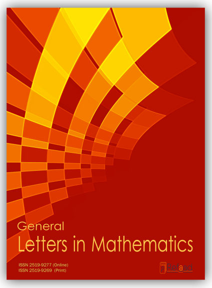

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On Soft Ordered Maps

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Abstract. In the current work, we define soft λ-continuous, soft λ-open, soft λ-closed and soft λ-homeomorphism maps via soft topological ordered spaces, where $\lambda \in \{I, D, B\}$. The relationships among these soft maps are shown with the help of examples and their main properties are studied. In this regard, the equivalent conditions for each one of these soft maps are investigated, and an enough condition for the equivalent between soft λ-open and soft λ-closed maps is given for each λ. Also, we discuss the interrelations between these soft maps and their counterparts on topological ordered spaces and clarify a significant role of extended soft topologies in this point. In the end, we point out under what conditions the initiated soft maps preserve some soft ordered separation axioms, and conclude the behaviors of these soft maps under some compositions.

Keywords: *Soft $I(D, B)$-continuous map, soft $I(D, B)$-open map, soft $I(D, B)$-homeomorphism map, soft ordered separation axioms*
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1 Introduction and Preliminaries

The concept of topological ordered spaces was first introduced and systematically studied by Nachbin [33]. He investigated the main properties of increasing and decreasing sets and verified some interesting results associated with the normally ordered spaces. Then McCartan [31], in 1968, carried out a detailed study on ordered separation axioms. He focused on characterizing T_i-ordered spaces ($i = 0, 1, 2$), regularly ordered and normality ordered spaces. The concepts of continuity, openness, closedness and homeomorphism between topological ordered spaces were presented and studied by Kumar [24]. For more investigation on topological ordered spaces, interested reader can find that from [11, 22, 25, 26, 27]. Das [14], in 2004, generalized ordered topologies to ordered supra topologies. Then many studies on supra topological ordered spaces were done (see, for example, [3, 5, 7, 8, 16, 17, 18]).

During recent past, the researchers endeavoured to approach uncertainty with some tools other than probability theory and fuzzy set theory. Soft set theory, was initiated by Molotdsov [32], has an important contribution towards the provision of such a mathematical tool to resolve the issues related to ambiguity and uncertainty in data based problems arising in mathematics or related disciplines. The ease of the definition of soft set made it more active to apply in many situations arising in information sciences, decision making, forecasting, demand analysis, mathematics and many other related disciplines as well. We refer reader for more studies on soft sets to ([1, 2, 4, 20, 28, 29, 30]).

Shabir and Naz [37] employed the notion of soft sets to initiate the concept of soft topological spaces. They formulated and studied the basic notions on soft topological spaces such as soft interior points of a soft set, soft subspaces and soft separation axioms. Later on, numerous studies were done on soft topological notions and their

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features (see, for example, [6, 10, 12, 15, 23, 34, 35, 36]). Recently, El-Shafei et al. [19] established two new soft relations, namely partial belong and total non belong. Depend on them, they introduced partial soft separation axioms, namely p-soft T_i-spaces for $i = 0, 1, 2, 3, 4$. With regard to our topic, Al-shami et al. [9] initiated a concept of soft topological ordered spaces. They gave the fundamental properties of a soft partially ordered sets and then defined p-soft T_i-ordered spaces for $i = 0, 1, 2, 3, 4$.

Since soft continuity, soft openness, soft closedness and soft homeomorphism are very important concepts in soft topology, it is natural to think about its soft order topology analogue. So the aim here, is to introduce and study new types of soft maps via soft topological ordered spaces, namely soft λ-continuous, soft λ-open, soft λ-closed and soft λ-homeomorphism maps, where $\lambda \in \{I, D, B\}$. The purpose of establishing various examples is twofold, one is to show the relationships among these soft maps introduced herein and the other is to illustrate that soft λ-continuous, soft λ-open, soft λ-closed and soft λ-homeomorphism maps are strictly stronger than soft continuous, soft open, soft closed and soft homeomorphism maps, respectively, for each λ. Also, we completely describe each one of the initiated soft maps and clarify a significant role of extended soft topologies on studying the interrelations between these soft maps and their counterparts on topological ordered spaces. Ultimately, we point out the interrelations between these soft maps and some soft ordered separation axioms and probe the behaviors of these maps under some compositions.

The following portion of this section is devoted to recalling some definitions and results that will be needed in the sequel.

Definition 1.1. *[32] An ordered pair (G, E) is said to be a soft set over X if G is a map of a set of parameters E into 2^X.*
For short, we use the notation G_E instead of (G, E) and we write it as $G_E = \{(e, G(e)) : e \in E$ and $G(e) \in 2^X\}$.

