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Controllability of Fractional-order Dynamical System with Time Varying-Delays

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Abstract. In this study, controllability of a class of fractional-order dynamical system with time-varying delays is considered. First, a representation of the solution is given using Laplace transform and Mittage-leffler function. Then necessary and sufficient conditions for the controllability of the fractional order system are derived by using algebraic criterion.

Keywords: *Fractional order system, time varying delays, Laplace transform, Mittage-Leffler function, controllability, algebraic criteria*
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1. Introduction

Currently, qualitative properties such as observability, controllability, stability, stabilizability of fractional-order dynamical systems are the issues dealt with by scholars. A fundamental concept widely used in the analysis and design of control systems in modern control theory is Controllability. (see [9], [30]). Controllability is the possibility of steering a dynamical system from an arbitrary initial state to an arbitrary final (desired) state by using a set of admissible controls. Depending on the type of dynamical system, there are numerous types of definitions of controllability [9], [24], [30].

Numerous remarkable results concerning the controllability of different types of dynamical systems have been obtained in ([14], [23], [26], [24], [25], [28], [19] and references therein). Different techniques have been used in establishing controllability of linear and nonlinear dynamical systems. [44], [47], [28], [27], [22], [29], [3], used fixed-point techniques to derive sufficient conditions for the controllability of dynamical systems. [12], [1], [4], [5] used the measure of noncompactness of a set and fixed-point theorem due to Darbo in establishing controllability results of nonlinear systems with implicit derivatives. [20], [17] used algebraic methods. [43] used geometric approach, while [11], [32], [16] used functional analytic method.

The development of fractional-order calculus since the 17th century can be traced to the works of Leibnitz, Liouville, Riemann, Grunwald, and Letnikov [21], [36], [40]. Fractional-order calculus is an extension of the classical calculus in which the order of the derivatives and integrals are of arbitrary order. Recently, due to the important role fractional-order systems plays in various scientific fields such as mathematics, physics, engineering, biology, chemistry, etc [21], [42], [45], [39], [36], [31], a renewed interest by scholars has been devoted to its study. The most important aspects which essentially differentiates fractional-order models and integer-order models are the following. First for mechanical and physical processes, at a particular time, the integer-order derivatives indicate a variation of a certain attributes, while the fractional-order derivative involves the entire space. Second, for physical processes, integer order derivatives describe the local properties of a certain position, while fractional order derivatives describe the nonlocal properties, [10], [18]. However, in many real-world phenomena, fractional order models are more realistic and suitable to describe its behaviors than integer order models, for details see [33], [46], [35], [38], [18], [15].

The study of controllability of fractional-order dynamical systems provide important issues for many applied problems due to the use of fractional order derivatives and integrals in control theory leading to better results than the integer ones. In recent years, the controllability of fractional-order dynamical systems has attracted the attention of many researchers because of the critical role it plays in their analysis. [41] established different types of necessary and sufficient conditions for the relative controllability and relative constrained controllability for both with and without retarded state for linear fractional-order systems. In [6] sufficient condition for the controllability of nonlinear integrodifferential system with implicit fractional derivative is obtained using the notion of the measure of noncompactness of a set and Darbo's fixed point theorem, while [2] used Schauder's fixed point theorem to prove sufficient conditions for the controllability of linear and nonlinear fractional-order dynamical systems in finite dimensional spaces. [8] studied sufficient conditions for the relative controllability of nonlinear fractional-order dynamical systems defined in a finite dimensional space with time varying multiple and distributed delays in the control using Schauder,s fixed point theorem. [7] proved controllability of linear and nonlinear fractional damped dynamical system in finite dimensional spaces using Mittage-Leffler matrix function and iterative techniques. Motivated by this literature, this article seeks to prove sufficient conditions for the controllability of a class of fractional-order dynamical systems with time-varying delays in the state defined by

