]$.

If $\mu(x+y) < t < \min\{\mu(x), \mu(y)\}$, then $x, y \in [\mu]_t$ and $x+y \in [\mu]_t$,

but $(x+y)_t \overline{\in \vee q} \mu$, a contradiction.

Thus $\mu(x+y) \ge \min\{\mu(x), \mu(y), 0.5\}$

In similar way, $\mu(xy) \ge \min\{\mu(x), \mu(y), 0.5\}$.

Hence μ is an $(\in, \in \vee q)$ -fuzzy hemiring of H.

Definition 3.9: Let μ_1 and μ_2 be two $(\in, \in \vee q)$ -fuzzy hemiring of a hemiring $(H;+,\cdot)$. We define $\mu_1 \cap \mu_2$ by

$$(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x), 0.5\}.$$

Example 3.9 (a): Let $H = \{0,1\}$ be a hemiring with the following two operations:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

Define two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring H by

$\mu_1(0) = 0.6, \mu_1(1) = 0.7,$ and $\mu_2(0) = 0.4, \mu_2(1) = 0.7,$ respectively.

Then routine verification gives that, $(\mu_1 \cap \mu_2)(0) = 0.4,$ $(\mu_1 \cap \mu_2)(1) = 0.5.$

Theorem 3.10: Let μ_1 and μ_2 be two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring H. Then $\mu_1 \cap \mu_2$ is also $(\in, \in \vee q)$ -fuzzy hemiring of H.

Proof: Let $x, y \in \mu_1 \cap \mu_2$. Then $x, y \in \mu_1$ and μ_2

Now,

$$(\mu_1 \cap \mu_2)(x+y) = \min\{\mu_1(x+y), \mu_2(x+y), 0.5\}$$

$$\ge \min\{\min\{\mu_1(x), \mu_1(y), 0.5\}, \min\{\mu_2(x), \mu_2(y), 0.5\}, 0.5\}$$

$$= \min\{\min\{\mu_1(x), \mu_2(x), 0.5\}, \min\{\mu_1(y), \mu_2(y), 0.5\}, 0.5\}$$

$$= \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y), 0.5\}$$

Similarly, we can prove that $(\mu_1 \cap \mu_2)(xy) \geq \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y), 0.5\}$. Thus $\mu_1 \cap \mu_2$ is a fuzzy hemiring of H.

Definition 3.11: Let μ_1 and μ_2 be two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring $(H;+,\cdot)$. Then ***product*** of μ_1 and μ_2 is defined as

$$(\mu_1 \circ \mu_2)(x) = \max_{x=y\cdot z}\{\min\{\mu_1(y), \mu_2(z), 0.5\}\}.$$

If x can be expressed as $x = yz$ otherwise $(\mu_1 \circ \mu_2)(x) = 0$.

Example 3.11 (a): Let $H = \{0, a, b\}$ be a hemiring with the following two operations:

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	b
b	0	b	b

Define two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring H by

$\mu_1(0) = 0.2, \mu_1(a) = 0.5, \mu_1(b) = 0.6$ and $\mu_2(0) = 0.2, \mu_2(a) = 0.2, \mu_2(b) = 0.6$ respectively. Then routine verification gives that, $(\mu_1 \circ \mu_2)(0) = 0.2$, $(\mu_1 \circ \mu_2)(a) = 0.2$, $(\mu_1 \circ \mu_2)(b) = 0.5$.

Theorem 3.12: Let μ_1 and μ_2 be two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring $(H;+,\cdot)$. Then their product $\mu_1 \circ \mu_2$ is also $(\in, \in \vee q)$ -fuzzy hemiring of H.

Proof: Trivial

Theorem 3.13: Let λ, μ and ν be $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring $(H;+,\cdot)$. If $\lambda \leq \mu$, then $\lambda \circ \nu \leq \mu \circ \nu$ and $\nu \circ \lambda \leq \nu \circ \mu$.

Proof: Let $x \in H$. If x is not expressible as $x = yz$, then

$$(\lambda \circ \nu)(x) = 0 = (\mu \circ \nu)(x).$$

Otherwise, $(\lambda \circ \nu)(x) = \max_{x=yz}\{\min\{\lambda(y), \nu(z), 0.5\}\}$

$$\leq \max_{x=yz}\{\min\{\mu(y), \nu(z), 0.5\}\}$$

$$= (\mu \circ \nu)(x).$$

Hence $\lambda \circ \nu \leq \mu \circ \nu$.

Similarly, we can prove that $\nu \circ \lambda \leq \nu \circ \mu$.

Theorem 3.14: Let λ, μ and ν be $(\in, \in \vee q)$ fuzzy hemirings of a hemiring $(H;+,\cdot)$. Then $(\lambda \circ \mu) \circ \nu = \lambda \circ (\mu \circ \nu)$.

