



# On the Lower Estimations for Maximal $(\pm 1)$ -Determinants

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**Abstract.** It is suggested several lower estimations for the determinants of matrices consisting of elements  $+1$  and  $-1$ , improving the known Hadamard's lower estimation.

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## 1 Introduction

Let  $L^n$  be the set of all square matrices of the order  $n$  each element of them equals  $1$  or  $-1$ . We are going to consider the sequence

$$\Delta_n = \max_{M \in L_n} \det M, \quad n = 1, 2, 3, \dots$$

Hadamard established [1] the following two-sided estimation

$$\sqrt{n!} \leq \Delta_n \leq \sqrt{n^n}. \quad (1)$$

If  $M \in L^n$  and  $\det M = \sqrt{n^n}$  then the matrix  $M$  is called a *Hadamard matrix* or an *anallagmatic pavement of Sylvester* [2]. Such a matrix exists only if  $n = 1, 2$  or  $n \equiv 0 \pmod{4}$  [2]. Sufficiency of this condition is known as Paley's hypothesis [3]. Improvement of both estimations has been in the center of attention for many researchers. Some lower bounds for maximal  $(\pm 1)$ -determinants depending on  $n \pmod{4}$  are given in [4]. (The survey of results on the problem of Hadamard is given in [5], the list of references covering the period of 1867-2005 years can be found in [6]. Among later works it can be noted [7] where Hadamard type inequality is considered for rectangular matrices and [8] where factors of Hadamard matrices are calculated for some values of  $n$ .)

Here the question about upper estimations won't be touched upon. The aim of the paper is to consider the lower ones only. Everywhere all letters denote positive integers or matrices of an appropriate order if otherwise is not emphasized.

## 2 Preparatory lemmas

**The basic Lemma.**

$$\Delta_{p+q} \geq 2^{\min\{p, q\}} \Delta_p \Delta_q. \quad (2)$$

*Proof.* Let the matrices  $A \in L^p$  and  $B \in L^q$  be chosen so that  $\det A = \Delta_p$ ,  $\det B = \Delta_q$ . For definiteness we will suppose  $p \geq q$ . Now consider the block-matrix

$$M = \begin{pmatrix} A & C \\ -D & B \end{pmatrix}$$

where the block  $C$  of the order  $p \times q$  is such that its upper  $q$  strings make the block equal to  $B$  (if  $p > q$  other strings of  $C$  are unessential here), and the block  $D$  consists of  $q$  upper strings of  $A$ . Adding first  $q$  strings of the matrix  $M$  to its strings from  $(p+1)^{th}$  to  $(p+q)^{th}$  we get

$$\det M = \det \begin{pmatrix} A & C \\ O & 2B \end{pmatrix} = 2^q \Delta_p \Delta_q.$$

By the definition of  $\Delta_n$  the last equality implies (2).

Despite simpleness of the statement the basic Lemma allows to derive several kinds of lower estimations. The first corollary is  $\Delta_{n+1} \geq 2\Delta_n$ . Thus the sequence  $\Delta_n$  grows not slower than  $2^n$ . One can wait that as much  $p$  and  $q$  are closer than the inequality (2) is sharper. Realizing this idea will be based on the special form of the induction method. For that let us introduce the following quantity that will be called an estimation function

$$\delta_n(c) = \sqrt{2} \left( \frac{n}{c} \right)^{n/2},$$

where  $c$  is some real positive constant.

**Lemma on an inductive step.** *Let a sequence  $\varepsilon_n$  possess the following properties*

$$\varepsilon_{2n} \geq 2^n \varepsilon_n^2, \quad \varepsilon_{2n+1} \geq 2^n \varepsilon_n \varepsilon_{n+1}.$$

*Then  $\varepsilon_n \geq \delta_n(c)$  implies  $\varepsilon_{2n} \geq \delta_{2n}(c)$ ,  $\varepsilon_{2n+1} \geq \delta_{2n+1}(c)$ .*

*Proof.* Indeed under the conditions of the Lemma

$$\varepsilon_{2n} \geq 2^n [\delta_n(c)]^2 = 2(2n/c)^n \geq \delta_{2n}(c)$$

for an even subscript. Further for an odd subscript

$$\varepsilon_{2n+1} \geq 2^n \delta_n(c) \delta_{n+1}(c) = 2^{n+1} \left( \frac{n}{c} \right)^{n/2} \left( \frac{n+1}{c} \right)^{(n+1)/2}.$$

Thus it is enough to verify that the last expression is greater  $\delta_{2n+1}(c)$ . This affirmation follows from the inequality

$$\left( 1 + \frac{1}{n} \right)^{n+1} > \left( 1 + \frac{1}{2n} \right)^{2n+1}$$

that is well known.

