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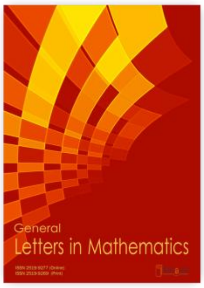

Hyers-Ulam-Rassias Stability Criteria of Nonlinear Differential Equations of Lane-Emden Type

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Abstract In this paper we establish Hyers-Ulam-Rassias stability and Hyers-Ulam Criteria for second order non-linear ordinary differential equations of Lane-Emden type; moreover two examples of such equations are considered.

Keywords: *Hyers-Ulam-Rassias Stability, Nonlinear Differential Equations, Lane-Emden Type.*
MSC: 34Cxx , 34K20, 37C75.

1 Introduction

Equations of the Lane-Emden type arise in mathematical physics and astrophysics. These Equations describe the temperature variations of a spherical gas cloud under the mutual attraction of its molecules and subject to the law of classical thermodynamics, [17].
The objective of this article is to investigate the Hyers-Ulam-Rassias Stability for the Lane-Emden type equation

$$u''(t) + \left(\frac{n-1}{t}\right)u' + g(t)f(u) = h(t) \tag{1.1}$$

and the nonlinear differential equation of second order

$$u''(t) + \left(\frac{n-1}{t}\right)u' + g(t)f(u) = 0 \tag{12}$$

with the initial conditions.

$$u(t_0) = u_0, u'(t_0) = u_1 \tag{1.3}$$

Moreover in this paper we consider the Hyers-Ulam-Rassias Stability for the Emden-Fowler equation

$$u''(t) + \left(\frac{2}{t}\right)u' + g(t)f(u) = 0 \tag{1.4}$$

Here n is a real parameter such that $n > 2$, $t \in [t_0, T]$, $0 \le t_0 < T < \infty$,
$h(t) : [0,\infty) \to R, g(t) : [0,\infty) \to R$ are continuous. Suppose that there is $L > 0$ such that

$$|f(u(t)) - f(v(t))| \le L|u - v| \tag{1.5}$$

In 1940, Ulam [27] posed the stability problem of functional equations. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [5] gave a partial solution to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians see [2,6-8,13, 18-20, 23]. More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left| F(t, y(t), y'(t), ..., y^{(n)}(t)) \right| \le \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$| y(t) - y_0(t) | \le K(\varepsilon)$$

and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0.$

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza see [16, 17]. Thereafter, Alsina and Ger [1] have studied the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers ([11, 28]) by using the method of integral factors. The results given in [9, 24, 25] have been generalized by Popa and Rus [21, 22] for the linear differential equations of nth order with constant coefficients. Gordji *et al.* [4] get sufficient conditions for Hyers-Ulam stability of the first order and the second order nonlinear partial differential equations. Lungu and Craciun [12] established results on the Ulam-Hyers stability and the generalized Ulam-Hyers-Rassias stability of nonlinear hyperbolic partial differential equations.

In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order see [3, 10, 14, 26].

2 Preliminaries

We introduce some definitions as follows:

Definition 2.1 We say that equation (1.1) (or (1.2) with $h(t) \equiv 0$) has the Hyers-Ulam-Rassias (HUR) stability with respect to φ if there exists a positive constant $k > 0$ with the following property: For each $u(t) \in C^{(2)}([t_0, \infty), (-\infty, \infty))$, if

$$\left| u''(t) + \left(\frac{n-1}{t} \right) u' + g(t) f(u) - h(t) \right| \le \varphi(t), \tag{2.1}$$

then there exists some $z_0(t)$ of the equation (1.4) such that

$$| u(t) - z_0(t) | \le k\varphi(t) \tag{2.2}$$

Definition 2.2 We say that equation (1.4) has the Hyers-Ulam-Rassias (HUR) stability with respect to φ if there exists a positive constant $k > 0$ with the following property: For each $u(t) \in C^{(2)}([t_0, \infty), (-\infty, \infty))$, if

$$\left| u''(t) + \left(\frac{2}{t} \right) u' + g(t) f(u) \right| \le \varphi(t), \tag{2.3}$$

then there exists some $z_0(t)$ of the equation (1.4) such that

$$| u(t) - z_0(t) | \leq k\varphi(t) \tag{2.4}$$

Lemma 2.1 (Gronwall's Inequality) Let u(t) and v(t) be nonnegative continuous functions on some interval $0 < t_0 \leq t \leq t_0 + a$. Also, let the function $f(t)$ be positive, continuous, and monotonically nondecreasing on $[t_0, t_0 + a]$ and satisfy the inequality

$$u(t) \leq f(t) + \int_{t_0}^{t} u(s)v(s)ds$$

then, there holds the inequality

$$u(t) \leq f(t)\exp\left(\int_{t_0}^{t} u(s)ds\right), \text{ for } t_0 \leq t \leq t_0 + a$$