Proof: Trivial

Definition 3.15: Let μ_1 and μ_2 be two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring H. We define the ***sum*** of μ_1 and μ_2, denoted by $\mu_1 + \mu_2$ by

$$(\mu_1 + \mu_2)(x) = \max_{x=y+z}\{\min\{\mu_1(y), \mu_2(z), 0.5\}\}.$$

Example 3.15 (a): Let $H = \{0, a, b\}$ be a hemiring with the following two operations:

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

·	0	a	b
0	0	0	0
a	0	a	b
b	0	b	B

Define two $(\in, \in \vee q)$ -fuzzy hemirings of a hemiring H by $\mu_1(0) = 0.2, \mu_1(a) = 0.5, \mu_1(b) = 0.6$ and $\mu_2(0) = 0.2, \mu_2(a) = 0.2, \mu_2(b) = 0.6$, respectively. Then routine verification gives that,

$(\mu_1 + \mu_2)(0) = 0.2,\ (\mu_1 + \mu_2)(a) = 0.2,\ (\mu_1 \circ \mu_2)(b) = 0.5.$

Theorem 3.16: Let μ_1 and μ_2 be two $(\in, \in \vee q)$-fuzzy hemirings of a hemiring $(H;+,\cdot)$. Then their sum $\mu_1 + \mu_2$ is also $(\in, \in \vee q)$ -fuzzy hemiring of H.

Proof: Trivial

Theorem 3.17: A fuzzy set μ of a hemiring $(H;+,\cdot)$. is an $(\in, \in \vee q)$-fuzzy hemiring of H iff $\mu + \mu \leq \mu$ and $\mu \circ \mu \leq \mu.$

Proof: Trivial

4 $(\bar{\in}, \bar{\in} \vee \bar{q})$ -Fuzzy Hemiring

Definition 4.1.: A fuzzy set μ of a hemiring $(H;+,\cdot)$. is called an $(\bar{\in}, \bar{\in} \vee \bar{q})$ ***-fuzzy hemiring*** of H if $\forall t, r \in]0,1]$ and $\forall x, y \in H$, the following conditions are hold:

$$[FH3a]: (x+y)_{\min\{t,r\}} \bar{\in} \mu \Rightarrow x_t \bar{\in} \vee \bar{q} \mu \ or\ y_t \bar{\in} \vee \bar{q} \mu;$$

$$[FH4a]: (xy)_{\min\{t,r\}} \bar{\in} \mu \Rightarrow x_t \bar{\in} \vee \bar{q} \mu \ or\ y_t \bar{\in} \vee \bar{q} \mu.$$

Example 4.1 (a): Let $H = \{0, a\}$ is a hemiring where addition and multiplication are defined as follows:

+	0	a
0	0	a
a	a	a

·	0	a
0	0	0
a	0	A

Consider the fuzzy sets $\mu(0) = 0.3$ and $\mu(a) = 0.4$. Then by routine calculation gives that, μ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ -fuzzy hemiring.

Theorem 4.2: The conditions $[FH3a]$ and $[FH4a]$ in definition 4.1 are equivalent to the following conditions $[FH3b]$ and $([FH4b]$ respectively.

$$[FH3b]: \max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\};$$

$$[FH4b]: \max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}.$$

Proof: First we prove $[FH3a]$ is equivalent to $[FH3b]$.

$[FH3a] \Rightarrow [FH3b]$: Suppose $[FH3b]$ does not hold.

Then there exists $x, y \in H$ such that $\max\{\mu(x+y), 0.5\} < \min\{\mu(x), \mu(y)\}$. Then we can choose $t \in]0.5,1]$ such that, $\max\{\mu(x+y), 0.5\} < t \leq \min\{\mu(x), \mu(y)\}$.

This implies $(x+y)_t \bar{\in} \mu$ but $x_t \in \wedge q$ and $y_t \in \wedge q$, which is a contradiction.

So $[FH3b]$ holds.

$[FH3b] \Rightarrow [FH3a]$:

Let $x, y \in H$ and $t, r \in]0,1]$ be such that $(x+y)_{\min\{t,r\}} \overline{\in} \mu.$

Then $\mu(x+y) < \min\{t, r)\}.$

If $\max\{\mu(x+y), 0.5\} = \mu(x+y),$ then,

$\min\{\mu(x), \mu(y)\} \le \mu(x+y) < \min\{t, r\} \Rightarrow \mu(x) < t$ or $\mu(y) < t \Rightarrow x_t \overline{\in} \mu$

or $y_r \overline{\in} \mu \Rightarrow x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu.$

If $\max\{\mu(x+y), 0.5\} = 0.5,$ then $\min\{\mu(x), \mu(y)\} \le 0.5.$

Suppose $x_t \in \mu$ and $y_r \in \mu$, then $\mu(x) \ge t$ and $\mu(y) \ge r$.

Thus $t \le \mu(x) < 0.5$ or $r \le \mu(y) < 0.5 \Rightarrow x_t \overline{q} \mu$ or $y_r \overline{q} \mu \Rightarrow x_t \overline{\in} \vee \overline{q} \mu$

Or $y_r \overline{\in} \vee \overline{q} \mu.$ Thus prove $[FH3a]$.

