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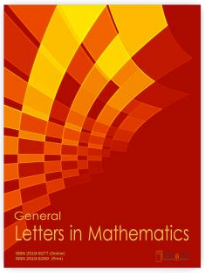

Characterizations of Interval-Valued Intuitionistic Fuzzy n-Fold Positive Implicative Deal of BCK-Algebras

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Abstract Using the concept of interval-valued intuitionistic fuzzy set, the notion of interval-valued intuitionistic fuzzy n-fold BCK-ideals and interval-valued intuitionistic fuzzy n-fold positive implicative ideas are introduced, in BCK-Algebras and investigate some of its related properties. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideal are investigated.

Keywords: Interval-valued intuitionistic fuzzy n-fold positive implicative ideal, Interval-valued intuitionistic fuzzy n-fold BCK-ideal.
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1 Introduction :

For the general development of BCK-algebras, the ideal theory plays an important role. In [7] 1966 by K. Iseki and Y. Imai introduced a new notion called a BCK-algebras, and then many researchers have investigate various properties of this algebra. Hung and Chen [6] introduced the notion of n-fold implicative ideals, n-fold (weak) commutative ideals, n-fold positive implicative ideals and investigate some of its properties. Jun and Kim [9], introduced the notions of n-fold fuzzy positive implicative ideals in BCK-algebras and investigate some of its related properties. Satyanarayana and Durga Prasad [14], introduced the intuitionistic fuzzification of n-fold BCK-ideal and n-fold positive implicative ideal in BCK-algebras and some of its related properties are investigated.

In this paper, we apply the concept of interval-valued intuitionistic fuzzy set to n-fold BCK-ideals and n-fold positive implicative ideals in BCK-algebras and introduced the notion of interval-valued intuitionistic fuzzy n-fold BCK-ideal and interval-valued intuitionistic fuzzy n-fold positive implicative ideal and we gave relations between these notions and investigate some of its related properties. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideals are investigated.

2 Preliminaries :

Definition 2.1. Let X be a set with a binary operation "$*$" and a constant "0". Then $(X, *, 0)$ is called BCK-algebra, if it satisfies the following conditions.

(BCK-1) $((x * y) * (x * z)) * (z * y)=0$,
(BCK-2) $(x * (x * y)) * y=0$,
(BCK-3) $x * x=0$
(BCK-4) $0 * x=0$
(BCK-5) $x * y=0$ and $y * x=0$ imply $x = y$, for all $x, y, z \in X$

We can define a binary relation $\leq$ on X by letting $x \leq y$ if and only if $x * y = 0$. Then $(X, \leq)$ is a partially ordered set with least element "0" and $(X, *, 0)$ is a BCK-algebra if and only if, it satisfies the following: For all $x, y, z \in X$

(i) $((x * y) * (x * z)) \leq (z * y)$, **(ii)** $(x * (x * y)) \leq y$, **(iii)** $x \leq x$, **(iv)** $0 \leq x$
(v) $x \leq y$ and $y \leq x$ imply that $x = y$,

In a BCK-algebra $(X, *, 0)$, we have the following properties:

(P1) $x * 0 = x$, (P2) $x * y \leq x$, (P3) $(x * y) * z = (x * z) * y$, (P4) $(x * z) * (y * z) \leq x * y$,
(P5) $x * (x * (x * y)) = x * y$, (P6) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$,
(P7) $x * y \leq z$ implies $x * z \leq y$, for all $x, y, z \in X$.

Throughout this paper X will always mean a BCK-algebra unless otherwise specified.
A non-empty sub-set I of X is said to be sub-algebra of X if for $x, y \in I \Rightarrow x * y \in I$
A non-empty subset I of X is called an ideal of X if (I_1) $0 \in I$ (I_2) $x * y$ and $y \in I \Rightarrow x \in I$ for every $x, y \in X$, is said to be an n-fold positive implicative ideal of X if (I_1) and (I_3) there exists a fixed $n \in X$ Such that $(x * y) * z^n \in$ and $y * z^n \in I \Rightarrow x * z^n \in I$ for every $x, y, z \in X$, is said to be an n-fold BCK-ideal of X if (I_1) and (I_3) there exists a fixed $n \in X$ Such that $(x * y^{n+1}) * z \in I$ and $z \in I \Rightarrow x * y^n \in I$ for every $x, y, z \in X$. For any elements x and y of X, $x * y^n$ denotes $(.......((x * y) * y) * ....) * y$ in which 'y' occurs n-times.