3 Main results on Hyers-Ulam-Rassias Stability

Theorem 3.1 Suppose that $u(t) \in C^{(2)}([t_0, T], (-\infty, \infty))$ satisfies the inequality (2.1) with initial conditions (1.3). Let $\varphi(t) : [t_0, \infty) \to (0, \infty)$ be a continuous function such that

$$\int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\varphi(r)drds \leq C\varphi(t), \quad \forall t \geq t_0. \tag{3.1}$$

If (1.5) holds, then the solution of (1.1) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $u(t)$ be an approximate solution of the initial value problem (1.1),(1.3). We will show that there exists a function $z_0(t) \in C^{(2)}([t_0, \infty), (-\infty, \infty))$ satisfying (1.1) and (1.3) such that

$$| u(t) - z_0(t) | \leq k\varphi(t)$$

The inequality (2.1) implies that

$$-\varphi(t) \leq u''(t) + \left(\frac{n-1}{t}\right)u' + g(t)f(u) - h(t) \leq \varphi(t), \tag{3.2}$$

Multiplying (3.2) by t^{n-1}, we get

$$-t^{n-1}\varphi(t) \leq \left(t^{n-1}u'(t)\right)' + t^{n-1}g(t)f(u) - t^{n-1}h(t) \leq t^{n-1}\varphi(t), \tag{3.3}$$

Integrating (3.3) from t_0 to t, we have

$$-\int_{t_0}^{t} s^{n-1}\varphi(s)ds \leq t^{n-1}u'(t) - {t_0}^{n-1}u_1 + \int_{t_0}^{t} s^{n-1}g(s)f(u(s))ds - \int_{t_0}^{t} s^{n-1}h(s)ds$$

$$\leq \int_{t_0}^{t} s^{n-1}\varphi(s)ds,$$

Or, equivalently we get

$$-t^{1-n}\int_{t_0}^{t} s^{n-1}\varphi(s)ds \le u'(t) - t^{1-n}{t_0}^{n-1}u_1 + t^{1-n}\int_{t_0}^{t} s^{n-1}g(s)f(u(s))ds$$

$$- t^{1-n}\int_{t_0}^{t} s^{n-1}h(s)ds \le t^{1-n}\int_{t_0}^{t} s^{n-1}\varphi(s)ds,$$

Again integrating the last inequality from t_0 to t , we obtain

$$\Bigg| u(t) - u_0 - t^{2-n}{t_0}^{n-1}u_1 + t_0u_1 + \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$

$$-\int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds\Bigg| \le \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}\varphi(r)drds$$

It is clear to see that

$$z_0(t) = u_0 - t_0u_1 + t^{2-n}{t_0}^{n-1}u_1 - \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(z_0(r))drds$$

$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds$$

satisfies the Eq. (1.1) with the initial condition (1.3).
Now consider the difference

$$| u(t) - z_0(t) | \le \Bigg| u(t) - u_0 - t^{2-n}{t_0}^{n-1}u_1 + t_0u_1 + \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$

$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds\Bigg| + \Big| \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$

$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(z_0(r))drds \Big|$$

$$\le C\varphi(t) + \int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\,|g(r)||f(u(r)) - f(z_0(r))|\,drds$$

$$\le C\varphi(t) + LM\int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\,|u(r) - z_0(r)|\,drds, \tag{3.4}$$

where $M = \max_{t_0 \le t \le T} |g(t)|$.