$$\begin{aligned} {}^{C}D_t^{\alpha}x(t) &= A_0(t)x(t) + A_1(t)x(t-\sigma(t)) + B(t)u(t) \\ x(0) &= x_0, \quad \forall t \in [-\sigma_M, 0] \end{aligned} \tag{1}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional-order derivative of order α, $\alpha \in (0,1)$, D the classical differential operator, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, $A_0(t), A_1(t)\, and\, B(t)$ are matrices of bounded variation of appropriate dimension, $\sigma(t)$ is time-varying delay, $x_0 \in C([-\sigma, 0], R^n)$ is an admissible initial state and $C([-\sigma_M, 0], R^n)$ is a Banach space of continuous function $\phi : [-\sigma_M, 0] \to R^n$ which converges uniformly and equipped with standard supremum norm $\|.\|$ given by

$$\|\varphi\| = \sup_{-\sigma_{\max} \le \theta \le 0} |\varphi(\theta)|$$

$I = [0, T]$, where $T > 0$ is the time interval over which the system's behavior is defined.

2. Preliminaries

First we recall some basic definitions, notations, lemmas, theorems and preliminary facts that will be used later in the work, (see [21], [34], [37], [40] for details).

Definition 2.1 Let the function $f : \mathbb{R}^+ \to \mathbb{R}$, then the Caputo fractional-order derivative of order α is defined as

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds, \ \alpha \in (n-1, n), \ \forall n \in \mathbb{N}$$

where $\Gamma(.)$ is the Euler's gamma function, and $D_t^\alpha f(t) = \frac{d^n f(t)}{dt^n}$ *if* $\alpha = n$. The nth derivative of order n of the function $f(t)$ with respect to t is denoted by $\frac{d^n f(t)}{dt^n}$. The Caputo fractional derivative and ordinary derivatives have quite similar properties. Also, the initial conditions for both Caputo fractional differential equations and ordinary differential equations take on the same form, but in real applications those of Caputo derivative are usually used since it has a clear physical meaning.

Definition 2.2 The two parameter Mittage-Leffler function $E_{\alpha,\beta}$ *for* $\alpha, \beta > 0$ is a function defined by the following series

$$E_{\alpha,\beta}(z) = \sum_{\rho=0}^{\infty} \frac{z^\rho}{\Gamma(\alpha\rho + \beta)}, \ z \in \mathbb{C}$$

According to [34], for arbitrary n-th order square matrix A and $\alpha > 0$, based on the definition of Mittage-Leffler function, the formula for the pseudo-transition matrix $\phi_0(t)$ of the linear fractional-order system ${}^c D^\alpha x(t) = A(t)x(t)$ is given as

$$\phi_0(t) = E_{\alpha,\beta}(At^\alpha) = \sum_{\rho=0}^{\infty} \frac{A^\alpha t^{\alpha\rho}}{\Gamma(\rho\alpha + 1)}$$

and then we set

$$\phi(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = t^{\alpha-1} \sum_{\rho=0}^{\infty} \frac{A^\alpha t^{\alpha\rho}}{\Gamma((\rho+1)\alpha)}$$

For $\alpha = 1$, the pseudo-transition matrix is equivalent to the classical transition matrix of ordinary differential equations

$$\phi_0(t) = \sum_{\rho=0}^{\infty} \frac{A^\rho t^\rho}{\Gamma(\rho+1)} = \sum_{\rho=0}^{\infty} \frac{(At)^\rho}{\rho} = e^{At}$$

This implies that the pseudo-transition matrix $\phi_0(t)$ is also known as matrix $\alpha -$exponential function and is denoted by $\phi_0(t) = e_\alpha^{At}$ [21]. In the space of the n-th order matrix with real elements, it is convergent.

Definition 2.3: Let $f(t)$ be an n-dimensional vector-valued function, then the Laplace transform of a function is defined as

$$F(s) = L[f(t)] = \int_0^t e^{-st} f(t)dt, \ \ s \in \mathbb{C} \quad (2)$$

Note: if $0 < \alpha < 1$, then

$$L[(D^\alpha f)(t)] = s^\alpha L[f(t)] - s^{\alpha-1} f(0) \quad (3)$$

(see [11]).