Definition 1.2. *[30] A soft set G_E over X is called a null soft set, denoting by $\widetilde{\emptyset}$, if $G(e) = \emptyset$ for each $e \in E$; and it is called an absolute soft set, denoting by $\widetilde{X}$, if $G(e) = X$ for each $e \in E$.*

Definition 1.3. *[4] The relative complement of a soft set G_E is denoted by G^c_E, where $G^c : E \to 2^X$ is a mapping defined by $G^c(e) = X \setminus G(e)$ for each $e \in E$.*
In this connection, it is worth noting that $x \notin G_E$ does not imply that $x \in G^c_E$.

Definition 1.4. *[4] A soft set G_A is a soft subset of a soft set F_B, denoted by $G_A \subseteq F_B$, if $A \subseteq B$ and $G(a) \subseteq F(a)$ for all $a \in A$.*
The soft sets G_A and F_B are soft equal if each of them is a soft subset of the other.

Definition 1.5. *[37] For $x \in X$ and a soft set G_E over X, we say that $x \in G_E$ if $x \in G(e)$ for each $e \in E$.*

Definition 1.6. *[37] A collection τ of soft sets over X under a parameters set E is said to be a soft topology on X if the following three conditions hold:*

(i) *$\widetilde{X}$ and $\widetilde{\emptyset}$ belong to τ.*

(ii) *The soft intersection of finite members in τ belongs to τ.*

(iii) *The soft union of any members in τ belongs to τ.*

The triple (X, τ, E) is called a soft topological space. Every member of τ is called soft open and its relative complement is called soft closed. The soft closure of a soft set H_E (briefly $cl(H_E)$) is the smallest soft closed set that contains H_E; and the soft interior of a soft set H_E (briefly $int(H_E)$) is the largest soft open set that is contained in H_E.
A collection of all soft subsets of the absolute soft set $\widetilde{X}$ is denoted by $S(X_E)$.

Proposition 1. *[37] Let (X, τ, E) be a soft topological space. Then $\tau_e = \{G(e) : G_E \in \tau\}$ defines a topology on X for each $e \in E$.*

Proposition 2. *[34] Consider (X, τ, E) is a soft topological space and τ_e is a topology on X as in the above proposition. Then $\tau^\star = \{G_E : G(e) \in \tau_e$ for each $e \in E\}$ is a soft topology on X finer than τ.*

Remark 1.1. *Henceforth, we called τ_e and $\tau^\star$, which given in the two proposition above, a parametric topology and an extended soft topology, respectively.*

We modify the definition of a soft map given in [38] to be as follows.

Definition 1.7. *Consider $f : X \to Y$ and $\phi : A \to B$ are two maps and let $f_\phi : S(X_A) \to S(Y_B)$ be a soft map. Let G_K and H_L be soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. The image of G_K and pre-image of H_L are defined as follows:*

(i) *$f_\phi(G_K) = (f_\phi(G))_B$ is a soft subset of $S(Y_B)$ such that*

$$f_\phi(G)(b) = \begin{cases} \bigcup_{a\in\phi^{-1}(b)\bigcap K} f(G(a)) & : \phi^{-1}(b)\bigcap K \neq \emptyset \\ \emptyset & : \phi^{-1}(b)\bigcap K = \emptyset \end{cases}$$

for each $b \in B$.

(ii) *$f_\phi^{-1}(H_L) = (f_\phi^{-1}(H))_A$ is a soft subset of $S(X_A)$ such that*

$$f_\phi^{-1}(H)(a) = \begin{cases} f^{-1}(H(\phi(a))) & : \phi(a) \in L \\ \emptyset & : \phi(a) \notin L \end{cases}$$

for each $a \in A$.

Remark 1.2. *Henceforward, a soft map $f_\phi : S(X_A) \to S(Y_B)$ means that we have a map f of the universe set X into the universe set Y and a map ϕ of the set of parameters A into the set of parameters B*

Definition 1.8. *[38] A soft map $f_\phi : S(X_A) \to S(Y_B)$ is said to be injective (resp. surjective, bijective) if f and ϕ are injective (resp. surjective, bijective).*

Proposition 3. *[34] Consider $f_\phi : S(X_A) \to S(Y_B)$ is a soft map and let G_A and H_B be two soft subsets of $S(X_A)$ and $S(Y_B)$, respectively. Then we have the following results:*

(i) *$G_A \widetilde{\subseteq} f_\phi^{-1} f_\phi(G_A)$ and the equality relation holds if f_ϕ is injective.*