An interval-valued intuitionistic fuzzy set (i-v IFS, shortly) "$\tilde{A}$" over X is an object having the form $\tilde{A} = \{(x, \tilde{\mu}_A, \tilde{\lambda}_A) : x \in X\}$, where $\tilde{\mu}_A(x) : X \to D[0,1]$ and $\tilde{\lambda}_A(x) : X \to D[0,1]$, the intervals $\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(x)$ denotes the intervals of the degree of membership and the degree of the non-membership of the element x to the set $\tilde{A}$, where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\tilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)]$ for all $x \in X$ with the condition $[0,0] \leq \tilde{\mu}_A(x) + \tilde{\lambda}_A(x) \leq [1,1]$ for all $x \in X$. For the sake of simplicity, we use the symbol $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$.

Definition. 2.2. An i-v IFS $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy ideal of X, if it satisfies

(i-v IF1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$
(i-v IF2) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\}$
(i-v IF3) $\tilde{\lambda}_A(x) \leq \min\{\tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y)\}$ for all $x, y \in X$.

Theorem.2.3. An intuitionistic fuzzy sub-algebra $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy ideal of X if and only if for $x, y, z \in X, x * y \leq z \Rightarrow \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}$ and $\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z)\}$.

Theorem 2.4. Let $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval valued intuitionistic fuzzy ideal of X.
If $x \leq y$ in X, then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$, $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)$, that is, $\tilde{\mu}_A$ is order-reversing and $\tilde{\lambda}_A$ is order-preserving.

3 Interval-valued intuitionistic fuzzy n-fold BCK- ideals of BCK-algebras :

Definition 3.1. An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X , if it satisfies

(i-vBCKI^{n}1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ and there exists a fixed $n \in N$ such that

(i-vBCKI^n 2) $\tilde{\mu}_A(x * y^n) \geq \min\{\tilde{\mu}_A((x * y^{n+1}) * z), \tilde{\mu}_A(z)\}$

(i-vBCKI^n 3) $\tilde{\lambda}_A(x * y^n) \leq \max\{\tilde{\lambda}_A((x * y^{n+1}) * z), \tilde{\lambda}_A(z)\}$ for all $x, y, z \in X$.

Theorem 3.2

Every interval-valued intuitionistic fuzzy n-fold BCK- ideal of X is an interval-valued intuitionistic fuzzy deal of X.

Proof:

Put $y = 0$ in (i-vBCKI^{n}2) and (i-vBCKI^{n}3) we get the proof of the result.

The following example shows that the converse of theorem 3.2 may not be true.

Example 3.3

Let $X = N \cup \{0\}$, where N is the set of natural numbers , in which the operation" $*$ " is defined by $x * y = \max\{0, \ x - y\}$ for all $x, y \in X$ .Then X is a BCK-algebra.

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X given by

$\tilde{\mu}_A(0) = [0.7, 0.8] > [0.3, 0.4] = \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) = [0.1, 0.2] < [0.3, 0.4] = \tilde{\lambda}_A(x)$,

for all $x(\neq 0) \in X$ .Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy ideal of X but it is not an interval-valued intuitionistic fuzzy 2-fold BCK-ideal of X, because

$\tilde{\mu}_A(5 * 2^2) = \tilde{\mu}_A(1) = [0.3, 0.4] < [0.7, 0.8] = \tilde{\mu}_A(0) = \min\{\tilde{\mu}_A((5 * 2^3)) * 0), \tilde{\mu}_A(0)\}$ and

$\lambda_A(5 * 2^2) = \tilde{\lambda}_A(1) = [0.4, 0.45] > [0.1, 0.2] = \tilde{\lambda}_A(0) = \max\{\tilde{\lambda}_A((5 * 2^3)) * 0), \tilde{\lambda}_A(0)\}$

We give a condition for an interval-valued intuitionistic fuzzy ideal to be an intuitionistic fuzzy n-fold BCK-ideal

Proposition 3.4

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X if and only if it satisfies the following inequalities $\tilde{\mu}_A(x * y^n) \geq \tilde{\mu}_A(x * y^{n+1})$ and $\tilde{\lambda}_A(x * y^n) \leq \tilde{\lambda}_A(x * y^{n+1})$ for all $x, y \in X$.