Lemma 2.1: Let $\mathbb{C}$ be the complex plane, for any $\alpha > 0, \beta > 0$ *and* $A \in \mathbb{C}^{n\times n}$, then

$$L\left[t^{\beta-1} E_{\alpha,\beta}(At^\alpha)\right] = s^{\alpha-\beta}(s^\alpha I - A)^{-1}, \ \ \Re(s) > \|A\|^{\frac{1}{\alpha}} \quad (4)$$

where $\Re(s)$ represents the real part of the complex number s, I denotes the identity matrix, and the inverse Laplace transform is given by following formula

$$L^{-1}\left[s^{\alpha-1}\left(s^{\alpha}I-A\right)^{-1}\right]=\phi_0(t)$$
$$L^{-1}\left[\left(s^{\alpha}I-A\right)^{-1}\right]=\phi(t)$$

(see [6])

Lemma 2.2 Let $0<\alpha<1$; If $f:[0,T]\to \mathbb{R}^n$ is continuous and exponentially bounded, then the solution of equ. (1) can be presented as

$$x(t)=\phi_0(t)x(0)+\int_0^t \phi(t-s)\times\left(Ax\left(s-\sigma(s)\right)+Bu(s)\right)ds,\qquad t\in[0,T]$$

and

$$x(t)=x(0),\ t\in[-\sigma_m,0]$$

Proof:

Taking the Laplace transform of both sides of equ. (1) and using definition 2.2 and lemma 2.1, gives

$$s^{\alpha}L\left[x(t)\right]-s^{\alpha-1}x(0)=A_0L\left[x(t)\right]+L\left[A_1x\left(t-\sigma(t)\right)+Bu(t)\right] \tag{5}$$

Hence, equ. (5) can be written as

$$\left(s^{\alpha}I-A_0\right)L\left[x(t)\right]=s^{\alpha-1}x(0)+L\left[Ax\left(t-\sigma(t)\right)+Bu(t)\right] \tag{6}$$

and

$$L\left[x(t)\right]=\left(s^{\alpha}I-A_0\right)^{-1}s^{\alpha-1}x(0)+\left(s^{\alpha}I-A_0\right)^{-1}L\left[Ax\left(t-\sigma(t)\right)+Bu(t)\right]$$
$$=L\left[\phi_0(t)x(0)\right]+L\left[\phi(t)\right]L\left[Ax\left(t-\sigma(t)\right)+Bu(t)\right]$$

Now applying the convolution theorem of the Laplace transform we obtain

$$L\left[x(t)\right]=L\left[\phi_0(t)x(0)\right]+L\left[\int_0^t \phi(t-s)\left(Ax\left(s-\sigma(s)\right)\right)+Bu(s)ds\right] \tag{7}$$

taking the inverse Laplace transform of both sides of equ. (7), we obtain

$$x(t)=\phi_0(t)x(0)+\int_0^t \phi(t-s)\times\left(Ax\left(s-\sigma(s)\right)+Bu(s)\right)ds \tag{8}$$

as required.

3. Main result

For the fractional order equ. (1), we prove the necessary and sufficient conditions for the controllability criteria in this section.

Definition 3.1

The equ. (1) is said to be controllable on the time interval $[0,T]$ if for every $x_0,x_1\in\mathbb{R}^n$, there exist a control $u(t)$ such that the solution of the equ. (1) satisfies the condition $x(0)=x_0$ *and* $x(T)=x_1$

Theorem 3.1

The fractional order equ. (1) is relatively controllable on the time interval $[0,T]$ if and only if the $n\times n$ dimensional Grammian matrix

$$W(0,T)=\int_0^T \phi(T-s)BB^*\phi^*(T-s)ds \tag{9}$$

is nonsingular where * denotes the matrix transpose and $\phi^*(t)=E_{\alpha,\alpha}\left(A^*t^{\alpha}\right)$