Applying Gronwall's inequality to (3.4) we infer that

$$
\begin{aligned}
| u(t) - z_0(t) | &\le C\varphi(t)\exp\left(LM\int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\, drds \right) \\
&\le C\varphi(t)\exp\left(\frac{LM}{n}\int_{t_0}^{t} s^{1-n}(s^n - t_0{}^n)\ ds \right) \\
&\le C\varphi(t)\exp\left(\frac{LM}{n}\int_{t_0}^{t} s^{1-n}s^n\ \ ds \right) = C\varphi(t)\exp\left(\frac{L}{n}\int_{t_0}^{t} s\ ds \right) \\
&\le C\varphi(t)\exp\left(\frac{LM\left(t^2 - t_0{}^2\right)}{2n} \right) \le C\varphi(t)\exp\left(\frac{LMT^2}{2n} \right) \equiv k\varphi(t)
\end{aligned}
$$

Consequently we have

$$| u(t) - z_0(t) | \le k\varphi(t)$$

which completes the proof of Theorem 3.1.

Corollary 3.1 Replacing $\varphi(t)$ by ε in the inequality (3.2) we can get Hyers-Ulam stability for Eq. (1.1) in the interval $0 < t_0 \le t \le T$, i.e. if $u(t) \in C^{(2)}[t_0, T]$ satisfies (1.5) and

$$\left| u''(t) + \left(\frac{n-1}{t}\right)u' + g(t)f(u) - h(t) \right| \quad \le \varepsilon,$$

with the initial condition $u(t_0) = u_0, u'(t_0) = u_1$, then there exists some $z_0(t)$ of the equation (1.1) such that

$$| u(t) - z_0(t) | \le k\varepsilon .$$

The proof of Corollary 2.1 is quite similar to the proof of Theorem 2.1 and will therefore be omitted.

In the following theorem we establish the HUR stability for (1.1) in the interval $0 < t_0 \le t \le T \le \infty$

Theorem 3.2 Suppose that $u(t) \in C^{(2)}\left([t_0,\infty),(-\infty,\infty)\right)$ satisfies the inequality (2.1) with initial conditions (1.3). Let $\varphi(t) : [t_0,\infty) \to (0,\infty)$ be a continuous function such that

$$\int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}\varphi(r)drds \le C\varphi(t), \quad \forall t \ge t_0. \tag{3.4}$$

If there exists a number $a_0 > 0$ such that

$$|g(t)| \le \frac{a_0}{t^2}, \text{ for sufficiently large t.} \tag{3.5}$$

then the solution of (1.1) is stable in the sense of HUR as $t \to \infty$.

Proof. Let $\varepsilon > 0$ and $u(t)$ be an approximate solution of the initial value problem (1.1),(1.3). We wish to show that there exists a function

$$u(t) \in C^{(2)}\left([t_0,\infty),(-\infty,\infty)\right)$$

satisfying (1.1) and (1.3) such that

$$| u(t) - z_0(t) | \le k\varphi(t)$$

Then it follows from the inequality (2.1) that

$$-\varphi(t) \le u''(t) + \left(\frac{n-1}{t}\right)u' + g(t)f(u) - h(t) \le \varphi(t), \tag{3.6}$$

Multiplying (3.6) by t^{n-1} gives

$$-t^{n-1}\varphi(t) \le \left(t^{n-1}u'(t)\right)' + t^{n-1}g(t)f(u) - t^{n-1}h(t) \le t^{n-1}\varphi(t), \tag{3.7}$$

Integrating (3.7) with respect to t , we have

$$-\int_{t_0}^{t} s^{n-1}\varphi(s)ds \le t^{n-1}u'(t) - {t_0}^{n-1}u_1 + \int_{t_0}^{t} s^{n-1}g(s)f(u(s))ds - \int_{t_0}^{t} s^{n-1}h(s)ds$$
$$\le \int_{t_0}^{t} s^{n-1}\varphi(s)ds,$$

Dividing the last inequality by t^{n-1} we have

$$-t^{1-n}\int_{t_0}^{t} s^{n-1}\varphi(s)ds \le u'(t) - t^{1-n}{t_0}^{n-1}u_1 + t^{1-n}\int_{t_0}^{t} s^{n-1}g(s)f(u(s))ds$$
$$- t^{1-n}\int_{t_0}^{t} s^{n-1}h(s)ds \le t^{1-n}\int_{t_0}^{t} s^{n-1}\varphi(s)ds,$$

