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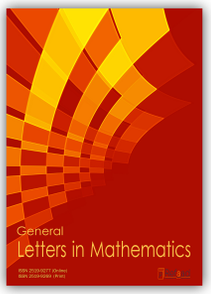

Oscillatory Behavior of Higher-Order Delay Differential Equations

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Abstract. This paper is concerned with asymptotic and oscillatory properties of the nonlinear higher-order differential equation with delay argument. Some examples are given to illustrate our main results .

Keywords: *Oscillation, higher-order, delay differential equations.*

1 Introduction and preliminaries

In this work, we study the oscillation of higher-order delay differential equation

$$\left[b\left(t\right)\left(x^{(n-1)}\left(t\right)\right)^{\gamma}\right]' + q\left(t\right)f\left(x\left(\tau\left(t\right)\right)\right) = 0 \qquad t \geq t_0. \tag{1}$$

We assume that the following assumptions hold:

(H_1) γ is a quotient of odd positive integers;

(H_2) $b \in C^1[t_0\ \infty)$, $b'\left(t\right) \geq 0$, $b\left(t\right) > 0$, $q, \tau \in C[t_0\ \infty)$, $f \in C\left(\mathbb{R}\ \mathbb{R}\right)$, and $-f\left(-xy\right) \geq f\left(xy\right) \geq f\left(x\right)f\left(y\right)$, for $xy > 0$, $q > 0$, $\tau\left(t\right) \leq t$, $\lim\limits_{t\to\infty}\tau\left(t\right) = \infty$.

$(H3)$ there exist constants $k > 0$ such that $f\left(u\right)/u^{\gamma} \geq k$, for $u \neq 0$.

By a solution of Eq. (1.1) we mean a function $x \in C^{(n-1)}[T_x\ \infty)$, $T_x \geq t_0$, which has the property $b\left(t\right)\left(x^{(n-1)}\left(t\right)\right)^{\gamma} \in C^1[T_x\ \infty)$, and satisfies Eq. (1.1) on $[T_x\ \infty)$.We consider only those solutions x of Eq.(1.1) which satisfy $\sup\{|x\left(t\right)| : t \geq T\} > 0$, for all $T > T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x\ \infty)$, and otherwise, it is called to be nonoscillatory. (1.1) is said to be oscillatory if all its solutions are oscillatory. In what follows, we present some related results that served as a motivation for the contents of this paper.

E.M.Elabbasy, et al.[9] study the a symptotic properties and oscillation of the nth-order delay differential equation

$$\left[r\left(t\right)\left(x^{(n-1)}\left(t\right)\right)^{\gamma}\right]' + \sum_{i=1}^{m} q_i\left(t\right)g\left(x\left(\tau_i\left(t\right)\right)\right) = 0 \qquad t \geq t_0.$$

C.Zhang, et al.[20] and Zhang, et al.[21] consider the oscillatory properties of the higher-order differential equation

$$\left[b\left(t\right)\left(x^{(n-1)}\left(t\right)\right)^{\gamma}\right]' + q\left(t\right)x^{\gamma}\left(\tau\left(t\right)\right) = 0 \qquad t \geq t_0. \tag{2}$$

under the conditions

$$\int_{t_\circ}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}\left(t\right)}dt = \infty,$$

and

$$\int_{t_\circ}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}(t)} d\dot{t} < \infty. \tag{3}$$

Our aim in the present paper is to employ the Riccatti technique to establish some conditions for the oscillation of all solutions of (1.1). Some examples are presented to illustrate our main results.

2 Main Results

In this section, we shall establish some oscillation criteria for (1.1). We begin with the following lemma.

Lemma 2.1. *Let $z \in (C^n[t_0\ \infty], \mathbb{R}^+)$ and assume that $z^{(n)}$ is of fixed sign and not identically zero on a subray of $[t_0 \infty]$. If, moreover, $z(t) > 0$, $z^{(n-1)}(t) z^{(n)}(t) \leq 0$ and $\lim_{t\to\infty} z(t) \neq 0$, then, for every $\lambda \in (0,\ 1)$, there exists $t_\lambda \geq t_\circ$ such that*

$$z(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| z^{(n-1)}(t) \right|, \text{ for } t \in [t_\lambda,\ \infty).$$