(ii) *$f_\phi f_\phi^{-1}(H_B) \widetilde{\subseteq} H_B$ and the equality relation holds if f_ϕ is surjective.*

Definition 1.9. *([34], [38]) A soft map $f_\phi : (X, \tau, A) \to (Y, \theta, B)$ is said to be:*

(i) *Soft continuous if the inverse image of each soft open subset of (Y, θ, B) is a soft open subset of (X, τ, A).*

(ii) *Soft open (resp. soft closed) if the image of each soft open (resp. soft closed) subset of (X, τ, A) is a soft open (resp. soft closed) subset of (Y, θ, B).*

(iii) *Soft homeomorphism if it is bijective, soft continuous and soft open.*

Definition 1.10. *([15], [34]) A soft set P_E over X is called soft point if there exists $e \in E$ and there exists $x \in X$ such that $P(e) = \{x\}$ and $P(\alpha) = \emptyset$ for each $\alpha \in E \setminus \{e\}$. A soft point will be shortly denoted by P_e^x and we say that $P_e^x \in G_E$, if $x \in G(e)$.*

Definition 1.11. *[19] For $x \in X$ and a soft set G_E over X, we say that $x \Subset G_E$ if $x \in G(e)$ for some $e \in E$.*

Proposition 4. *[19] Consider $x \in X$, $y \in Y$, $G_A \in S(X_A)$ and $H_B \in S(Y_B)$ and let $f_\phi : S(X_A) \to S(Y_B)$ be a soft map. Then we have the following results:*

(i) *If ϕ is surjective and $x \Subset G_A$, then $f(x) \Subset f_\phi(G_A)$.*

(ii) *If $y \Subset H_B$, then $x \Subset f_\phi^{-1}(H_B)$ for each $x \in f^{-1}(y)$..*

(iii) *If f is injective and $x \not\Subset G_A$, then $f(x) \not\Subset f_\phi(G_A)$.*

(iv) *If $y \not\Subset H_B$, then $x \not\Subset f_\phi^{-1}(H_B)$ for each $x \in f^{-1}(y)$.*

Definition 1.12. *[19] A soft topological space (X, τ, E) is called:*

(i) *p-soft T_0 if for every pair of distinct points $x, y \in X$, there is a soft open set G_E such that $x \Subset G_E$, $y \not\Subset G_E$ or $y \Subset G_E$, $x \not\Subset G_E$.*

(ii) *p-soft T_1 if for every pair of distinct points $x, y \in X$, there are soft open sets G_E and F_E such that $x \Subset G_E, y \not\Subset G_E$ and $y \Subset F_E$, $x \not\Subset F_E$.*

(iii) *p-soft T_2 if for every pair of distinct points $x, y \in X$, there are disjoint soft open sets G_E and F_E containing x and y, respectively.*

(iv) *p-soft regular if for every soft closed set H_E and $x \in X$ such that $x \notin H_E$, there are disjoint soft open sets G_E and F_E such that $H_E \subseteq G_E$ and $x \in F_E$.*

(vi) *[37] Soft normal if for every two disjoint soft closed sets H_{1_E} and H_{2_E}, there are two disjoint soft open sets G_E and F_E such that $H_{1_E} \widetilde{\subseteq} G_E$ and $H_{2_E} \widetilde{\subseteq} F_E$.*

(vii) *p-soft T_3 (resp. p-soft T_4) if it is both p-soft regular (resp. soft normal) and p-soft T_1.*

Definition 1.13. *[9] Let $\preceq$ be a partial order relation on a non-empty set X and let E be a set of parameters. A triple $(X, E, \preceq)$ is said to be a partially ordered soft set.*

Proposition 5. *[19] The following two results hold for a soft map $f_\phi : S(X_A) \to S(Y_B)$:*

(i) *The image of each soft point is soft point.*

(ii) *The inverse image of each soft point is soft point provided that f is bijective.*

Definition 1.14. *[9] We define an increasing soft operator $i : (S(X_E), \preceq) \to (S(X_E), \preceq)$ and a decreasing soft operator $d : (S(X_E), \preceq) \to (S(X_E), \preceq)$ for each soft subset G_E of $S(X_E)$ as follows:*

(i) *$i(G_E) = (iG)_E$, where iG is a mapping of E into X given by $iG(e) = i(G(e)) = \{x \in X : y \preceq x$ for some $y \in G(e)\}$.*

(ii) *$d(G_E) = (dG)_E$, where dG is a mapping of E into X given by $dG(e) = d(G(e)) = \{x \in X : x \preceq y$ for some $y \in G(e)\}$.*