If we integrate again the last inequality from t_0 to t , we obtain

$$\Bigg| u(t) - u_0 - t^{2-n}{t_0}^{n-1}u_1 + t_0u_1 + \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$
$$-\int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds\Bigg| \le \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}\varphi(r)drds$$

One can easily show that

$$z_0(t) = u_0 - t_0u_1 + t^{2-n}{t_0}^{n-1}u_1 - \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(z_0(r))drds$$
$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds$$

satisfies the Eq. (1.1) with the initial condition (1.3).
Now, let us estimate the difference

$$| u(t) - z_0(t) | \le \Bigg| u(t) - u_0 - t^{2-n}{t_0}^{n-1}u_1 + t_0u_1 + \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$
$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}h(r)drds\Bigg| + | \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(u(r))drds$$
$$- \int_{t_0}^{t} s^{1-n}\int_{t_0}^{s} r^{n-1}g(r)f(z_0(r))drds |$$
$$\le C\varphi(t) + \int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\,|g(r)||f(u(r)) - f(z_0(r))|\,drds$$

By applying the mean value theorem to the integral in the last inequality, and from (1.5) we have

$$| u(t) - z_0(t) | \le C\varphi(t) + L|g(t_*)|\int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\,|u(r) - z_0(r)|\,drds$$

Using Gronwall's inequality we get

$$| u(t) - z_0(t) | \leq C\varphi(t)\exp\left(L|g(t_*)| \int_{t_0}^{t} s^{1-n} \int_{t_0}^{s} r^{n-1}\, drds \right)$$

$$\leq C\varphi(t)\exp\left(\frac{L|g(t_*)|}{n} \int_{t_0}^{t} s^{1-n}(s^n - t_0{}^n)\; ds \right)$$

Now, since $|g(t)| \leq \frac{a_0}{t^2}$, for sufficiently large t it follows from the last inequality that

$$| u(t) - z_0(t) | \leq C\varphi(t)\exp\left(\frac{L}{nt^2} \int_{t_0}^{t} s^{1-n}s^n \; ds \right) = C\varphi(t)\exp\left(\frac{L}{nt^2} \int_{t_0}^{t} s\; ds \right)$$

$$\leq C\varphi(t)\exp\left(\frac{L\left(t^2 - t_0{}^2\right)}{2nt^2} \right) \leq C\varphi(t)\exp\left(\frac{L}{2n}\right) \equiv k\varphi(t)$$

which means that (2.2) holds true for all $t > 0$.

Remark 3.1 If we let in Theorems 2.1 and 2.2 $h(t) \equiv 0$ then we get the HUR stability for the equation (1.2).

Theorem 3.3 Suppose that $u(t) \in C^{(2)}([t_0, T], (-\infty, \infty))$ satisfies the inequality (2.3) with initial conditions (1.3). Let $\varphi(t) : [t_0, \infty) \to (0, \infty)$ be a continuous function such that

$$\int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2\varphi(r)drds \leq C\varphi(t), \quad \forall t \geq t_0. \tag{3.8}$$

If (1.5) holds, then the solution of (1.4) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $u(t)$ be an approximate solution of the initial value problem (1.4),(1.3). We will show that there exists a function $z_0(t) \in C^{(2)}([t_0, T], (-\infty, \infty))$ satisfying (1.4) and (1.3) such that

$$| u(t) - z_0(t) | \leq k\varphi(t)$$

From the inequality (2.3) we have

$$-\varphi(t) \leq u''(t) + \left(\frac{2}{t}\right)u' + g(t)f(u) \leq \varphi(t), \tag{3.9}$$

Multiplying (3.9) by t^2, we get

$$-t^2\varphi(t) \leq \left(t^2u'(t)\right)' + t^2g(t)f(u) \leq t^2\varphi(t), \tag{3.10}$$

Integrating (3.10) from t_0 to t, we obtain

$$-\int_{t_0}^{t} s^2\varphi(s)ds \leq t^2u'(t) - t_0{}^2u_1 + \int_{t_0}^{t} s^2g(s)f(u(s))ds \leq \int_{t_0}^{t} s^2\varphi(s)ds,$$