**Lemma on an inductive base.** *If there exists some  $\hat{n}$  that  $\varepsilon_n \geq \delta_n(c)$  for  $n = \hat{n}, \hat{n} + 1, \dots, 2\hat{n} - 1$  then  $\varepsilon_n \geq \delta_n(c)$  for all  $n, n \geq \hat{n}$ .*

For example consider  $\hat{n} = 3$  i.e. let us suppose the inequality  $\varepsilon_n \geq \delta_n(c)$  is true for  $n = 3, 4, 5$ . Then by the Lemma on an inductive step that is true for  $n = 6, 8, 10$  and  $n = 7, 9, 11$  as well. Therefore the inequality holds for  $n = 12, 14, \dots, 22$  and  $n = 13, 15, \dots, 23$  and etc. (Obviously, the proof can be finished by usual induction.)

### 3 Lower estimations for maximal $(\pm)$ -determinants

Now putting consequently  $p = q = n$  and  $p = n+1, q = n$  in (2) one can see that for  $\varepsilon_n = \Delta_n, c = 2$  the propositions of *Lemma on an inductive step* holds. Besides the propositions of the Lemma on an inductive base also hold if  $\hat{n} = 1$ . Thus we get the first lower estimation.

**Theorem 3.1.**

$$\Delta_n \geq \sqrt{2} \left( \frac{n}{2} \right)^{n/2}, \quad n = 1, 2, \dots \quad (3)$$

The estimation (3) has a compact form but it is far from the upper bound in (1) (cm. [6]). Moreover there shouldn't be a proper estimation by a regularly growing sequence because a) if  $n$  equals to a power of 2 then the right estimation (1) is exact and b) Paley's conjecture presupposing that  $\Delta_n = \sqrt{n^n}$  if and only if  $n$  divides 4 which is verified till  $n = 664$  [4].

The next estimation takes into account this factor to a certain extent. Let  $\delta_n = d_0 + d_1 + \dots + d_{n-1}$  where  $d_k$  denotes the number of units in the binary representation of  $k$ . (One can note that the sequence  $d_k$  has fractality like property: if  $D_n = \{d_0, d_1, \dots, d_{2^n-1}\}$  then  $D_{n+1} = D_n \cup (D_n + 1)$ .)

**Theorem 3.2.**

$$\Delta_n \geq 2^{\delta_n}, \quad n = 1, 2, \dots \quad (4)$$

*Proof.* By the definition of the numbers  $d_k$  one has  $d_{2n} = d_n$ ,  $d_{2n+1} = d_n + 1$ . Therefore

$$\delta_{2n} = 2\delta_n + n, \quad \delta_{2n+1} = \delta_n + \delta_{n+1} + n. \quad (5)$$

Thus *Lemma on an inductive step* can be applied (as the base of induction can be taken  $\hat{n} = 1$ ).

Now let us compare the estimations (3) and (4).

**Theorem 3.3.**

$$2^{\delta_n} \geq \sqrt{2} \left( \frac{n}{\sqrt{2}} \right)^{n/2}, \quad n = 6, 7, \dots \quad (6)$$

*Proof.* (6) is true for  $n = 6, 7, 8, 9, 10, 11$  that can be verified by simple calculations. Then after *the Lemma on an inductive base* can be applied putting  $\varepsilon_n = 2^{\delta_n}$  and  $c = \sqrt{2}$ .

Thus the estimation (4) is better than (3). Nevertheless it needs to be improved. It is easy to see that  $\Delta_n$  coincides with  $2^{\delta_n}$  if  $n = 2^m$  for some  $m$ ,  $m > 1$  but  $2^{\delta_n} < \Delta_n$  if  $n$  greater 4 and doesn't equal to a power of 2. The exact values of  $\Delta_n$  for small  $n$ 's are given in [4]. The beginning of the sequence looks

$$\begin{aligned} \Delta_1 &= 1, & \Delta_2 &= 2, & \Delta_3 &= 2^2, & \Delta_4 &= 2^4, \\ \Delta_5 &= 2^4 \cdot 3, & \Delta_6 &= 2^5 \cdot 5, & \Delta_7 &= 2^6 \cdot 3^2, & \Delta_8 &= 2^{12}. \end{aligned}$$

It is notable that every lower estimation (the more so exact value of  $\Delta_n$ ) generates a sequence of estimations. For example basing on the value  $\Delta_5$  and using *the basic Lemma* one can write down

$$\Delta_{10} \geq 2^5 \cdot (\Delta_5)^2 = 2^{12} \cdot 3^2, \quad \Delta_{20} \geq 2^{10} \cdot (\Delta_{10})^2 = 2^{36} \cdot 3^4$$

and etc. This sequence consists of 'needlekind waves'.

