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Local existence and uniqueness for a fractional SIRS model with Mittag–Leffler law

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Abstract

In this paper, we study an epidemic model with Atangana-Baleanu-Caputo fractional derivative. We obtain a special solution using an iterative scheme via Laplace transformation. Uniqueness and existence of a solution using the Banach fixed point theorem are studied. A detailed analysis of the stability of the special solution is presented. Finally, our generalized model in the derivative sense is solved numerically by the Adams-Bashforth-Moulton method.

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1. Introduction

The generalization of mathematical models in epidemiology has for the purpose providing a good description closer to reality by using the concept of derivatives or more concretely the concept of derivatives with fractional order. Since the fractional order can be any positive real α , one can choose the one that better fits available data [1]. Therefore, we can adjust the model to real data for better predict the future evolution of the disease [30, 34]. Moreover, virus propagation is typically discontinuous and some classical differential models cannot describe it in a proper way. In contrast, fractional systems deal naturally with such discontinuous properties [16, 31].

The virus propagation is similar to heat transmission or moistness penetrability in a porous medium, which can be exactly modelled by fractional calculus [26, 39]. Authors in [20, 21] gave a geometrical description of fractional calculus, concluding that the fractional order can be related with the fractal dimension. The relationship between fractal dimension and fractional calculus has been reported by several different authors: See [29, 35] and references therein. The fractional complex transform [22, 25]

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is an approximate transform of a fractal space (time) to a continuous one, and it is now widely used in fractional calculus [10, 33, 36].

Several definitions were proposed in the literature. Some authors have used the Caputo fractional derivative because of some useful properties provided by this derivative [24, 6, 38, 12], especially in the analysis of the spread of diseases [28, 2, 3, 18]. However, this derivative has some limitations for instance the kernel therein has a singularity. To solve this problem, Caputo and Fabrizio has proposed a derivative with fractional order that has a kernel with no singularity. Recently the derivative was used by few researchers to solve some real world problems, see for example [27, 4]. However, many other researchers testified that the Caputo–Fabrizio operator is nothing more than a filter with a fractional regulator. They based their argument upon the fact that the kernel used in this design is local and the associate integral is the average of the given function and its integral. In many solutions of the fractional differentiation based on the power law $x^{-\sigma}$, the Mittag-Leffler function is mostly present. The Mittag–Leffler function is of course a generalization of the exponential function. In addition, it is also a non-local kernel [5, 8, 9]. To solve the failures of the Caputo–Fabrizio derivative, the fractional derivative based on the Mittag–Leffler function was introduced and used in some new problems with great success [23, 41, 40, 19]. It is important to know that where the power based on $x^{-\sigma}$ function relax then the Mittag–Leffler function raise more complex problems.

Several research papers have been published using this new concept of fractional differentiation with Mittag-Leffler function. The results obtained in [5] revealed that, the new concept is more adequate for modeling real world problems to take into account the non-locality and also to have a memory effect.

Some infectious diseases confer temporally acquired immunity. This type of disease can be modeled by the SIRS model. The total population N is divided into three compartments with $N = S + I + R$, where S is the number of susceptible, I is the number of infectious individuals and R is the number of individuals recovered [32].

The aim of this work is to show, by applying the Picard iteration method, the existence and uniqueness of the solution to the Atangana-Baleanu-Caputo fractional derivative for a SIRS epidemic model with saturated treatment and an incidence function presented in [32] and generalizing the classical bilinear incidence rate, the saturated incidence rate, the Beddington–DeAngelis functional response, introduced in [15, 11] and the Crowley–Martin functional response given in [14].

Picard iteration has more theoretical value than practical one. Finding approximate solution using this method is almost impractical for complicated function of second member of a fractional differential equation. For that, in this work we will use a numerical method called Adams-Bashforth-Moulton [7].

The paper is organized as follows. In Section 2, we give the definitions of the new fractional derivative with non-singular and non-local kernel. The existence of solution for our epidemic model via Picard-Lindelöf method is investigated in Section 3. In Section 4, we will study the stability analysis of the numerical scheme obtained by Picard’s iteration method. Finally, in Section 5, some numerical results obtained at different instances of fractional order are presented.

2. Basic properties of new fractional derivative

In this section, we present the definitions of the new fractional derivative with non-singular and non-local kernel ([5]).

Definition 2.1. Let $f \in H^1(0, T)$, $T > 0$, $0 \leq \alpha < 1$ then, the Atangana–Baleanu fractional derivative in Caputo sense is given as:

$${}^{\text{ABC}}_0 D_t^\alpha(f(t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha \left[-\alpha \frac{(t-s)^\alpha}{1-\alpha} \right] ds. \quad (2.1)$$

where the kernel E_α is the Mittag–Leffler function of one parameter and $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$ ([13]).