We are now ready to state and prove the main results. For convenience, we denote

$$\begin{aligned} \pi(s) &:= \int_{t_\circ}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}(s)} ds,\ \delta'(t) := \max\{0,\ \delta'(t)\}, \\ \sigma(t) &= \frac{\int_t^\infty (\mu - t)^{(n-4)} \pi(\mu)\, d\mu}{(n-4)!}. \end{aligned}$$

Theorem 2.2. *Let $n \geq 4$. Assume that (1.3) holds. Further, assume that for some constant $\lambda \in (0,\ 1)$, the differential equation*

$$y'(t) + q(t) f\left(\frac{\lambda}{(n-1)! b^{1/\gamma}(\tau(t))} \tau^{n-1}(t) \right) f\left(y^{1/\gamma}(\tau(t)) \right) = 0, \tag{4}$$

is oscillatory. If

$$\lim_{t\to\infty} \sup_{t_\circ}^{\infty} \left[kq(t) \left(\frac{\tau^3(t)}{t^3} \right)^\gamma \delta(s) - \frac{(\delta'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1} (\delta(s))^{2\gamma+1} \sigma^\gamma(s)} \right] ds = \infty, \tag{5}$$

holds. Then every solution of (1.1) is oscillatory.

Proof. Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we can assume that $x(t) > 0$. It follows from (1.1) that there exist two possible cases:

case1 : $x^{(n-1)}(t) > 0,\ x^{(n)}(t) < 0,\ \left(b \left(x^{(n-1)} \right)^\gamma \right)'(t) \leq 0.$

case 2 : $x^{(n-2)}(t) > 0,\ x^{(n-1)}(t) < 0,\ \left(b \left(x^{(n-1)} \right)^\gamma \right)'(t) \leq 0.$

for $t > t_1$, t_1 is large enough.

Assume that case(1) holds. From Lemma 2.1, we have

$$x(\tau(t)) \geq \frac{\lambda \tau^{n-1}(t)}{(n-1)! b^{1/\gamma}(t)} \left(b^{1/\gamma}(t) x^{(n-1)} \tau(t) \right), \tag{6}$$

for every $\lambda \in (0,\ 1)$. Using (2.3) in Eq. (1.1), we see that

$y(t) = b(t) \left[x^{(n-1)}(t) \right]^\gamma$ is a positive solution of the differential inequality

$$y'(t) + q(t) f\left(\frac{\lambda}{(n-1)! b^{1/\gamma}(\tau(t))} \tau^{n-1}(t) \right) f\left(y^{1/\gamma}(\tau(t)) \right) \leq 0,$$

By Theorem 1 in [18], we conclude that the corresponding equation (1.1) also has a positive solution. This contradiction.

Assume that case (2) holds. Noting that $b(t) \left(x^{(n-1)}(t) \right)^\gamma$ is decreasing, we obtain

$$\begin{aligned} b^{\frac{1}{\gamma}}(s) x^{(n-1)}(s) &\leq b^{\frac{1}{\gamma}}(t) x^{(n-1)}(t),\ s \geq t \geq t_1, \\ x^{(n-1)}(s) &\leq b^{\frac{1}{\gamma}}(t) x^{(n-1)}(t) b^{\frac{-1}{\gamma}}(s). \end{aligned}$$

Integrating again from t to v, we get

$$x^{(n-2)}(t) - x^{(n-2)}(v) \geq -b^{\frac{1}{\gamma}}(t)\, x^{(n-1)}(t) \int_t^v b^{\frac{-1}{\gamma}}(s)\, ds.$$

Letting $v \to \infty$; we obtain

$$x^{(n-2)}(t) \geq -b^{\frac{1}{\gamma}}(t)\, x^{\left(n-1\right)}(t)\, \pi(t). \tag{7}$$

Integrating from t to ∞, we get

$$-x^{\left(n-3\right)}(t) \geq -b^{\frac{1}{\gamma}}(t)\, x^{\left(n-1\right)}(t) \int_t^\infty \pi(s)\, ds. \tag{8}$$

Similarly, integrating the above inequality from t to ∞ a total of (n - 4) times, we find

$$-x'(t) \geq \frac{-b^{\frac{1}{\gamma}}(t)\, x^{(n-1)}(t)}{(n-4)!} \int_t^\infty (\mu - t)^{(n-4)}\, \pi(\mu)\, d\mu. \tag{9}$$