If the values

$$\Delta_4 = 2^4, \quad \Delta_5 = 2^4 \cdot 3$$

are taken as 'a perturbing wave' then one gets the sequence of estimations in the form of 'lakunary waves' as

$$\mathbf{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \dots}$$

Here bold numbers single out indexes for those  $\Delta_n$  which will be estimated via the previous 'wave', using *the basic Lemma* in the form

$$\Delta_{2n} \geq 2^n (\Delta_n) \quad \Delta_{2n+1} \geq 2^n \Delta_n \Delta_{n+1}. \quad (7)$$

For example  $\Delta_9 \geq 2^4 \Delta_4 \Delta_5 \geq 2^{12} \cdot 3$ .

It will be obtained the estimation sequence without 'lakunas' if one takes as the initiating 'wave' the values

$$\Delta_4 = 2^4, \quad \Delta_5 = 2^4 \cdot 3, \quad \Delta_6 = 2^5 \cdot 5, \quad \Delta_7 = 2^6 \cdot 3^2, \quad \Delta_8 = 2^{12} \quad (8)$$

because 'the consequent waves' are running across intervals of integers  $n, n + 1, \dots, 2n$  for  $n = 4, 8, 16, \dots$ . For example the next 'wave' will be

$$\begin{aligned} \Delta_8 &= 2^{12}, & \Delta_9 &\geq 2^4 \Delta_4 \Delta_5 \geq 2^{12} \cdot 3, & \Delta_{10} &\geq 2^5 \cdot 3^2, & \Delta_{11} &\geq 2^{14} \cdot 3 \cdot 5, \\ \Delta_{12} &\geq 2^{16} \cdot 5^2, & \Delta_{13} &\geq 2^{17} \cdot 3^2 \cdot 5, & \Delta_{14} &\geq 2^{19} \cdot 3^4, & \Delta_{15} &\geq 2^{25} \cdot 3^2, & \Delta_{16} &= 2^{32}. \end{aligned}$$

Let us introduce the sequences  $\alpha_n, \beta_n, \gamma_n$  defined by the following way

$$\begin{aligned} \alpha_{2n+l} &= \alpha_{n+[l/2]} + \alpha_{n+[(l+1)/2]} + n + \left\lfloor \frac{l}{2} \right\rfloor, \\ \beta_{2n+l} &= \beta_{n+[l/2]} + \beta_{n+[(l+1)/2]}, \\ \gamma_{2n+l} &= \gamma_{n+[l/2]} + \gamma_{n+[(l+1)/2]}. \end{aligned} \quad (9)$$

( $l = 0, 1, 2, \dots, 2n$ ; square brackets mean integral part of a fraction) with the beginning values (taken from (8))

$$\begin{aligned} \alpha_4 &= 4, & \alpha_5 &= 4, & \alpha_6 &= 4, & \alpha_7 &= 6, & \alpha_8 &= 12, \\ \beta_4 &= 0, & \beta_5 &= 1, & \beta_6 &= 0, & \beta_7 &= 2, & \beta_8 &= 0, \\ \gamma_4 &= 0, & \gamma_5 &= 0, & \gamma_6 &= 1, & \gamma_7 &= 0, & \gamma_8 &= 0. \end{aligned}$$

Obviously the sequence  $\alpha_n$  is increasing.

**Theorem 3.4.** *If  $n \geq 4$  then*

$$\Delta_n \geq 2^{\alpha_n} 3^{\beta_n} 5^{\gamma_n}. \quad (10)$$

*Proof* follows of (7) and (9) immediately.

The estimation (9) is better than (4). Indeed using the inequalities  $\log 3 > 1.5$  and  $\log 5 > 2$  one can show the following statement.

**Theorem 3.5.** *If  $n \geq 4$  then*

$$2^{\alpha_n} 3^{\beta_n} 5^{\gamma_n} > \left(\frac{3}{\sqrt{8}}\right)^{\beta_n} \left(\frac{5}{4}\right)^{\gamma_n} 2^{\delta_n}.$$

## 4 Conclusion

More proper lower estimations of the kind (10) can be obtained basing on further exact values of  $\Delta_n$  for example those from  $n = 8$  till  $n = 16$  and etc. In the other hand if some new prime say 7 occurs a divisor of Hadamard matrices  $M$  of an order  $n$  for some  $n$  than (10) can be improved adding to the right side the multiplier  $7^{\epsilon_n}$  where  $\epsilon_n$  will satisfy the same recurrent relations as  $\beta_n$  and  $\gamma_n$ . Maybe this would be the most that could be brought out from (2).

In the end an open question can be formulated now: *for which  $n$ 's does the equality a)  $\Delta_n = 2^{\alpha_n} 3^{\beta_n}$ ; b)  $\Delta_n = 2^{\alpha_n} 3^{\beta_n} 5^{\gamma_n}$  hold?*

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