Definition 1.15. *[9] A soft subset G_E of a partially ordered soft set $(X, E, \preceq)$ is said to be increasing (resp. decreasing) if $G_E = i(G_E)$(resp. $G_E = d(G_E)$).*

Theorem 1.1. *[9] If a soft map $f_\phi : (S(X_A), \preceq_1) \to (S(Y_B), \preceq_2)$ is increasing, then the inverse image of each increasing (resp. decreasing) soft subset of $\widetilde{Y}$ is an increasing (resp. a decreasing) soft subset of $\widetilde{X}$.*

Definition 1.16. *[9] A quadrable system $(X, \tau, E, \preceq)$ is said to be a soft topological ordered space, where (X, τ, E) is a soft topological space and $(X, E, \preceq)$ is a partially ordered soft set. Through this work, we use the two notations $(X, \tau, E, \preceq_1)$ and $(Y, \theta, F, \preceq_2)$ to refer to soft topological ordered spaces.*

Definition 1.17. *[9] A soft topological ordered space $(X, \tau, E, \preceq)$ is said to be:*

(i) *Lower (resp. Upper) p-soft T_1-ordered if for every $x \npreceq y$ in X, there exists an increasing (resp. a decreasing) soft neighborhood W_E of x (resp. y) such that $y \notin W_E$ (resp. $x \notin W_E$).*

(ii) *p-soft T_0-ordered if it is lower soft T_1-ordered or upper soft T_1-ordered.*

(iii) *p-soft T_1-ordered if it is lower soft T_1-ordered and upper soft T_1-ordered.*

(iv) *p-soft T_2-ordered if for every $x \npreceq y$ in X, there exist disjoint soft neighborhoods W_E and V_E of x and y, respectively, such that W_E is increasing and V_E is decreasing.*

(v) *Lower (resp. Upper) p-soft regularly ordered if for each decreasing (resp. increasing) soft closed set H_E and $x \in X$ such that $x \notin H_E$, there exist disjoint soft neighbourhoods W_E of H_E and V_E of x such that W_E is decreasing (resp. increasing) and V_E is increasing (resp. decreasing).*

(vi) *p-soft regularly ordered if it is both lower p-soft regularly ordered and upper p-soft regularly ordered.*

(vii) *Lower (resp. Upper) p-soft T_3-ordered if it is both lower (resp. upper) p-soft T_1-ordered and lower (resp. upper) p-soft regularly ordered.*

(viii) *p-soft T_3-ordered if it is both lower p-soft T_3-ordered and upper p-soft T_3-ordered.*

(ix) *Soft normally ordered if for each disjoint soft closed sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint soft neighbourhoods W_E of F_E and V_E of H_E such that W_E is increasing and V_E is decreasing.*

(x) *p-soft T_4-ordered if it is soft normally ordered and p-soft T_1-ordered.*

Definition 1.18. *[24] A map $f : (X, \tau, \preceq_1) \to (Y, \theta, \preceq_2)$ is said to be:*

(i) *I (resp. D, B) -continuous if the inverse image of each open set is I (resp. D, B) -open.*

(ii) *I (resp. D, B) -open if the image of each open set is I (resp. D, B) -open.*

(iii) *I (resp. D, B) -closed if the image of each open set is I (resp. D, B) -closed.*

(iv) *I (resp. D, B) -homeomorphism if it is bijective, I (resp. D, B) -continuous and I (resp. D, B) -open.*

Throughout this manuscript $\lambda \in \{I, D, B\}$.

2 Soft λ-continuous maps

In this section, we give some kinds of soft continuity via soft topological ordered spaces and illustrate the relationships among them with the help of examples. We then characterize each one of the introduced soft maps and study the pre image of p-soft T_i-spaces ($i = 0, 1, 2$) under these types of soft continuous maps.

Definition 2.1. *A soft subset H_E of $(X, \tau, E, \preceq_1)$ is said to be balancing if it is increasing and decreasing.*

Definition 2.2. *A soft subset H_E of $(X, \tau, E, \preceq_1)$ is said to be:*

(i) *Soft I (resp. Soft D, Soft B) -open if it is soft open and increasing (resp. decreasing, balancing).*

(ii) *Soft I (resp. Soft D, Soft B) -closed if it is soft closed and increasing (resp. decreasing, balancing).*

Definition 2.3. *A soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is said to be:*

(i) *Soft I (resp. Soft D, Soft B) -continuous at $P_e^x \in \widetilde{X}$ if for each soft open set H_F containing $f_\phi(P_e^x)$, there exists a soft I (resp. soft D, soft B) -open set G_E containing P_e^x such that $f_\phi(G_E) \widetilde{\subseteq} H_F$.*