Define the function $\omega(t)$ by

$$\omega(t) := \delta(t)\, \frac{b(t)\left(x^{(n-1)}\right)^\gamma(t)}{(x(t))^\gamma}. \tag{10}$$

Then $\omega(t) < 0$ for $t \geq t_1$ and

$$\begin{aligned} \omega'(t) &= \delta'(t)\, \frac{b(t)\left(x^{(n-1)}\right)^\gamma(t)}{(x(t))^\gamma} + \delta(t)\, \frac{\left(b\left(x^{(n-1)}\right)^\gamma\right)'(t)}{(x(t))^\gamma} \\ &\quad -\gamma\delta(t)\, \frac{(x)'(t)\, b(t)\left(x^{(n-1)}\right)^\gamma(t)}{(x)^{\gamma+1}(t)}. \end{aligned} \tag{11}$$

By the Kiguradze ,we find $x(t) \geq (t/3)\, x'(t)$ and, hence

$$\frac{x(\tau_i(t))}{x(t)} \geq \frac{\tau_i^3(t)}{t^3}. \tag{12}$$

It follows from (1.1) and (2.6) , we get

$$\begin{aligned} \omega'(t) &\leq -kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma \delta(t) + \frac{\delta'(t)}{\delta(t)}\omega(t) \\ &\quad -\gamma\delta(t)\, \frac{\int_t^\infty (\mu - t)^{(n-4)}\, \pi(\mu)\, d\mu}{(n-4)!}\omega^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \tag{13}$$

$$\begin{aligned} \omega'(t) &\leq -kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma \delta(t) + \frac{\delta'(t)}{\delta(t)}\omega(t) \\ &\quad -\gamma\delta(t)\, \sigma(t)\, \omega^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \tag{14}$$

Define now

$$C := \gamma\delta(t)\, \sigma(t),\ D := \frac{\delta'(t)}{\delta(t)},\ y := \omega(t).$$

Appling the inequality

$$Dy - Cy^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{D^{\gamma+1}}{C^\gamma},\ C,\ D > 0, \tag{15}$$

we find

$$\frac{\delta'(t)}{\delta(t)}\omega(t) - \gamma\, \delta(t)\, \sigma(t)\, \omega(t)^{\frac{\gamma+1}{\gamma}} \leq \frac{(\delta'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(t))^{2\gamma+1}\sigma^\gamma(t)}. \tag{16}$$

Hence, we obtain

$$\omega'(t) \leq -kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^{\gamma}\delta(t) - \frac{(\delta'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(t))^{2\gamma+1}\sigma^{\gamma}(t)}.$$

Integrating from t_1 to t, we get

$$\int_{t_1}^{t}\left[kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^{\gamma}\delta(s) - \frac{(\delta'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(s))^{2\gamma+1}\sigma^{\gamma}(s)}\right]ds \leq \omega(t_1),$$

which contradicts (2.2).

Theorem 2.1 is proved. □

Corollary 2.3. *Let $n \geq 4$. Assume that (1.3) holds. If*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{\infty} q(s) f\left(\frac{\lambda}{(n-1)!b^{1/\gamma}(\tau(s))}\tau^{n-1}(s)\right)ds > \frac{((n-1)!)^{\gamma}}{e}, \tag{17}$$

and

$$\lim_{t\to\infty}\sup_{t_\circ}^{\infty}\left[kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^{\gamma}\delta(s) - \frac{(\delta'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(s))^{2\gamma+1}\sigma^{\gamma}(s)}\right]ds = \infty, \tag{18}$$

holds for some constant $\lambda \in (0,1)$. Then every solution of is oscillatory or tends to zero.

Corollary 2.4. *Let $n \geq 4$. Assume that (1.3) holds. and let β be the ratio of two odd positive integers with $\beta < \gamma$. If*

$$\lim_{t\to\infty}\sup\int_{\tau(t)}^{\infty} q(s)\frac{\left(\tau^{n-1}(s)\right)^{\beta}}{b^{\frac{\beta}{\gamma}}(\tau(s))} > 0, \tag{19}$$

then the differential equation

$$\left[r(t)\left(x^{(n-1)}(t)\right)^{\gamma}\right]' + q(t)x^{\beta}(\tau(t)) = 0 \qquad t \geq t_\circ, \tag{20}$$

is oscillatory.

3 Examples

We give the following example to illustrate our main results.