Similarly we can prove that $[FH4a] \Leftrightarrow [FH4b].$

Theorem 4.3: A fuzzy set μ of a hemiring $(H;+,\cdot)$. is an $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy hemiring of H iff for any $t \in]0.5,1], \mu_t (\ne \phi)$ is a sub-hemiring of H.

Proof: It is immediate consequence of the theorem 3.6.

5 Conclusion

Fuzzy algebraic structures play an important role in many branches of applied mathematics and engineering sciences. In order to broad fuzzy algebraic structures in this article we introduce $(\in, \in \vee q)$-fuzzy hemiring and $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy hemiring and some of its structures are investigated. These structures will be helpful to broad the applications of fuzzy hemirings in mathematics and engineering sciences.

References

[1] L. A. Zadeh, Fuzzy sets, Inform. Control, 8, (1965), pp. 338-353.

[2] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35, (1971), pp. 512-517.

[3] S. K. Bhakat, P. Das, $(\in, \in \vee q)$ -fuzzy subgroups, Fuzzy Sets and Systems, 80, (1996), pp. 359-368.

[4] P. M. Pu, Y. M. Liu, Fuzzy topology I: Neighourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76, (1980), pp. 571-599.

[5] X. Yuan, C. Zhang and Y. Ren, Generalized fuzzy groups and many-valued implications, Fuzzy Sets and Systems, 138, (2003), pp. 205-211.

[6] Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc., 42, (2005), pp. 703-711.

[7] B. Davvaz, $(\in, \in \vee q)$ -fuzzy subnear-rings and ideals, Soft Computing, 10, (2006), pp. 206-211.

[8] B. Davvaz and P. Corsini, On (α, β) - Fuzzy H_v Ideals of H_v Rings, Iranian Journal of Fuzzy Systems, 5 (2), (2008), pp. 35-47.

[9] Y. B. Jun, Generalization of $(\in, \in \vee q)$ -fuzzy sub algebras in BCK/BCI- algebras, Comput. Math. Appl., 58, (2009), pp. 1383-1390.

[10] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$ -fuzzy bi-ideals, Bull. Malays. Math. Sci. Soc., 32, (2009), pp. 391-408.

[11] Y. Yunqiang and D. Xu, The $(\in, \in \vee q)$ -fuzzy subsemigroups and ideals of an $(\in, \in \vee q)$ -fuzzy semigroup, Southeast Asian Bull. Of math., 33, (2009), pp. 391-400.

[12] Anwar Zeb et.al Characterization of ternary semigroups in terms of $(\in, \in \vee qk)$ ideals, Mathematica Aeterna, 2, (2012), pp. 227 – 246.

[13] X. Ma, J.M. Zhan, B. Davvaz, Y.B. Jun, Some kinds of $(\in, \in \vee q)$ -interval-valued fuzzy ideals of BCI-algebras, Inform. Sci., 178, (2008), pp. 3738-3754.

[14] W.A. Dudek, M. Shabir, M. Irfan Ali, $(\in, \in \vee q)$ -fuzzy ideals of hemirings, Comput. Math. Appl., 58, (2009), pp. 310-321.

[15] Young Bae Jun, Note on "(α, β) -fuzzy ideals of hemirings", Comput. Math. Appl., 59, (2010), pp. 2582-2586.

[16] Y. Yin, J. Zhan and X. Huang, A New way to fuzzy h-ideals of hemirings, Iranian Journal of Fuzzy Systems, 8, (2011), pp.81-101.

[17] X. Ma, J. Zhan and K. P. Shum, Generalized Fuzzy h- ideal of Hemirings, Bull. Malays. Math. Sci. Soc., 34, (2011), pp. 561-574.

[18] T. Mahmood, Characterizations of h- Hemiregular and h-Semisimple hemirings by interval Valued $(\bar{\in}, \bar{\in} \vee \bar{q})$ - Fuzzy h- Ideals, The Nucleus, 51, (2014), pp. 19-27.

[19] G. Mohanraj and E. Prabu, Generalized Fuzzy Right H- ideal of Hemirings, Intern. J. Fuzzy Mathematical Archive, 7(2), (2015), pp. 147-155.

[20] T. Mahmood and M. Shabir, Ideal Redefined to Characterize Hemirings, World Applied Sciences Journal, 18(12), (2012), pp. 1870-1882.

[21] M. Y. Ali, K. R. Chowdhury, A. Sultana, N. K. Mitra, Some Structures of Hemirings, Pure and Applied Mathematics Journal, l 6(1), (2017), pp. 45-50.

[22] M. Y. Ali, K. R. Chowdhury, A. Sultana, N. K. Mitra, T-Fuzzy Left h-Ideals of Hemirings, Asian Research Journal of Mathematics, 4(1), (2017), pp. 1-14.

[23] M. Y. Ali, K. R. Chowdhury, A. Sultana, N. K. Mitra, Fuzzy hemirings Asian Journal of Mathematics and computer Research, 20(2), (2017), pp. 96-103.