Proof:

Put $z = 0$ in (i-v BCKI^{n}2) and (i-v BCKI^{n}3), we get

$\tilde{\mu}_A(x * y^n) \geq \min\{\tilde{\mu}_A((x * y^{n+1}) * 0), \tilde{\mu}_A(0)\} = \min\{\tilde{\mu}_A(x * y^{n+1}), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(x * y^{n+1}\}$

and $\tilde{\lambda}_A(x * y^n) \leq \max\{\tilde{\lambda}_A((x * y^{n+1}) * 0), \tilde{\lambda}_A(0\} = \max\{\tilde{\lambda}_A(x * y^{n+1}), \tilde{\lambda}_A(0)\} = \tilde{\lambda}_A(x * y^{n+1}\}$

Therefore, $\tilde{\mu}_A(x * y^n) \geq \tilde{\mu}_A(x * y^{n+1})$ and $\tilde{\lambda}_A(x * y^n) \leq \tilde{\lambda}_A(x * y^{n+1})$ for all $x, y \in X$.

Conversely, Since (i-vIF2) and (i-vIF3) we get

$$\tilde{\mu}_A(x * y^n) \geq \tilde{\mu}_A(x * y^{n+1}) \geq \min\{\tilde{\mu}_A((x * y^{n+1}) * z), \tilde{\mu}_A(z)\} \text{ and}$$

$$\tilde{\lambda}_A(x*y^n) \le \tilde{\lambda}_A(x*y^{n+1}) \le \max\{\tilde{\lambda}_A((x*y^{n+1})*z), \tilde{\lambda}_A(z)\} \text{ for all } x,y,z \in X.$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X

Corollary 3. 5

Every interval-valued intuitionistic fuzzy n-fold BCK-ideal $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ of X satisfies the inequalities $\tilde{\mu}_A(x*y^n) \ge \tilde{\mu}_A(x*y^{n+k})$ and $\tilde{\lambda}_A(x*y^n) \le \tilde{\lambda}_A(x*y^{n+k})$ for all $x,y \in X$. and $k \in N$

Proof:

Using the proposition 3.4, the proof is straightforward by Induction

Theorem 3.6.

An interval-valued intuitionistic fuzzy set $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X if and only if the non-empty upper $\tilde{s}$-level cut $U(\tilde{\mu}_A; \tilde{s})$ and the non-empty lower $\tilde{t}$-level cut $L(\tilde{\lambda}_A; \tilde{t})$ are n-fold BCK-ideals of X for any $\tilde{s}, \tilde{t} \in D[0,1]$.

Proof: The proof is straight forward.

4 Interval-valued intuitionistic fuzzy n-fold positive implicative ideals of BCK-algebras :

Definition 4.1

An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal (i-vIFPI^n-ideal) of X if it satisfies

(i-vIFPI^{n}1) $\tilde{\mu}_A(0) \ge \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(0) \le \tilde{\lambda}_A(x)$ and there exists a fixed $n \in N$ such that

(i-vIFPI^{n}2) $\tilde{\mu}_A(x*z^n) \ge \min\{\tilde{\mu}_A((x*y)*z^n), \tilde{\mu}_A(y*z^n)\}$

(i-vIFPI^{n}3) $\tilde{\lambda}_A(x*z^n) \le \max\{\tilde{\lambda}_A((x*y)*z^n), \tilde{\lambda}_A(y*z^n)\}$ for all $x,y,z \in X$.

Example 4.2

Let $X = \{0,1,2\}$ be a BCK-algebra with the following Cayley table

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define an i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in X by

$$\tilde{\mu}_A(0) = [0.75, 0.8], \tilde{\mu}_A(1) = [0.6, 0.7], \tilde{\mu}_A(2) = [0.2, 0.3] \text{ and}$$

$$\tilde{\lambda}_A(0) = [0.1, 0.15], \tilde{\lambda}_A(1) = [0.2, 0.25], \tilde{\mu}_A(2) = [0.4, 0.45]$$

Then $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionist fuzzy n-fold positive implicative ideal of X for all $n \in N$

Theorem 4.3

Every interval-valued intuitionistic fuzzy n-fold positive implicative ideal X must be interval-valued intuitionistic fuzzy idea of X.

Proof:

Put $z = 0$ in (i-vIFPI^{n}2) and (i-vIFPI^{n}3), we get the proof of the result.

The following example shows that the converse of the Theorem 4.3 not true in general.