Or, equivalently

$$-t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds \leq u'(t) - t^{-2}t_0{}^{n-1}u_1 + t^{-2}\int_{t_0}^{t} s^2g(s)f(u(s))ds$$

$$\leq t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds,$$

Again integrating the last inequality from t_0 to t, we obtain

$$\left| u(t) - u_0 - t^{-1}{t_0}^2 u_1 + t_0 u_1 + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(u(r)) dr ds \right| \leq \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \varphi(r) dr ds$$

Clearly we see that

$$z_0(t) = u_0 - t_0 u_1 + t^{-1}{t_0}^2 u_1 - \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(z_0(r)) dr ds$$

satisfies the Eq. (1.4) with the initial condition (1.3).
Now consider the difference

$$\begin{aligned} | u(t) - z_0(t) | &\leq \left| u(t) - u_0 - t^{-1}{t_0}^2 u_1 + t_0 u_1 + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(u(r)) dr ds \right| \\ &+ \left| \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(u(r)) dr ds - \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(z_0(r)) dr ds \right| \\ &\leq C\varphi(t) + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \left| g(r) \right| \left| f(u(r)) - f(z_0(r)) \right| dr ds \\ &\leq C\varphi(t) + LM \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \left| u(r) - z_0(r) \right| dr ds, \end{aligned}$$

where $M = \max_{t_0 \leq t \leq T} |g(t)|$.

Applying Gronwall's inequality to above inequality we obtain that

$$\begin{aligned} | u(t) - z_0(t) | &\leq C\varphi(t) \exp\left(LM \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \, dr ds \right) \\ &\leq C\varphi(t) \exp\left(\frac{LM}{3} \int_{t_0}^{t} s^{-2} (s^3 - {t_0}^3) \, ds \right) \\ &\leq C\varphi(t) \exp\left(\frac{LM}{3} \int_{t_0}^{t} s^{-2} s^3 \, ds \right) = C\varphi(t) \exp\left(\frac{LM}{3} \int_{t_0}^{t} s \, ds \right) \\ &\leq C\varphi(t) \exp\left(\frac{LM \left(t^2 - {t_0}^2 \right)}{6} \right) \leq C\varphi(t) \exp\left(\frac{LMT^2}{6} \right) \equiv k\varphi(t) \end{aligned}$$

Thus

$$| u(t) - z_0(t) | \leq k\varphi(t)$$

which completes the proof.

Corollary 3.2 Replacing $\varphi(t)$ by ε in the inequality (3.9) we can get Hyers-Ulam stability for Eq. (1.4) in the interval $0 < t_0 \leq t \leq T$, i.e. if $u(t) \in C^{(2)}[t_0, T]$ satisfies (1.5) and

$$\left| u''(t) + \left(\frac{2}{t} \right) u' + g(t) f(u) \right| \leq \varepsilon,$$

with the initial condition $u(t_0)=u_0, u'(t_0)=u_1$, then there exists some $z_0(t)$ of the equation (1.4) such that

$$|\, u(t)-z_0(t)\,|\le k\varepsilon .$$

Since the proof of Corollary 3.2 is very similar to the proof of Theorem 3.3 then it can be omitted.

In the following theorem we establish the HUR stability for (1.4),(1.3) in the interval $0\le t_0 < t < T \le \infty$

Theorem 3.4 Suppose that $u(t)\in C^{(2)}([t_0,\infty),(-\infty,\infty))$ satisfies the inequality (2.3) with initial conditions (1.3). Let $\varphi(t):[t_0,\infty)\to(0,\infty)$ be a continuous function such that

$$\int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2\varphi(r)drds \le C\varphi(t), \quad \forall t\ge t_0. \tag{3.11}$$

If there exists a number $a_0 > 0$ such that

$$|g(t)|\le \frac{a_0}{t^2}, \text{ for sufficiently large t.} \tag{3.12}$$

then the solution of (1.4) is stable in the sense of HUR as $t\to\infty$.