Definition 2.2. The fractional integral of order α of a new fractional derivative is defined as:

$${}^{\text{AB}}I_t^\alpha(f(t)) = \frac{1-\alpha}{\text{B}(\alpha)}f(t) + \frac{\alpha}{\text{B}(\alpha)\Gamma(\alpha)}\int_0^t f(u)(t-u)^{\alpha-1}du. \tag{2.2}$$

Here $\Gamma(\cdot)$ is the Euler Gamma function which is defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

Remark 2.3. When $\alpha = 0$, the initial function is obtained and when $\alpha = 1$, the ordinary integral is obtained.

3. Existence and uniqueness of solution

In this section, we extend the SIRS model to a fractional-order model, and we shall recall that the reason for the extension has been presented in the introduction. Nevertheless, it is important noting that the concept of local derivative that is used to describe the rate of change has failed to model accurately some complex real-world problems. Due to this failure, the concept of fractional differentiation based on the convolution of $x^{-\sigma}$ was introduced, and also failed in some cases due to the disc of convergence of this function. The Mittag-Leffler function, that is the more generalized version, can therefore be used in order to handle more physical problems.

We consider the following system

$$\begin{aligned} {}^{\text{ABC}}D_t^\alpha S(t) &= \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1 + k_1 S(t) + k_2 I(t) + k_3 S(t)I(t)} + \lambda R(t), \\ {}^{\text{ABC}}D_t^\alpha I(t) &= \frac{\beta S(t)I(t)}{1 + k_1 S(t) + k_2 I(t) + k_3 S(t)I(t)} - (\mu + d + r)I(t) - \frac{\xi I(t)}{1 + \gamma I(t)}, \\ {}^{\text{ABC}}D_t^\alpha R(t) &= rI(t) + \frac{\xi I(t)}{1 + \gamma I(t)} - (\mu + \lambda)R(t), \end{aligned} \tag{3.1}$$

subject to initial conditions:

$$S(0) \geq 0, \quad I(0) \geq 0, \quad R(0) \geq 0. \tag{3.2}$$

The positive constants $\Lambda, \beta, \mu, r, d, \lambda$ are the recruitment rate of the population, the infection rate, the natural death rate, the recovery rate of the infective individuals, the death rate due to disease, the rate that recovered individuals lose immunity and return to the susceptible class, respectively. While contacting with infected individuals, the susceptibles become infected at the incidence rate $\beta SI/(1 + k_1 S + k_2 I + k_3 SI)$, with k_1, k_2 and k_3 non-negative constants [32]. Through treatment, the infected individuals recover at a saturated treatment function $\xi I(t)/(1 + \gamma I(t))$, where ξ is positive and γ is nonnegative and $1/(1 + \gamma I(t))$ describes the reverse effect of the infected being delayed for treatment. When $\gamma = 0$, the saturated treatment function stays to the linear one [37].

3.1. Iterative scheme with Laplace transform

Theorem 3.1. For $\alpha \in [0, 1]$, the following time fractional ordinary differential equation

$${}^{\text{ABC}}D_t^\alpha(f(t)) = u(t), \tag{3.3}$$

has a unique solution, namely

$$f(t) = f(0) + \frac{1-\alpha}{\text{B}(\alpha)}u(t) + \frac{\alpha}{\text{B}(\alpha)\Gamma(\alpha)}\int_0^t u(p)(t-p)^{\alpha-1}dp. \tag{3.4}$$

Proof. Using the Laplace transformation on both sides of the equation (3.3):

$$L[{}_0^{ABC}D_t^\alpha(f(t))](p) = L[u(t)](p), \quad p > 0$$

and according to [8, Theorem 3], we have that

$$\frac{B(\alpha)}{1-\alpha} \frac{p^\alpha L\{f(t)\}(p) - p^{\alpha-1}f(0)}{p^\alpha + \frac{\alpha}{1-\alpha}} = L\{u(t)\}(p),$$

which is equivalent,

$$L\{f(t)\}(p) = \frac{1}{p}f(0) + \frac{1-\alpha}{B(\alpha)}L\{u(t)\}(p) + \frac{\alpha}{p^\alpha B(\alpha)}L\{u(t)\}(p).$$

Now, we use the inverse Laplace transform

$$f(t) = f(0) + \frac{1-\alpha}{B(\alpha)}u(t) + L^{-1}\left\{\frac{\alpha}{p^\alpha B(\alpha)}L\{u(t)\}(p)\right\}(t).$$

