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Explicit Expression for a First Integral for a Class of Two-dimensional Differential System

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Abstract. In this paper we are intersted in studying the existence of a first integral and to the curves which are formed by the trajectories of the two-dimensional differential systems of the form

$$\begin{cases} x' = P(x,y) + x\left(\lambda x \exp\left(\frac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\frac{R(x,y)}{S(x,y)}\right)\right), \\ y' = Q(x,y) + y\left(\lambda x \exp\left(\frac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\frac{R(x,y)}{S(x,y)}\right)\right), \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $M(x,y)$, $N(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree a, a, b, b, c, c respectively and λ, $\beta \in \mathbb{R}$. Concrete examples exhibiting the applicability of our result are introduced.

Keywords: *Differential system, First integral, Curves.*
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1 Introduction

We consider two-dimensional autonomous systems of differential equations of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = F(x(t), y(t)), \\ y' = \dfrac{dy}{dt} = G(x(t), y(t)), \end{cases} \tag{1}$$

where $F(x,y)$ and $G(x,y)$ are reals functions. In the qualitative theory of planar dynamical systems [1, 9, 10, 16], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [15]. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly [2, 4, 7, 13]. There exist three main open problems in the qualitative theory of real planar differential systems, the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals. For more details about first integral see for instance [3, 5, 8, 11, 12, 14], see the

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references quoted in those articles. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1)

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} F(x,y) + \frac{\partial H(x,y)}{\partial y} G(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

In this paper we are intersted in studying the existence of a first integral and to the curves which are formed by the trajectories of the two-dimensional differential systems of the form

$$\begin{cases} x' = P(x,y) + x\left(\lambda x \exp\left(\frac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\frac{R(x,y)}{S(x,y)}\right)\right), \\ y' = Q(x,y) + y\left(\lambda x \exp\left(\frac{M(x,y)}{N(x,y)}\right) + \beta y \exp\left(\frac{R(x,y)}{S(x,y)}\right)\right), \end{cases} \tag{2}$$

where $P(x,y)$, $Q(x,y)$, $M(x,y)$, $N(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree a, a, b, b, c, c respectively and λ, $\beta \in \mathbb{R}$.

We define the trigonometric functions

$f_1(\theta) = \lambda(\cos\theta)\exp\left(\frac{M(\cos\theta,\sin\theta)}{N(\cos\theta,\sin\theta)}\right) + \beta(\sin\theta)\exp\left(\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)}\right)$,

$f_2(\theta) = P(\cos\theta,\sin\theta)\cos\theta + Q(\cos\theta,\sin\theta)\sin\theta$,

$f_3(\theta) = (\cos\theta)Q(\cos\theta,\sin\theta) - (\sin\theta)P(\cos\theta,\sin\theta)$.

2 Main result

Our main result on the integrability of differential system (2) is the following :

Theorem 1. *Consider a differential system (2), then the following statements hold.*

(1) *If $f_3(\theta)N(\cos\theta,\sin\theta)S(\cos\theta,\sin\theta) \neq 0$ and $a \neq 2$, then system (2) has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{a-2}{2}} \exp\left((2-a)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) - \\ &\quad (a-2)\int_0^{\arctan\frac{y}{x}} \exp\left((2-a)\int_0^{w} A(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where $A(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as

$$x^2+y^2 = \left(\begin{array}{c} h\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ (a-2)\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((2-a)\int_0^{w} A(\omega)\,d\omega\right) B(w)\,dw \end{array}\right)^{\frac{2}{a-2}},$$

where $h \in \mathbb{R}$.

(2) *If $f_3(\theta)N(\cos\theta,\sin\theta)S(\cos\theta,\sin\theta) \neq 0$ and $a = 2$, then system (2) has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} (A(\omega)+B(\omega))\,d\omega\right),$$

and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} (A(\omega)+B(\omega))\,d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

(3) *If $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then system (2) has the first integral $H = \frac{y}{x}$, and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$.*

Proof. In order to prove our results we write the polynomial differential system (2) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} r' = f_1(\theta)\, r^2 + f_2(\theta)\, r^a, \\ \theta' = f_3(\theta)\, r^{a-1}, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $f_3(\theta)\, N(\cos\theta, \sin\theta)\, S(\cos\theta, \sin\theta) \neq 0$ and $a \neq 2$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = A(\theta)\, r + B(\theta)\, r^{3-a}, \tag{4}$$

where $A(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$ and $B(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, which is a Bernoulli equation.

By introducing the standard change of variables $\rho = r^{a-2}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = (a-2)\left(A(\theta)\,\rho + B(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned} \rho(\theta) &= \exp\left((a-2)\int_0^\theta A(\omega)\, d\omega\right) \\ &\quad \left(\mu + (a-2)\int_0^\theta \exp\left((2-a)\int_0^w A(\omega)\, d\omega\right) B(w)\, dw\right), \end{aligned}$$

where $\mu \in \mathbb{R}$, which has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{a-2}{2}} \exp\left((2-a)\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) - \\ &\quad (a-2)\int_0^{\arctan\frac{y}{x}} \exp\left((2-a)\int_0^w A(\omega)\, d\omega\right) B(w)\, dw. \end{aligned}$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$x^2 + y^2 = \left(\begin{array}{c} h\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) + \\ (a-2)\exp\left((a-2)\int_0^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) \\ \int_0^{\arctan\frac{y}{x}} \exp\left((2-a)\int_0^w A(\omega)\, d\omega\right) B(w)\, dw \end{array}\right)^{\frac{2}{a-2}},$$

where $h \in \mathbb{R}$.