Proof:

Suppose $W(0,T)$ is nonsingular, this means that there exists a well-defined inverse matrix $W^{-1}(0,T)$. For any initial complete state $x(0)\in C\left([-\sigma_M,0],\mathbb{R}^n\right)$ we take the following control function

$$\tilde{u}(t) = B^*\phi^*(T-t)W^{-1}(0,T)\left[\tilde{x} - \phi_0(t)x(0) - \int_0^T \phi(T-s)Ax(s-\sigma(s))ds\right] \quad (10)$$

Substituting $t = b$ in equ. (8) we get

$$x(b) = \phi_0(b)x(0) + \int_0^b \phi(b-s)Ax(s-\sigma(s))ds + \int_0^b \phi(b-s)Bu(s)ds \quad (11)$$

Now inserting equ. (10) in equ. (11) we obtain

$$\begin{aligned} x(b) &= \phi_0(b)x(0) + \int_0^b \phi(b-s)Ax(s-\sigma(s))ds \\ &+ \int_0^b \phi(b-s)BB^*\phi^*(b-t)W^{-1}(0,b)\left[\tilde{x} - \phi_0(b)x(0) - \int_0^b \phi(b-s)Ax(s-\sigma(s))ds\right] = \tilde{x} \end{aligned} \quad (12)$$

By definition (3.1) the equ. (1) is relatively controllable in $[0,b]$.

We prove the necessary condition by contradiction. On the other hand, suppose that equ. (1) is relatively controllable, but the Grammian matrix $W(0,T)$ is singular. Then there exists a nonzero vector y such that

$$y^*W(0,T)y = 0$$

That is

$$\int_0^b y^*\phi(b-s)BB^*\phi^*(b-s)ds = 0 \quad (13)$$

Thus, for each $t \in [0,b]$, we have

$$y^*\phi(b-t)B = 0 \quad (14)$$

Since the system is controllable. By assumption, there exists an input u such that it drives the initial state x_0 to the origin in the interval $[0,b]$. It follows that

$$\begin{aligned} x(b) &= x(b, x_0, u_0) \\ &= \phi_0(b)x(0) + \int_0^b \phi(b-s)Ax(b-\sigma(s))ds + \int_0^b \phi(b-s)Bu_0(s)ds = 0 \end{aligned} \quad (15)$$

Moreover, there exists a control $\tilde{u}(t)$ that drives the initial state x_0 to the state $\tilde{x}$, hence

$$\begin{aligned} \tilde{x} &= x(b, x_0, \tilde{u}) \\ &= \phi_0(b)x(0) + \int_0^b \phi(b-s)Ax(s-\sigma(s))ds + \int_0^b \phi(b-s)B\tilde{u}(s)ds = 0 \end{aligned} \quad (16)$$

Combining equs. (15) and (16) gives

$$\tilde{x} - \int_0^b \phi(b-s)B\left[\tilde{u}(s) - u_0(s)\right]ds = 0 \quad (17)$$

Multiplying both sides of equ. (17) and using equ. (14) by $\tilde{x}'$ gives

$$\tilde{x}\tilde{x}' - \tilde{x}'\int_0^b \phi(b-s)B\left[\tilde{u}(s) - u_0(s)\right]ds = 0 \quad (18)$$

By equ. (14), the second term of equ. (18) is zero, hence $\tilde{x}\tilde{x}' = 0$. Thus $\tilde{x} = 0$, this contradicts the assumption. Therefore, the Grammian matrix $W(0,T)$ is nonsingular, hence the desired result.

The next theorem is a necessary and sufficient condition for the relative controllability of equ. (1) that is based on the matrices A and B. this is an algebraic criterion similar to the famous Kalman rank condition, see [30].