We have

$$\frac{\alpha}{p^\alpha B(\alpha)} = \frac{\alpha}{B(\alpha)}L\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}(p).$$

Let $F(p) = \frac{\alpha}{B(\alpha)}L\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}(p)$ and $G(p) = L\{u(t)\}(p)$, then applying the convolution theorem, we obtain

$$\begin{aligned} L^{-1}\left\{\frac{\alpha}{p^\alpha B(\alpha)}L\{u(t)\}(p)\right\}(t) &= L^{-1}\left\{F(p) \times G(p)\right\}(t) = \frac{\alpha}{B(\alpha)}\left(u(t) * \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)(p) \\ &= \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t u(p)(t-p)^{\alpha-1}dp. \end{aligned}$$

Hence the result. □

Using the above theorem, our system is equivalent to the following

$$\begin{aligned} S(t) - S(0) &= \frac{1-\alpha}{B(\alpha)}\left\{\Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} + \lambda R(t)\right\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-p)^{\alpha-1}\left\{\Lambda - \mu S(p) - \frac{\beta S(p)I(p)}{1+k_1S(p)+k_2I(p)+k_3S(p)I(p)} + \lambda R(p)\right\}dp, \\ I(t) - I(0) &= \frac{1-\alpha}{B(\alpha)}\left\{\frac{\beta S(t)I(t)}{1+k_1S(t)+k_2I(t)+k_3S(t)I(t)} - (\mu+d+r)I(t) - \frac{\xi I(t)}{1+\gamma I(t)}\right\} \\ &\quad - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-p)^{\alpha-1}\left\{(\mu+d+r)I(p) + \frac{\xi I(p)}{1+\gamma I(p)} - \frac{\beta S(p)I(p)}{1+k_1S(p)+k_2I(p)+k_3S(p)I(p)}\right\}dp, \\ R(t) - R(0) &= \frac{1-\alpha}{B(\alpha)}\left\{rI(t) + \frac{\xi I(t)}{1+\gamma I(t)} - (\mu+\lambda)R(t)\right\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-p)^{\alpha-1}\left\{rI(p) + \frac{\xi I(p)}{1+\gamma I(p)} - (\mu+\lambda)R(p)\right\}dp. \end{aligned} \tag{3.5}$$

The iterative scheme of the system (3.5) is given by:

$$S_0(t) = S(0); \quad I_0(t) = I(0); \quad R_0(t) = R(0). \tag{3.6}$$

$$\begin{aligned} S_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \left\{ \Lambda - \mu S_n(t) - \frac{\beta S_n(t) I_n(t)}{1+k_1 S_n(t)+k_2 I_n(t)+k_3 S_n(t) I_n(t)} + \lambda R_n(t) \right\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-p)^{\alpha-1} \left\{ \Lambda - \mu S_n(p) \right. \\ &\quad \left. - \frac{\beta S_n(p) I_n(p)}{1+k_1 S_n(p)+k_2 I_n(p)+k_3 S_n(p) I_n(p)} + \lambda R_n(p) \right\} dp, \\ I_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \left\{ \frac{\beta S_n(t) I_n(t)}{1+k_1 S_n(t)+k_2 I_n(t)+k_3 S_n(t) I_n(t)} - (\mu+d+r) I_n(t) \right. \\ &\quad \left. - \frac{\xi I_n(t)}{1+\gamma I_n(t)} \right\} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-p)^{\alpha-1} \left\{ (\mu+d+r) I_n(p) \right. \\ &\quad \left. + \frac{\xi I_n(p)}{1+\gamma I_n(p)} - \frac{\beta S_n(p) I_n(p)}{1+k_1 S_n(p)+k_2 I_n(p)+k_3 S_n(p) I_n(p)} \right\} dp, \\ R_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \left\{ r I_n(t) + \frac{\xi I_n(t)}{1+\gamma I_n(t)} - (\mu+\lambda) R_n(t) \right\} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-p)^{\alpha-1} \left\{ r I_n(p) + \frac{\xi I_n(p)}{1+\gamma I_n(p)} - (\mu+\lambda) R_n(p) \right\} dp. \end{aligned} \tag{3.7}$$

If we take the limit with greater than n, we expect to obtain the exact solution.