Hence statement (1) of Theorem 1 is proved.

Suppose now that $f_3(\theta)\, N(\cos\theta, \sin\theta)\, S(\cos\theta, \sin\theta) \neq 0$ and $a = 2$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = \left(A(\theta) + B(\theta)\right) r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta) = \mu \exp\left(\int_0^\theta \left(A(\omega) + B(\omega)\right) d\omega\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right) d\omega\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

Hence statement (2) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories.

This completes the proof of statement (3) of Theorem 1. □

3 Examples

The following examples are given to illustrate our result

Example 1 If we take $\lambda = 1$, $\beta = -2$, $P(x,y) = 2x - 3y$, $Q(x,y) = 3x + 2y$, $M(x,y) = x^2 + 2y^2$, $N(x,y) = x^2 + y^2$, $R(x,y) = x^4 + 3x^2y^2 + y^4$ and $S(x,y) = x^4 + 2x^2y^2 + y^4$, then system (2) reads

$$\begin{cases} x' = 2x - 3y + x\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \\ y' = 3x + 2y + y\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \end{cases} \tag{7}$$

the differential system (7) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = \left((\cos\theta)\exp\left(1+\sin^2\theta\right) - 2(\sin\theta)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4\theta\right)\right) r^2 + 2r, \\ \theta' = 3, \end{cases}$$

here $f_1(\theta) = (\cos\theta)\exp\left(1+\sin^2\theta\right) - 2(\sin\theta)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4\theta\right)$, $f_2(\theta) = 2$ and $f_3(\theta) = 3$, it is the case (1) of the Theorem 1.

The differential system (7) has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{-1}{2}} \exp\left(\frac{2}{3}\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{2}{3}w\right) B(w)\, dw,$$

where $B(w) = \dfrac{(\cos w)\exp\left(1+\sin^2 w\right) - 2(\sin w)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4w\right)}{3}$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (7), in cartesian coordinates are written as

$$x^2+y^2 = \left[\left(h + \int_0^{\arctan\frac{y}{x}} \exp\left(\frac{2}{3}w\right) B(w)\, dw\right) \exp\left(\frac{-2}{3}\arctan\frac{y}{x}\right)\right]^{-2},$$

where $h \in \mathbb{R}$.

Example 2 If we take $\lambda = 1$, $\beta = -2$, $P(x,y) = 5x^2 + 2xy$, $Q(x,y) = -2xy + 5y^2$, $M(x,y) = x^2 + 2y^2$, $N(x,y) = x^2 + y^2$, $R(x,y) = y$ and $S(x,y) = x$, then system (2) reads

$$\begin{cases} x' = 5x^2 + 2xy + x\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{y}{x}\right)\right), \\ y' = -2xy + 5y^2 + y\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{y}{x}\right)\right), \end{cases} \tag{8}$$

the differential system (8) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = \left(7\cos^3\theta + 3\sin^3\theta - 2\left(\sin\theta\right)\left(-1+\exp\tan\theta\right) + \left(\cos\theta\right)\left(-2+\exp\left(1+\sin^2\theta\right)\right)\right) r^2, \\ \theta' = \left(3\cos\theta\sin^2\theta - 7\cos^2\theta\sin\theta\right) r, \end{cases}$$

then the differential system (8) has the first integral

$$H\left(x,y\right) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} \left(A\left(\omega\right)+B\left(\omega\right)\right) d\omega\right)$$

where $A\left(\omega\right)+B\left(\omega\right) = \dfrac{7\cos^3\omega + 3\sin^3\omega - 2\left(\sin\omega\right)\left(-1+\exp\tan\omega\right) + \left(\cos\omega\right)\left(-2+\exp\left(1+\sin^2\omega\right)\right)}{3\cos\omega\sin^2\omega - 7\cos^2\omega\sin\omega}$, it is the case (2) of the Theorem 1,

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (8), in cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan\frac{y}{x}} \left(A\left(\omega\right)+B\left(\omega\right)\right) d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

Example 3 If we take $\lambda = 1$, $\beta = -2$, $P\left(x,y\right) = x^3 + xy^2$, $Q\left(x,y\right) = y^3 + yx^2$, $M\left(x,y\right) = x^2 + 2y^2$, $N\left(x,y\right) = x^2 + y^2$, $R\left(x,y\right) = x^4 + 3x^2y^2 + y^4$ and $S\left(x,y\right) = x^4 + 2x^2y^2 + y^4$, then system (2) reads

$$\begin{cases} x' = x^3 + xy^2 + x\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \\ y' = y^3 + yx^2 + y\left(x\exp\left(\frac{x^2+2y^2}{x^2+y^2}\right) - 2y\exp\left(\frac{x^4+3x^2y^2+y^4}{x^4+2x^2y^2+y^4}\right)\right), \end{cases} \tag{9}$$

the differential system (9) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = r' = r^3 + \left(\left(\cos\theta\right)\exp\left(1+\sin^2\theta\right) - 2\left(\sin\theta\right)\exp\left(\frac{9}{8} - \frac{1}{8}\cos 4\theta\right)\right) r^2, \\ \theta' = 0, \end{cases} \tag{10}$$

it is the case (3) of the Theorem 1, then from system (10) it follows that $\theta' = 0$. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (9), in Cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories.

4 Conclusion

The elementary method used in this paper seems to be fruitful to investigate planar differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories, this is a one of the classical tools in the classification of all trajectories of dynamical systems.

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