Proof. Let $\varepsilon>0$ and $u(t)$ be an approximate solution of the initial value problem (1.4),(1.3). We will show that there exists a function $z_0(t)\in C^{(2)}([t_0,\infty),(-\infty,\infty))$ satisfying (1.4) and (1.3) such that

$$|\, u(t)-z_0(t)\,|\le k\varphi(t)$$

From (2.3) the inequality it follows

$$-\varphi(t)\le u''(t)+\left(\frac{2}{t}\right)u'+g(t)f(u)\le\varphi(t), \tag{3.13}$$

The inequality (3.13) can be written as

$$-t^2\varphi(t)\le\left(t^2u'(t)\right)'+t^2g(t)f(u)\le t^2\varphi(t), \tag{3.14}$$

Integrating (3.14) successively twice from t_0 to t , one has

$$-t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds \le u'(t)-t^{-2}{t_0}^2u_1+t^{-2}\int_{t_0}^{t} s^2g(s)f(u(s))ds$$
$$\le t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds,$$

and

$$\left|u(t)-u_0-t^{-1}{t_0}^2u_1+t_0u_1+\int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2g(r)f(u(r))drds\right|$$
$$\le \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2\varphi(r)drds$$

It can be easily shown that

$$z_0(t)=u_0-t_0u_1+t^{-1}{t_0}^2u_1-\int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2g(r)f(z_0(r))drds$$

satisfies the Eq. (1.4) with the initial condition (1.3).
Now consider the difference

$$| u(t) - z_0(t) | \le \left| u(t) - u_0 - t^{-1}{t_0}^2 u_1 + t_0 u_1 + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(u(r)) dr ds \right|$$
$$+ \left| \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(u(r)) dr ds - \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 g(r) f(z_0(r)) dr ds \right|$$
$$\le C\varphi(t) + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 |g(r)| |f(u(r)) - f(z_0(r))| dr ds$$

By applying the mean value theorem to the integral in the last inequality and from (1.5), we have

$$| u(t) - z_0(t) | \le C\varphi(t) + L|g(t_*)| \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 |u(r) - z_0(r)| dr ds$$

Using Gronwall's inequality yields

$$| u(t) - z_0(t) | \le C\varphi(t) \exp\left(L|g(t_*)| \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \, dr ds \right)$$
$$\le C\varphi(t) \exp\left(\frac{L|g(t_*)|}{n} \int_{t_0}^{t} s^{-2} (s^2 - {t_0}^2) \, ds \right)$$

In view of condition (3.12) it follows that for sufficiently large t, say $t \ge t_0$

$$| u(t) - z_0(t) | \le C\varphi(t) \exp\left(\frac{L}{3t^2} \int_{t_0}^{t} s^{-2} s^3 \, ds \right) = C\varphi(t) \exp\left(\frac{L}{3t^2} \int_{t_0}^{t} s \, ds \right)$$
$$\le C\varphi(t) \exp\left(\frac{L\left(t^2 - {t_0}^2\right)}{6t^2} \right) \le C\varphi(t) \exp\left(\frac{L}{6} \right) \equiv k\varphi(t)$$

which establishes the HUR stability for arbitrarily large t.

4 Special Cases of Equation (1.4)

In this section we consider HUR stability for the most typical of Lane-Emden equation

$$u''(t) + \left(\frac{2}{t}\right) u' + u^m = 0, m \ge 1 \tag{4.1}$$

and the white dwarf equation

$$u''(t) + \left(\frac{2}{t}\right) u' + [u(u+1)]^{3/2} = 0, \tag{4.2}$$

with the initial conditions

$$u(t_0) = u_0, u'(t_0) = u_1 \tag{4.3}$$

Theorem 4.1 Suppose that $u(t) \in C^{(2)}([t_0, T], (-\infty, \infty))$ satisfies the inequality (3.1) with initial conditions (4.3). If $\varphi(t) : [t_0, \infty) \to (0, \infty)$ is a continuous function such that

$$\int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \varphi(r) dr ds \le C\varphi(t), \quad \forall t \ge t_0. \tag{4.4}$$

then the solution of the equation (4.1) is stable in the sense of HUR.

Proof. Let $\varepsilon > 0$ and $u(t)$ be an approximate solution of the initial value problem (4.1),(4.3). We will show that there exists a function $z_0(t) \in C^{(2)}([t_0, T], (-\infty, \infty))$

satisfying (4.1) and (4.3) such that

$$| u(t) - z_0(t) | \leq k\varphi(t)$$

Since $u(t)$ is an approximate solution of Eq. (4.1), then we get the inequality

$$-\varphi(t) \leq u''(t) + \left(\frac{2}{t}\right)u' + u^m \leq \varphi(t), \tag{4.5}$$