3.2. Existence of solution via Picard–Lindelöf method

We define the following operators:

$$\begin{aligned} f_1(t, \Omega(t)) &= \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{1+k_1 S(t)+k_2 I(t)+k_3 S(t) I(t)} + \lambda R(t), \\ f_2(t, \Omega(t)) &= \frac{\beta S(t) I(t)}{1+k_1 S(t)+k_2 I(t)+k_3 S(t) I(t)} - (\mu+d+r) I(t) - \frac{\xi I(t)}{1+\gamma I(t)}, \\ f_3(t, \Omega(t)) &= r I(t) + \frac{\xi I(t)}{1+\gamma I(t)} - (\mu+\lambda) R(t), \end{aligned} \tag{3.8}$$

and the matrix form of system (3.1) subject to conditions (3.2) :

$${}^{\Lambda BC}D_t^\alpha \Omega(t) = F(t, \Omega(t)), \quad \Omega(0) = \Omega_0, \tag{3.9}$$

where $\Omega(t) = (S(t), I(t), R(t))$, $\Omega_0 = (S_0, I_0, R_0)$
and $F(t, \Omega(t)) = (f_1(t, \Omega(t)), f_2(t, \Omega(t)), f_3(t, \Omega(t)))$.

Lemma 3.2. *The function F is Lipschitz continuous on $[0, T] \times \mathbf{B}(\Omega_0, \rho)$, with*

$$[0, T] \times \mathbf{B}(\Omega_0, \rho) = \{(t, \Omega(t)) \in [0, T] \times \mathbb{R}_+^3 / \sup_{t \in [0, T]} \|\Omega(t) - \Omega_0\|_1 \leq \rho\},$$

i.e., there exists a constant $L \in \mathbb{R}_+$, $\forall (t, \Omega_1(t)), (t, \Omega_2(t)) \in [0, T] \times \mathbf{B}(\Omega_0, \rho)$

$$\|F(t, \Omega_1(t)) - F(t, \Omega_2(t))\|_1 \leq L \|\Omega_1(t) - \Omega_2(t)\|_1, \tag{3.10}$$

with $\|\Omega(t)\|_1 = \sum_{i=1}^3 |\Omega_i(t)|$ is the Manhattan norm.

Proof. We shall prove that F satisfies the Lipschitz condition in the second argument Ω .

$$\begin{aligned}
 & \|F(t, \Omega_1(t)) - F(t, \Omega_2(t))\|_1 = |f_1(t, \Omega_1(t)) - f_1(t, \Omega_2(t))| + |f_2(t, \Omega_1(t)) \\
 & - f_2(t, \Omega_2(t))| + |f_3(t, \Omega_1) - f_3(t, \Omega_2)| \\
 & = \left| \Lambda - \mu S_1(t) - \frac{\beta S_1(t) I_1(t)}{1 + k_1 S_1(t) + k_2 I_1(t) + k_3 S_1(t) I_1(t)} + \lambda R_1(t) \right. \\
 & - \left. (\Lambda - \mu S_2(t) - \frac{\beta S_2(t) I_2(t)}{1 + k_1 S_2(t) + k_2 I_2(t) + k_3 S_2(t) I_2(t)} + \lambda R_2(t)) \right| \\
 & + \left| \frac{\beta S_1(t) I_1(t)}{1 + k_1 S_1(t) + k_2 I_1(t) + k_3 S_1(t) I_1(t)} - (\mu + d + r) I_1(t) - \frac{\xi I_1(t)}{1 + \gamma I_1(t)} \right. \\
 & - \left. \left(\frac{\beta S_2(t) I_2(t)}{1 + k_1 S_2(t) + k_2 I_2(t) + k_3 S_2(t) I_2(t)} - (\mu + d + r) I_2(t) - \frac{\xi I_2(t)}{1 + \gamma I_2(t)} \right) \right| \\
 & + \left| r I_1(t) + \frac{\xi I_1(t)}{1 + \gamma I_1(t)} - (\mu + \lambda) R_1(t) - (r I_2(t) + \frac{\xi I_2(t)}{1 + \gamma I_2(t)} - (\mu + \lambda) R_2(t)) \right|.
 \end{aligned} \tag{3.11}$$

We reduce the next two fractions to the same denominator

$$D = (1 + k_1 S_1(t) + k_2 I_1(t) + k_3 S_1(t) I_1(t)) (1 + k_1 S_2(t) + k_2 I_2(t) + k_3 S_2(t) I_2(t)).$$