Example 4.4

Let $X = N \cup \{0\}$, where N is the set of natural numbers , in which the operation $*$ is defined by $x * y = \max\{0, \; x - y\}$ for all $x, y \in X$. Then X is a BCK-algebra.

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X given by

$$\tilde{\mu}_A(0) = [0.7, 0.8] > [0.3,0.4] = \tilde{\mu}_A(x) \text{ and } \tilde{\lambda}_A(0) = [0.1,0.2] < [0.4, 0.45] = \tilde{\lambda}_A(x),$$

for all $x(\neq 0) \in X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy ideal of X but $\tilde{A}$ is not an interval-valued intuitionistic fuzzy 2-fold Positive implicative-ideal of X, because

$$\mu_A(13 * 5^2) = \mu_A(3) = [0.3,0.4] < [0.7, 0.8] = \tilde{\mu}_A(0)$$

$$= \min\{\tilde{\mu}_A((13 * 3) * 5^2)), \; \tilde{\mu}_A(3 * 5^2)\}$$

$$\tilde{\lambda}_A(13 * 5^2) = \tilde{\lambda}_A(3) = [0.4,0.45] > [0.1,0.2] = \lambda_A(0)$$

$$= \max\{\lambda_A((13 * 3) * 5^2)), \; \lambda_A(3 * 5^2)\}$$

Theorem 4.5

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it satisfies the inequalities $\tilde{\mu}_A((x * z^n) * (y * z^n)) \geq \tilde{\mu}_A((x * y) * z^n)$ and $\tilde{\lambda}_A((x * z^n) * (y * z^n)) \leq \tilde{\lambda}_A((x * y) * z^n)$ for all $x, y, z \in X$.

Proof:

Let $x, y, z \in X$ and $a = x * (y * z^n)$ and

$b = x * y$. Since, for all $x, y, z \in X$, $((x*(y*z^n))*(x*y))*z^n) \leq (y*(y*z^n))*z^n$, By BCK-1 and Theorem 2.4, we have $\tilde{\mu}_A(((x * (y * z^n)) * (x * y)) * z^n)) \geq \tilde{\mu}_A((y * (y * z^n)) * z^n)$ and

$$\tilde{\lambda}_A(((x * (y * z^n)) * (x * y)) * z^n)) \leq \tilde{\lambda}_A((y * (y * z^n)) * z^n)$$

Then $\tilde{\mu}_A((a * b) * z^n) = \tilde{\mu}_A(((x * (y * z^n)) * (x * y)) * z^n) \; \geq \tilde{\mu}_A((y * (y * z^n)) * z^n)$

$$= \tilde{\mu}_A((y * z^n) * (y * z^n)) \qquad \text{[By P3]}$$

$$= \tilde{\mu}_A(0). \qquad \text{[By BCK-3]}$$

and so $\tilde{\mu}_A((a * b) * z^n) = \tilde{\mu}_A(0)$.

And $\tilde{\lambda}_A((a * b) * z^n) = \tilde{\lambda}_A(((x * (y * z^n)) * (x * y)) * z^n) \; \leq \tilde{\lambda}_A((y * (y * z^n)) * z^n))$

$$= \tilde{\lambda}_A((y * z^n) * (y * z^n)) \qquad \text{[By P3]}$$

$$= \tilde{\lambda}_A(0). \qquad \text{[By BCK-3]}$$

and so $\tilde{\lambda}_A((a*b)*z^n)=\tilde{\lambda}_A(0)$.

Using (P3), (i-vIFPI^{n}2) and (i-vIFPI^{n}3) we obtain

$$\begin{aligned}\tilde{\mu}_A((x*z^n)*(y*z^n)) &= \tilde{\mu}_A((x*(y*z^n))*z^n)=\tilde{\mu}_A(a*z^n)\\ &\geq \min\{\tilde{\mu}_A((a*b)*z^n),\tilde{\mu}_A(b*z^n)\}\\ &= \min\{\tilde{\mu}_A(0),\tilde{\mu}_A(b*z^n)\}\\ &= \tilde{\mu}_A(b*z^n)=\tilde{\mu}_A((x*y)*z^n) \text{ and}\end{aligned}$$

$$\begin{aligned}\tilde{\lambda}_A((x*z^n)*(y*z^n)) &= \tilde{\lambda}_A((x*(y*z^n))*z^n)=\tilde{\lambda}_A(a*z^n)\\ &\leq \max\{\tilde{\lambda}_A((a*b)*z^n),\tilde{\lambda}_A(b*z^n)\}\\ &= \max\{\tilde{\lambda}_A(0),\tilde{\lambda}_A(b*z^n)\}\\ &= \tilde{\lambda}_A(b*z^n)=\tilde{\lambda}_A((x*y)*z^n)\end{aligned}$$

Thus $\tilde{\mu}_A((x*z^n)*(y*z^n))\geq\tilde{\mu}_A((x*y)*z^n)$ and

$\tilde{\lambda}_A((x*z^n)*(y*z^n))\leq\tilde{\lambda}_A((x*y)*z^n)$ for all $x,y\in X$.