Consider a differential equation

$$\left(t^6(x'''(t))^3\right)' + \frac{\beta+t^2+t}{t}x^3\left(\frac{t}{2}\right) = 0,\ t \geq 1, \tag{21}$$

where $\beta > 0$ is a constant. Let

$$\begin{aligned}
\gamma &= 3,\ n=4,\ b(t)=t^6>0,\ b'(t)=6t^5\geq 0,\ b\in C^1[t_\circ\ \infty),\\
q(t) &= \frac{\beta+t^2+t}{t}>0,\ q\in C[t_\circ\ \infty),\\
\tau(t) &= \frac{t}{2}\leq t,\ \lim_{t\to\infty}\frac{t}{2}=\infty,\ \tau(t)\in C[t_\circ\ \infty),
\end{aligned}$$

we see (H_1), (H_2) and(H_3) holds. Then, we find

$$\pi(s) := \int_t^{\infty}\frac{1}{b^{\frac{1}{\gamma}}(s)}ds = \int_t^{\infty}\frac{1}{(s^6)^{\frac{1}{3}}}ds = \frac{1}{t},$$

we now set $\delta(s) = 1$, It is easy to see that all conditions of Corollary 2.1 hold. Hence every solution of (3.1) is oscillatory.

Example 3.1. *Consider a differential equation*

$$\left(t^3\left(x'''(t)\right)\right)' + \frac{1}{t}\left(x^2+x\right)(\alpha t) = 0,\ t \geq 1, \tag{22}$$

where $0 < \alpha < 1$ is a constant. Let

$$\begin{aligned} \gamma &= 1,\ n=4,\ b(t)=t^3>0,\ b'(t)=9t^8 \geq 0,\ b \in C^1[t_0\ \infty), \\ q(t) &= \frac{1}{t} > 0,\ q \in C[t_0\ \infty), \\ \tau(t) &= \alpha t \leq t,\ \lim \alpha t = \infty,\ \tau(t) \in C[t_0\ \infty), \end{aligned}$$

we see (H_1), (H_2) and (H_3) holds. Then, we find

$$\pi(s) := \int_t^\infty \frac{1}{b^{\frac{1}{\gamma}}(s)} ds = \int_t^\infty \frac{1}{s^3} ds = \frac{1}{2s^2}$$

IF we now set $\delta(s) = 1$. It is easy to see that all conditions of Corollary 2.1 hold. Hence every solution of (3.2) is oscillatory. However,theresults of [21] cannot confirm this conclusion.

Example 3.2. *Consider a differential equation*

$$\left(e^t x'''(t)\right)' + 2\sqrt{10}e^{\arcsin\frac{\sqrt{10}}{10}+t}x\left(t - \arcsin\frac{\sqrt{10}}{10}\right) = 0, \tag{23}$$

$$\begin{aligned} \text{Let } \gamma &= 1,\ n=4,\ b(t)=e^t>0,\ b'(t)=e^t \geq 0,\ b \in C^1[t_0\ \infty]. \\ q(t) &= 2\sqrt{10}e^{\arcsin\frac{\sqrt{10}}{10}+t},\ q \in C[t_0\ \infty]. \\ \tau(t) &= t - \arcsin\frac{\sqrt{10}}{10} \leq t,\ \lim_{t\to\infty} t - \arcsin\frac{\sqrt{10}}{10} = \infty,\ \tau(t) \in C[t_0\ \infty], \end{aligned}$$

we see (H_1), (H_2) and (H_3) holds. Then, we get

$$\begin{aligned} \pi(s) &:= \int_t^\infty \frac{1}{b^{\frac{1}{\gamma}}(s)} ds = \int_t^\infty \frac{1}{(e^s)} ds = e^{-t}, \\ \sigma(t) &= \frac{\int_t^\infty (\mu - t)^{(n-4)} \pi(\mu)\, d\mu}{(n-4)!} = \int_t^\infty e^{-s} ds = e^{-t}. \end{aligned}$$

IF we now set $\delta(s) = 1$,we get

$$\lim_{t\to\infty} \sup_{t_\circ}^{\infty} \left[kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma \delta(s) - \frac{(\delta'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(s))^{2\gamma+1}\sigma^\gamma(s)} \right] ds = \infty.$$

Then we can easily see that all assumptions of Corollary 2.1 are satisfied. Hence (3.3) is oscillatory.

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