We note that $D > 1$, hence:

$$\begin{aligned}
 & \left| \frac{\beta S_1(t) I_1(t)}{1 + k_1 S_1(t) + k_2 I_1(t) + k_3 S_1(t) I_1(t)} - \frac{\beta S_2(t) I_2(t)}{1 + k_1 S_2(t) + k_2 I_2(t) + k_3 S_2(t) I_2(t)} \right| \\
 & \leq \beta |S_1(t) I_1(t) - S_2(t) I_2(t)| + \beta k_1 |S_1(t)| |S_2(t)| |I_1(t) - I_2(t)| \\
 & + \beta k_2 |I_1(t)| |I_2(t)| |S_1(t) - S_2(t)| \\
 & \leq \beta |S_1(t) I_1(t) - S_2(t) I_1(t)| + \beta |S_2(t) I_1(t) - S_2(t) I_2(t)| \\
 & + \beta k_1 |S_1(t)| |S_2(t)| |I_1(t) - I_2(t)| + \beta k_2 |I_1(t)| |I_2(t)| |S_1(t) - S_2(t)| \\
 & \leq \beta |I_2(t)| |S_1(t) - S_2(t)| + \beta |S_1(t)| |I_1(t) - I_2(t)| \\
 & + \beta k_1 |S_1(t)| |S_2(t)| |I_1(t) - I_2(t)| + \beta k_2 |I_1(t)| |I_2(t)| |S_1(t) - S_2(t)|.
 \end{aligned} \tag{3.12}$$

On the same manner, we can prove that

$$\left| \frac{\xi I_1(t)}{1 + \gamma I_1(t)} - \frac{\xi I_2(t)}{1 + \gamma I_2(t)} \right| \leq \xi |I_1(t) - I_2(t)|. \tag{3.13}$$

Then

$$\begin{aligned}
 & \|F(t, \Omega_1(t)) - F(t, \Omega_2(t))\|_1 \\
 & \leq \mu |S_1(t) - S_2(t)| + \lambda |R_1(t) - R_2(t)| + \beta |I_2(t)| |S_1(t) - S_2(t)| \\
 & + \beta |S_1(t)| |I_1(t) - I_2(t)| \\
 & + \beta k_1 |S_1(t)| |S_2(t)| |I_1(t) - I_2(t)| + \beta k_2 |I_1(t)| |I_2(t)| |S_1(t) - S_2(t)| \\
 & + \beta |I_2(t)| |S_1(t) - S_2(t)| + \beta |S_1(t)| |I_1(t) - I_2(t)| \\
 & + \beta k_1 |S_1(t)| |S_2(t)| |I_1(t) - I_2(t)| + \beta k_2 |I_1(t)| |I_2(t)| |S_1(t) - S_2(t)| \\
 & + (\mu + d + r) |I_1(t) - I_2(t)| + \xi |I_1(t) - I_2(t)| \\
 & + r |I_1(t) - I_2(t)| + (\mu + \lambda) |R_1(t) - R_2(t)| + \xi |I_1(t) - I_2(t)| \\
 & \leq (\mu + 2\beta |I_2(t)| + 2\beta k_2 |I_1(t)| |I_2(t)|) |S_1(t) - S_2(t)| \\
 & + (2\beta |S_1(t)| + 2\beta k_1 |S_1(t)| |S_2(t)| + \mu + d + 2r + 2\xi) |I_1(t) - I_2(t)| \\
 & + (2\lambda + \mu) |R_1(t) - R_2(t)|
 \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\leq (\mu + 2\beta(\rho + I_0) + 2\beta k_2(\rho + I_0)^2)|S_1(t) - S_2(t)| + (2\beta(\rho + S_0) \\ &+ 2\beta k_1(\rho + S_0)^2 + \mu + d + 2r + 2\xi)|I_1(t) - I_2(t)| + (2\lambda + \mu)|R_1(t) - R_2(t)| \\ &\leq L\|\Omega_1(t) - \Omega_2(t)\|_1, \end{aligned}$$

where

$$L = \text{Max}\{(\mu + 2\beta(\rho + I_0) + 2\beta k_2(\rho + I_0)^2), (2\beta(\rho + S_0) + 2\beta k_1(\rho + S_0)^2 + \mu + d + 2r + 2\xi), (2\lambda + \mu)\}. \tag{3.15}$$

It is clear that $L > 0$, then F is Lipschitz continuous in the second argument. □

Theorem 3.3. *Let $W = \{\Omega \in (C^0[0, \delta])^3 : \Omega(0) = \Omega_0, \Omega(t) \in \mathbf{B}(\Omega_0, \rho)\}$ be the space of continuous functions which is complete with norm $\|\Omega\|_W = \sup_{t \in [0, \delta]} \|\Omega(t)\|_1$. Let $\|F(t, \Omega)\|_W \leq \bar{N}$ for $(t, \Omega) \in [0, \delta] \times \mathbf{B}(\Omega_0, \rho)$. Then the system (3.1) subject to conditions (3.2) has a unique solution on $[0, \delta]$ with $\delta > 0$ and*

$$\delta < \min\left\{T, \left(\frac{\rho B(\alpha)\Gamma(\alpha)}{\bar{N}} + \alpha\Gamma(\alpha) - \Gamma(\alpha)\right)^{\alpha-1}, \left(\frac{B(\alpha)\Gamma(\alpha)}{L} + \alpha\Gamma(\alpha) - \Gamma(\alpha)\right)^{\alpha-1}\right\}.$$