Covertly , For any $x,y,z\in X$. Using (i-vIF-2) and (i-vIF-3), we obtain

$\tilde{\mu}_A(x*z^n)\geq\min\{\tilde{\mu}_A((x*z^n)*(y*z^n)),\tilde{\mu}_A(y*z^n)\}\geq\min\{\tilde{\mu}_A((x*y)*z^n),\tilde{\mu}_A(y*z^n)\}$ and

$\tilde{\lambda}_A(x*z^n)\leq\max\{\tilde{\lambda}_A((x*z^n)*(y*z^n)),\tilde{\lambda}_A(y*z^n)\}\leq\max\{\tilde{\lambda}_A((x*y)*z^n),\tilde{\lambda}_A(y*z^n)\}$ for all $x,y,z\in X$. Thus $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionist fuzzy n-fold positive implicative ideal of X.

Proposition 4.6

Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X then it satisfies the inequalities

$\tilde{\mu}_A(x*y^n)\geq\tilde{\mu}_A((x*y)*y^n)$ and $\tilde{\lambda}_A(x*y^n)\leq\tilde{\lambda}_A((x*y)*y^n)$ for all $x,y\in X$.

Proof:

Put $z=y$ in (i-vIFPI^{n}2) and (i-vIFPI^{n}3), we get the proof of the result.

Proposition 4.7

Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionist fuzzy set of X. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it is an intuitionistic fuzzy n-fold BCK-ideal of X.

Proof:

Putting $z=y$ in (i-vIFPI^{n}2) and (i-vIFPI^{n}3), we get

$$\begin{aligned}\tilde{\mu}_A(x*y^n)\geq\min\{\tilde{\mu}_A((x*y)*y^n),\tilde{\mu}_A(y*y^n)\} &= \min\{\tilde{\mu}_A(x*y^{n+1}),\tilde{\mu}_A(0)\}\}\\ &= \tilde{\mu}_A(x*y^{n+1}) \text{ and}\end{aligned}$$

$$\tilde{\lambda}_A(x*y^n) \le \max\{\tilde{\lambda}_A((x*y)*y^n), \tilde{\lambda}_A(y*y^n) = \max\{\tilde{\mu}_A(x*y^{n+1}), \tilde{\lambda}_A(0)\}\}$$
$$= \tilde{\lambda}_A(x*y^{n+1}).$$

Therefore, $\tilde{\mu}_A(x*y^n) \ge \tilde{\mu}_A(x*y^{n+1})$ and $\tilde{\lambda}_A(x*y^n) \le \tilde{\lambda}_A(x*y^{n+1})$ for all $x, y \in X$.

By proposition 3.4, $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X.
Conversely, It follows from (P3) and (P4) that

$$\tilde{\mu}_A((x*z^{2n})*(y*z^n)) = \tilde{\mu}_A(((x*z^n)*z^n)*(y*z^n)) = \tilde{\mu}_A(((x*z^n)*(y*z^n))*z^n)$$
$$\ge \tilde{\mu}_A((x*y)*z^n).$$

$$\tilde{\lambda}_A((x*z^{2n})*(y*z^n)) = \tilde{\lambda}_A(((x*z^n)*z^n)*(y*z^n)) = \tilde{\lambda}_A(((x*z^n)*(y*z^n))*z^n)$$
$$\le \tilde{\lambda}_A((x*y)*z^n).$$

Using Corollary 3.5, (i-vIF2) and (i-vIF3) we get

$$\tilde{\mu}_A(x*z^n) \ge \tilde{\mu}_A(x*z^{2n}) \ge \min\{\tilde{\mu}_A((x*z^{2n})*(y*z^n)), \tilde{\mu}_A(y*z^n)\}$$
$$\ge \min\{\tilde{\mu}_A((x*y)*z^n), \tilde{\mu}_A(y*z^n)\}$$
$$\tilde{\lambda}_A(x*z^n) \le \tilde{\lambda}_A(x*z^{2n}) \le \max\{\tilde{\lambda}_A((x*z^{2n})*(y*z^n)), \tilde{\lambda}_A(y*z^n)\}$$
$$\le \max\{\tilde{\lambda}_A((x*y)*z^n), \tilde{\lambda}_A(y*z^n)\}.$$

Thus $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X.