Theorem 3.2: the fractional control system equ. (1) is controllable in the interval $[0,T]$ if and only if $rank\left[B, AB, A^2B, \ldots, A^{n-1}B\right] = n$

Proof:

According to Carley-Hamilton theorem $\phi(t) = t^{\alpha-1}E_{\alpha,\alpha}\left(At^{\alpha}\right)$ can be represented as

$$\phi(t)=t^{\alpha-1}\sum_{\rho=0}^{\infty}\frac{A^{\rho}t^{\alpha\rho}}{\Gamma(\rho\alpha+\alpha)}=\sum_{\rho=0}^{\infty}\frac{t^{\rho\alpha+\alpha-1}}{\Gamma(\rho\alpha+\alpha)}A^{\rho}=\sum_{\rho=0}^{n-1}G_{\rho}(t)A^{\rho} \tag{19}$$

For $b\in[0,t_1]$

$$\begin{aligned}x(b)&=\phi_0(b)x_0+\int_0^b\phi(b-s)\times\left[Ax(s-\sigma(s))+Bu(s)\right]ds\\&=\phi_0(b)x_0+\sum_{\rho=0}^{n-1}\int_0^b G_{\rho}(b-s)A^{k}\times\left[Ax(s-\sigma(s))+Bu(s)\right]ds\end{aligned} \tag{20}$$

Let

$$\psi=\phi_0(b)x_0+\sum_{\rho=0}^{n-1}\int_0^b G_{\rho}(b-s)A^{\rho}\times\left[Ax(s-\sigma(s))\right]ds \tag{21}$$

Then, subtracting equ. (21) from equ. (20) yields

$$\begin{aligned}x(b)-\psi&=\sum_{\rho=0}^{n-1}A^{\rho}B\int_0^b G_{\rho}(b-s)u(s)ds\\&=\left[B,AB,A^2B,\ldots,A^{n-1}B\right]\begin{bmatrix}c_1\\c_2\\c_3\\.\\.\\.\\c_{n-1}\end{bmatrix}\end{aligned} \tag{22}$$

where $c_{\rho}=\int_0^b G_{\rho}(b-s)u(s)ds,\ \rho=0,1,2,\ldots,n-1$. Note that for arbitrary $x(0)\in C\left([-\tau_M,0],\mathbb{R}^n\right)$ and $x(b)\in\mathbb{R}^n$, the necessary and sufficient condition to have a control $u(t)$ satisfy equ. (22) is that

$$rank\left[B,AB,A^2B,\ldots,A^{n-1}B\right]=n$$

For $b\in(t_i,t_{i+1})\ \ i=1.2,\ldots,k$

$$\begin{aligned}x(b)&=\phi_0(b)x_0+\sum_{j=1}^{i}I_j\left(x\left(\bar{t}_j\right)\right)+\int_0^b\phi(b-s)\times\left[Ax(s-\sigma(s))+Bu(s)\right]ds\\&=\phi_0(b)x_0+\sum_{\rho=0}^{n-1}\int_0^b G_{\rho}(b-s)A^{\rho}\times\left[Ax(s-\sigma(s))+Bu(s)\right]ds\end{aligned} \tag{23}$$

Combining equ. (21) and equ. (23) yields

$$\begin{aligned}x(b)-\psi-\sum_{j=1}^{i}I_j\left(x\left(\bar{t}_j\right)\right)&=\sum_{\rho=0}^{n-1}A^{\rho}B\int_0^b G_k(b-s)u(s)ds\\&=\left[B,AB,A^2B,\ldots,A^{n-1}B\right]\begin{bmatrix}c_0\\c_1\\.\\.\\.\\c_{n-1}\end{bmatrix}\end{aligned} \tag{24}$$

Note that for arbitrary $x_0\in C\left([-\sigma_M,0],\mathbb{R}^n\right)$ and $x(b)\in\mathbb{R}^n$, the necessary and sufficient condition to have a control $u(t)$ satisfying equ. (24) is that

$$rank\left[B, AB, A^2B, \ldots, A^{n-1}B\right] = n$$

This completes the proof.

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