Proof. The fixed-point theorem in Banach space W can be employed here. For that, the Picard’s operator Θ is defined between the functional space W into itself, as follows:

$$\Theta : W \rightarrow W, \tag{3.16}$$

such that

$$\Theta\Omega(t) = \Omega_0 + \frac{1 - \alpha}{B(\alpha)}F(t, \Omega(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t F(p, \Omega(p))(t - p)^{\alpha-1} dp. \tag{3.17}$$

We show that Θ maps W to W . Due to the fact that there is no disease that is able to kill the whole world population, we can suppose that $\Omega(t) \in W$. We can also assume that the solution is bounded within a period of time since the number of targeted population is finite. Next, we show that $\Theta\Omega(t) \in W$.

$$\begin{aligned} \|\Theta\Omega - \Omega_0\|_W &= \sup_{t \in [0, \delta]} \left\| \frac{1 - \alpha}{B(\alpha)}F(t, \Omega(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t F(p, \Omega(p))(t - p)^{\alpha-1} dp \right\|_1 \\ &\leq \frac{1 - \alpha}{B(\alpha)} \times \sup_{t \in [0, \delta]} \|F(t, \Omega(t))\|_1 + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \\ &\quad \times \sup_{t \in [0, \delta]} \int_0^t \|F(p, \Omega(p))\|_1 (t - p)^{\alpha-1} dp \\ &\leq \frac{1 - \alpha}{B(\alpha)} \times \sup_{t \in [0, \delta]} \|F(t, \Omega(t))\|_1 + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \\ &\quad \times \int_0^\delta \sup_{p \in [0, \delta]} \|F(p, \Omega(p))\|_1 (t - p)^{\alpha-1} dp \\ &\leq \frac{(1 - \alpha)}{B(\alpha)}\bar{N} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\bar{N} \int_0^\delta (t - p)^{\alpha-1} dp \\ &\leq \frac{(1 - \alpha)}{B(\alpha)}\bar{N} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\bar{N} \int_0^\delta (t - p)^{\alpha-1} dp \\ &\leq \frac{(1 - \alpha)}{B(\alpha)}\bar{N} + \frac{\delta^\alpha}{B(\alpha)\Gamma(\alpha)}\bar{N}. \end{aligned} \tag{3.18}$$

Then, it is necessary that $\frac{(1-\alpha)}{B(\alpha)}\bar{N} + \frac{\delta^\alpha}{B(\alpha)\Gamma(\alpha)}\bar{N} \leq \rho$, this implies :

$$\delta < \left(\frac{\rho B(\alpha)\Gamma(\alpha)}{\bar{N}} + \alpha\Gamma(\alpha) - \Gamma(\alpha) \right)^{\alpha^{-1}} \tag{3.19}$$

for Θ maps W into itself.

Using the definition of the operator defined in (3.17), we deduce the following

$$\begin{aligned} \|\Theta\Omega_1 - \Theta\Omega_2\|_W &= \sup_{t \in [0, \delta]} \left\| \frac{1-\alpha}{B(\alpha)} \{F(t, \Omega_1(t)) - F(t, \Omega_2(t))\} \right. \\ &\quad \left. + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t \{F(p, \Omega_1(p)) - F(p, \Omega_2(p))\} (t-p)^{\alpha-1} dp \right\|_1 \\ &\leq \frac{1-\alpha}{B(\alpha)} \times \sup_{t \in [0, \delta]} \|F(t, \Omega_1(t)) - F(t, \Omega_2(t))\|_1 \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \times \sup_{t \in [0, \delta]} \int_0^t \|F(p, \Omega_1(p)) - F(p, \Omega_2(p))\|_1 (t-p)^{\alpha-1} dp. \end{aligned} \tag{3.20}$$