Theorem 4.9

Let $\tilde{A} = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IF set of X, Then the following conditions are equivalent.

(i) $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X.

(ii) The non-empty sets $\bigcup(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\lambda}_A; [t_1, t_2])$ are interval valued n-fold positive implicative ideals of X, for all $[s_1, s_2], [t_1, t_2] \in D[0,1]$

Proof:

The proof is straight forward.

Theorem 4.10

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X, then the following Conditions are equivalent:

(i) $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal.

(ii) $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of X.

(iii) $\tilde{\mu}_A(x*y^n) \ge \tilde{\mu}_A(x*y^{n+1})$ and $\tilde{\lambda}_A(x*y^n) \le \tilde{\lambda}_A(x*y^{n+1})$ for all $x, y \in X$.

(iv) $\tilde{\mu}_A((x*z^n)*(y*z^n)) \ge \tilde{\mu}_A((x*y)*z^n)$ and $\tilde{\lambda}_A((x*z^n)*(y*z^n)) \le \tilde{\lambda}_A((x*y)*z^n)$ for all $x, y, z \in X$.

(v) $U(\tilde{\mu}_A; [s_1, s_2])$ and $L(\tilde{\lambda}_A; [t_1, t_2])$ are n-fold positive implicative ideals of X for all $[s_1, s_2], [t_1, t_2] \in D[0,1]$.

Proof:
The proof is follows from the Proposition 4.7.Proposition 3.4, Theorem 4.5 and Theorem 4.9

Theorem 4.11

If $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X then

(i) for $x,y,a,b\in X$, $((x*y)*y^n)*a\le b$ imply $\mu_A(x*y^n)\ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\}$ and

$\tilde{\lambda}_A(x*y^n)\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}$

(ii) for $x,y,z,a,b\in X$, $((x*y)*z^n)*a\le b$ imply $\tilde{\mu}_A((x*z^n)*(y*z^n))\ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\}$

and $\tilde{\lambda}_A((x*z^n)*(y*z^n))\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}$

Proof:

(i) Let $x,y,z\in X$ be such that $((x*y)*y^n)*a\le b$ .Using Theorem 2.3,

We have $\tilde{\mu}_A((x*y)*y^n)\ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\}$ and $\tilde{\lambda}_A((x*y)*y^n)\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}$.

Put $z=y$ in (i-vIFPI^{n}2) and (i-vIFPI^{n}3) we get

$$\tilde{\mu}_A(x*y^n)\ge \min\{\tilde{\mu}_A((x*y)*y^n),\tilde{\mu}_A(y*y^n)\} = \min\{\tilde{\mu}_A((x*y)*y^n),\tilde{\mu}_A(0)\}$$
$$=\tilde{\mu}_A((x*y)*y^n) \ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\} \text{ and}$$
$$\tilde{\lambda}_A(x*y^n)\le \max\{\tilde{\lambda}_A((x*y)*y^n),\tilde{\lambda}_A(y*y^n)\} = \max\{\tilde{\lambda}_A((x*y)*y^n),\tilde{\lambda}_A(0)\}$$
$$=\tilde{\lambda}_A((x*y)*y^n)\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}.$$

Therefore, $\tilde{\mu}_A(x*y^n)\ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\}$ and $\tilde{\lambda}_A(x*y^n)\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}$

(ii) Let $x,y,z\in X$ be such that $((x*y)*z^n)*a\le b$. It follows from Theorems 4.5, Theorems 4.6 and By (i).

We obtain $\tilde{\mu}_A((x*z^n)*(y*z^n))\ge \tilde{\mu}_A((x*y)*z^n)\ge \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(b)\}$ and

$\tilde{\lambda}_A((x*z^n)*(y*z^n))\le \tilde{\lambda}_A((x*y)*z^n)\le \max\{\tilde{\lambda}_A(a),\tilde{\lambda}_A(b)\}$

This completes the proof.

Theorem 4.16.