Multiplying (4.4) by t^2, we get

$$-t^2\varphi(t) \leq \left(t^2u'(t)\right)' + t^2u^m \leq t^2\varphi(t), \tag{4.6}$$

Integrating (4.5) from t_0 to t, we have

$$-\int_{t_0}^{t} s^2\varphi(s)ds \leq t^2u'(t) - t_0^{\,2}u_1 + \int_{t_0}^{t} s^2u^m(s)ds \leq \int_{t_0}^{t} s^2\varphi(s)ds,$$

Or, equivalently

$$-t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds \leq u'(t) - t^{-2}t_0^{\,n-1}u_1 + t^{-2}\int_{t_0}^{t} s^2u^m(s)ds$$

$$\leq t^{-2}\int_{t_0}^{t} s^2\varphi(s)ds,$$

Again integrating the last inequality from t_0 to t, we obtain

$$\left|u(t) - u_0 - t^{-1}t_0^{\,2}u_1 + t_0u_1 + \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2u^m(r)drds\right|$$

$$\leq \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2\varphi(r)drds$$

It is clear to see that

$$z_0(t) = u_0 - t_0u_1 + t^{-1}t_0^{\,2}u_1 - \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2{z_0}^m(r)drds$$

satisfies the Eq. (4.1) with the initial condition (4.3).
Let us consider the difference

$$| u(t) - z_0(t) | \leq \left|u(t) - u_0 - t^{-1}t_0^{\,2}u_1 + t_0u_1 + \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2u^m(r)drds\right|$$

$$+\left|\int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2u^m(r)drds - \int_{t_0}^{t} s^{-2}\int_{t_0}^{s} r^2{z_0}^m(r)drds\right|$$

$$\leq C\varphi(t) + \int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2\,|g(r)||u^m(r) - {z_0}^m(r)|\,drds$$

Now, since $u(t) \in C^{(2)}\left([t_0,T],(-\infty,\infty)\right)$, then $\left|\frac{d(u^m)}{du}\right|$ is bounded and hence the function u^m satisfies Lipschitz condition in u for all $t \in [t_0,T]$ and $m \geq 1$.
Consequently

$$| u(t) - z_0(t) | \leq C\varphi(t) + L\int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \, |u(r) - z_0(r)| \, drds,$$

From the Gronwall's inequality, we obtain

$$\begin{aligned} | u(t) - z_0(t) | &\leq C\varphi(t)\exp\left(L\int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \, drds \right) \\ &\leq C\varphi(t)\exp\left(\frac{L}{3}\int_{t_0}^{t} s^{-2}(s^3 - {t_0}^3) \, ds \right) \\ &\leq C\varphi(t)\exp\left(\frac{L}{3}\int_{t_0}^{t} s^{-2}s^3 \, ds \right) = C\varphi(t)\exp\left(\frac{L}{3}\int_{t_0}^{t} s \, ds \right) \\ &\leq C\varphi(t)\exp\left(\frac{L\left(t^2 - {t_0}^2\right)}{6} \right) \leq C\varphi(t)\exp\left(\frac{LT^2}{6} \right) \end{aligned}$$

Thus

$$| u(t) - z_0(t) | \leq k\varphi(t)$$

i.e. equation (4.1) is HUR stable.

Corollary 4.1 Replacing $\varphi(t)$ by ε in the inequality (4.5) we get Hyers-Ulam stability for Eq. (4.1) in the interval $0 < t_0 \leq t \leq T$.

Theorem 4.2 Suppose that $u(t) \in C^{(2)}[[t_0, T], (-\infty, \infty)]$ satisfies the inequality (4.2) with initial conditions (4.3). If $\varphi(t) : [t_0, \infty) \to (0, \infty)$ is a continuous function such that

$$\int_{t_0}^{t} s^{-2} \int_{t_0}^{s} r^2 \varphi(r) drds \leq C\varphi(t), \quad \forall t \geq t_0. \tag{4.7}$$

then the solution of the equation (4.2) is stable in the sense of HUR.
It should be noted here that the proof of Theorem 4.2 is very similar to the proof of Theorem 4.1 and it will therefore be omitted.

5 Conclusion

In this work, the problem of the HUR Stability of solutions of nonlinear differential equations of Lane-Emden type has been investigated and solved using the direct method. To illustrate the results we provided the most two typical examples satisfying the assumptions of the proved theorems.

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