In Lemma 3.2, we have shown that F is Lipschitzian with respect to the second argument. Then,

$$\begin{aligned} \|\Theta\Omega_1 - \Theta\Omega_2\|_W &\leq \frac{(1-\alpha)L}{B(\alpha)} \times \sup_{t \in [0, \delta]} \|\Omega_1(t) - \Omega_2(t)\|_1 \\ &\quad + \frac{\alpha L}{B(\alpha)\Gamma(\alpha)} \int_0^\delta \|\Omega_1(p) - \Omega_2(p)\|_1 (t-p)^{\alpha-1} dp \\ &\leq \frac{(1-\alpha)L}{B(\alpha)} \times \sup_{t \in [0, \delta]} \|\Omega_1(t) - \Omega_2(t)\|_1 \\ &\quad + \frac{\alpha L}{B(\alpha)\Gamma(\alpha)} \int_0^\delta \sup_{p \in [0, \delta]} \|\Omega_1(p) - \Omega_2(p)\|_1 (t-p)^{\alpha-1} dp \\ &\leq \frac{(1-\alpha)L}{B(\alpha)} \|\Omega_1 - \Omega_2\|_W + \frac{\alpha L}{B(\alpha)\Gamma(\alpha)} \int_0^\delta \|\Omega_1 - \Omega_2\|_W (t-p)^{\alpha-1} dp \\ &\leq \left(\frac{(1-\alpha)L}{B(\alpha)} + \frac{\delta^\alpha L}{B(\alpha)\Gamma(\alpha)} \right) \|\Omega_1 - \Omega_2\|_W. \end{aligned} \tag{3.21}$$

Thus, the defined operator Θ is a contraction with a unique fixed point $\Omega \in W$ if $L \left(\frac{(1-\alpha)}{B(\alpha)} + \frac{\delta^\alpha}{B(\alpha)\Gamma(\alpha)} \right) < 1$ which implies that

$$\delta < \left(\frac{B(\alpha)\Gamma(\alpha)}{L} + \alpha\Gamma(\alpha) - \Gamma(\alpha) \right)^{\alpha^{-1}}. \tag{3.22}$$

This shows that the system under investigation has a unique solution. □

4. Stability analysis of Picard’s iteration method

In the following we study the stability of our iterative scheme proposed in (3.7) to show its convergence.

Definition 4.1. (see[42]) Let $(X, \|\cdot\|)$ be a Banach space and Θ a self-map of X . Let $y_{n+1} = g(\Theta, y_n)$ be particular recursive procedure. Suppose that $F_P(\Theta)$ is the fixed-point set of Θ and has at least one element and that y_n converges to a point $p \in F_P(\Theta)$. Let $\{x_n\} \subseteq X$ and define $e_n = \|x_{n+1} - g(\Theta, x_n)\|$. If $\lim_{n \rightarrow +\infty} e_n = 0$ implies that $\lim_{n \rightarrow +\infty} x_n = p$, then the iteration method $y_{n+1} = g(\Theta, y_n)$ is said to be Θ -stable.

Remark 4.2. (see[42]) Without any loss of generality, we must assume that $\{x_n\}$ is bounded. Otherwise, if $\{x_n\}$ is not bounded, then it cannot converge. If all conditions in Definition 4.1 are satisfied for $y_{n+1} = \Theta(y_n)$ which is known as Picard's iteration, consequently, the iteration will be Θ -stable.

We shall then state the following theorem.

Theorem 4.3. (see[42]) Let $(X, \|\cdot\|)$ be a Banach space and Θ a self-map of X satisfying

$$\|\Theta(x) - \Theta(y)\| \leq K\|\Theta(x) - x\| + k\|x - y\| \quad (4.1)$$

for all x, y in X where $0 \leq K, 0 \leq k < 1$. Suppose that Θ has a fixed point. Then, Θ is Picard's Θ -stable.

Theorem 4.4. The system (3.7) is Θ -stable in Banach space W if it satisfies the condition (3.22).

Proof. Let $n, m \in \mathbb{N}$, we put $\Omega_n = (S_n, I_n, R_n)$ and $\Theta(\Omega_n) = (S_{n+1}, I_{n+1}, R_{n+1})$. In order to show that P admits a fixed point, we introduce the norm on both sides and we have

$$\begin{aligned} \|\Theta(\Omega_n(t)) - \Theta(\Omega_m(t))\|_1 &\leq \frac{1-\alpha}{B(\alpha)} \|F(t, \Omega_n(t)) - F(t, \Omega_m(t))\|_1 \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t \|F(t, \Omega_n(t)) - F(t, \Omega_m(t))\|_1 (t-p)^{\alpha-1} dp. \end{aligned} \quad (4.2)$$

Then, the same arguments as before lead to

$$\|\Theta(\Omega_n) - \Theta(\Omega_m)\|_W \leq L \left(\frac{1-\alpha}{B(\alpha)} + \frac{\delta^\alpha}{B(\alpha)\Gamma(\alpha)} \right) \|\Omega_n - \Omega_m\|_W. \quad (4.3)$$

If condition (3.22) is satisfied, then the condition (4.1) of Theorem 4.3 holds. This completes the proof. \square