(Extension property for an interval-valued intuitionistic fuzzy n-fold positive implicative ideal). Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ and $\tilde{B}=(\tilde{\mu}_B,\tilde{\lambda}_B)$ be an interval-valued intuitionistic fuzzy ideals of X such that $\tilde{A}(0)=\tilde{B}(0)$ and $\tilde{A}\subseteq\tilde{B}$, that is, $\tilde{\mu}_A(0)=\tilde{\mu}_B(0)$, $\tilde{\lambda}_A(0)=\tilde{\lambda}_B(0)$ and $\tilde{\mu}_A(x)\le\tilde{\mu}_B(x)$, $\tilde{\lambda}_A(x)\ge\tilde{\lambda}_B(x)$, for all $x\in X$. If $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X then so is $\tilde{B}$.

Proof:

It is sufficient to show that $B=(X,\mu_B,\lambda_B)$ satieties the inequalities $\tilde{\mu}_B(x*y^n)\ge\tilde{\mu}_B(x*y^{n+1})$ and $\tilde{\lambda}_B(x*y^n)\le\tilde{\lambda}_B(x*y^{n+1})$ for all $x,y\in X$.

Let $x, y \in X$ .Using (BCK-3), (P3) and Proposition 3.4, we get

$$\tilde{\mu}_B(0) = \tilde{\mu}_A(0) = \tilde{\mu}_A((x*(x*y^{n+1}))*y^{n+1}) \le \tilde{\mu}_A((x*(x*y^{n+1}))*y^n)$$
$$= \tilde{\mu}_A((x*y^n)*(x*y^{n+1}))$$
$$\le \tilde{\mu}_B((x*y^n)*(x*y^{n+1}) \text{ and } \tilde{\lambda}_B(0) = \tilde{\lambda}_A(0) = \tilde{\lambda}_A((x*(x*y^{n+1}))*y^{n+1})$$
$$\ge \tilde{\lambda}_A((x*(x*y^{n+1}))*y^n)$$
$$= \tilde{\lambda}_A((x*y^n)*(x*y^{n+1})).$$
$$\ge \tilde{\lambda}_B((x*y^n)*(x*y^{n+1}).$$

Therefore, $\tilde{\mu}_B(0) \le \tilde{\mu}_B((x*y^n)*(x*y^{n+1})$ and $\tilde{\lambda}_B(0) \ge \tilde{\lambda}_B((x*y^n)*(x*y^{n+1})$ for all $x, y \in X$. It follows from (i-vIF1), (i-vIF2) and (i-vIF3) that

$$\tilde{\mu}_B(x*y^n) \ge \min\{\tilde{\mu}_B((x*y^n)*(x*y^{n+1})), \tilde{\mu}_B(x*y^{n+1})\}$$
$$\ge \min\{\tilde{\mu}_B(0), \tilde{\mu}_B(x*y^{n+1})\} = \tilde{\mu}_B(x*y^{n+1})$$

And $\tilde{\lambda}_B(x*y^n) \le \max\{\tilde{\lambda}_B((x*y^n)*(x*y^{n+1})), \tilde{\lambda}_B(x*y^{n+1})\}$

$$\le \max\{\tilde{\lambda}_B(0), \lambda_B(x*y^{n+1})\} = \tilde{\lambda}_B(x*y^{n+1}).$$

Therefore, $\tilde{\mu}_B(x*y^n) \ge \tilde{\mu}_B(x*y^{n+1})$ and $\tilde{\lambda}_B(x*y^n) \le \tilde{\lambda}_B(x*y^{n+1})$ for all $x, y \in X$.

Thus $\tilde{B} = (\tilde{\mu}_B, \tilde{\lambda}_B)$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X.

5. Conclusion :

Every interval-valued intuitionistic fuzzy n-fold BCK- ideal and interval-valued intuitionistic fuzzy n-fold positive implicatice BCK- ideal of X is an interval-valued intuitionistic fuzzy deal of X. An interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X satisfies the inequalities

$$\tilde{\mu}_A((x*z^n)*(y*z^n)) \ge \tilde{\mu}_A((x*y)*z^n) \text{ and}$$
$$\tilde{\lambda}_A((x*z^n)*(y*z^n)) \le \tilde{\lambda}_A((x*y)*z^n) \text{ for all } x, y, z \in X.$$

Interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it is an intuitionistic fuzzy n-fold BCK-ideal of X. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideals are proved.

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