5. Numerical Simulations

We present numerical simulations of the special solution of our model using the Adams–Bashforth–Moulton method presented in [7] for different arbitrary values of fractional order α . The convergence and stability of this method apply at very long intervals. It is a reliable and yet fast alternative for the approximate evaluation of Mittag-Leffler functions[17]. Moreover, it is very useful for the numerical evaluation of other special functions arising in fractional calculus[17]. In our numerical computations, we consider the following values of the parameters: $\Lambda = 0.7$, $\mu = 0.1$, $\lambda = 0.02$, $\beta = 0.1$, $r = 0.02$, $d = 0.01$, $\xi = 0.07$, $\gamma = 0.1$, $k_1 = 0.1$, $k_2 = 0.02$, and $k_3 = 0.002$ with initial conditions $S(0) = 3.0$, $I(0) = 2.0$, $R(0) = 1.0$. The numerical results given in Figure 1 show numerical simulations of the special solution of our model as a function of time for different values of α .

Figure 1 shows the dynamics of the interactions between the compartments of the susceptible and the infected in a particular environment. We notice that when the number of susceptible people decreases the number of infected people increases.

6. Conclusion

Recently, the fractional derivative of Baleanu-Atangana in the sense of Caputo has been used for some applications in many scientific fields with great success. This fractional differentiation based on the Mittag-Leffler function. In this work, we extended the SIRS model with a saturated treatment and a nonlinear incidence function to the concept of fractional differentiation Atangana-Baleanu-Caputo. We have shown the existence and uniqueness of our model using the fixed point theorem. Analysis of the stability of the iterative scheme is validated via the Θ -stable approach. Finally, numerical simulations presented for different values of α .

Since our epidemic model contains saturated treatment function, then as a perspective we will study the influence of the parameter γ on the stability of our system which leads to the case of multiple equilibrium points and shows bifurcation phenomena.

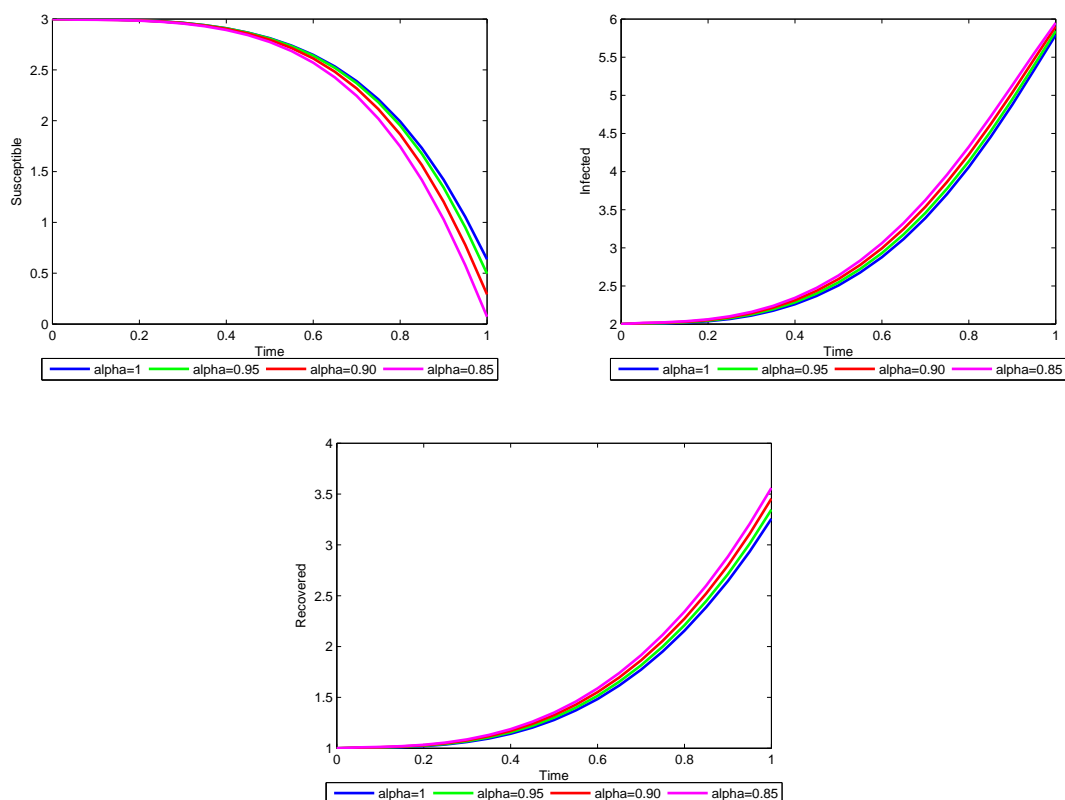


Figure 1: Nature of solution with respect to time for different values of α .

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