(ii) *Soft I (resp. Soft D, Soft B) -continuous at $x \in X$ if it is soft I (resp. soft D, soft B) -continuous at each P_e^x.*

(iii) *Soft I (resp. Soft D, Soft B) -continuous if it is soft I (resp. soft D, soft B) -continuous at each $x \in X$.*

Theorem 2.1. *A soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) -continuous if and only if the inverse image of each soft open subset of $\widetilde{Y}$ is a soft I (resp. soft D, soft B) -open subset of $\widetilde{X}$.*

Proof. We only present the proof in the case of f_ϕ is soft I-continuous and the cases between parenthesis can be made similarly.
To prove the necessary part, let G_F be a soft open subset of $\widetilde{Y}$. Since the proof is trivial if $f_\phi^{-1}(G_F) = \widetilde{\emptyset}$, then we consider $f^{-1}(G_F) \neq \widetilde{\emptyset}$. By choosing $P_e^x \in \widetilde{X}$ such that $P_e^x \in f_\phi^{-1}(G_F)$, we obtain $f_\phi(P_e^x) \in G_F$. So there exists a soft I-open set H_E containing P_e^x such that $f_\phi(H_E) \widetilde{\subseteq} G_F$. Since P_e^x is chosen arbitrary, then $f_\phi^{-1}(G_F) = \widetilde{\bigcup}_{P_e^x \in f_\phi^{-1}(G_F)} H_E$. Thus $f_\phi^{-1}(G_F)$ is a soft I-open subset of $\widetilde{X}$.
To prove the sufficient part, let G_F be a soft open subset of $\widetilde{Y}$ containing $f_\phi(P_e^x)$. Then $P_e^x \in f_\phi^{-1}(G_F)$. By hypothesis, $f_\phi^{-1}(G_F)$ is a soft I-open set. Since $f_\phi(f_\phi^{-1}(G_F)) \widetilde{\subseteq} G_F$, then f_ϕ is a soft I-continuous at $P_e^x \in X$ and since P_e^x is chosen arbitrary, then f_ϕ is soft I-continuous. □

Remark 2.1. *From Definition* (2.3)*, we can note the following:*

(i) *Every soft I (soft D, soft B) -continuous map is always soft continuous.*

(ii) *Every soft B-continuous map is soft I-continuous (soft D-continuous).*

To elucidate that a soft B-continuous map is real generalization of soft I-continuous and soft D-continuous maps, we construct the following two examples.

Example 2.1. *Let the two universe sets $X=\{h,i,j\}$, $Y=\{w,z\}$ and the two parameters sets $A=\{a_1,a_2\}$, $B=\{b_1,b_2\}$. Consider a map $\phi:A\to B$ is defined as, $\phi(a_m)=b_m$ for $m\in\{1,2\}$ and a map $f:X\to Y$ is defined as, $f(h)=w$ and $f(i)=f(j)=z$. We define a partial order relation on X as $\preceq=\triangle\bigcup\{(i,h),(h,j),(i,j)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau=\{\widetilde{\emptyset},\widetilde{X},F_A,G_A\}$ and $\theta=\{\widetilde{\emptyset},\widetilde{Y},H_B\}$, where $F_A=\{(a_1,X),(a_2,\{h\})\}$, $G_A=\{(a_1,\{h,i\}),(a_2,\{h\})\}$ and $H_B=\{(b_1,Y),(b_2,\{w\})\}$. Then one can readily check that a soft map $f_\phi:S(X_A)\to S(Y_B)$ is soft continuous. On the other hand, $f_\phi^{-1}(H_B)=F_A$ is neither a soft D-open nor a soft I-open set. Hence f_ϕ is not soft I (soft D, soft B) -continuous.*

Example 2.2. *In Example above, if we replace only the partial order relation by $\preceq=\triangle\bigcup\{(h,j),(i,j)\}$(resp. $\preceq=\triangle\bigcup\{(i,h)\}$), then the soft map f_ϕ is soft D-continuous (resp. soft I-continuous), but is not soft B-continuous.*

Definition 2.4. *For a soft subset H_E of $(X,\tau,E,\preceq)$, we define the following six operators:*

(i) *H_E^{io}(resp. H_E^{do},H_E^{bo}) is the largest soft I (resp. soft D, soft B) -open set that is contained in H_E.*

(ii) *H_E^{icl}(resp. H_E^{dcl},H_E^{bcl}) is the smallest soft I (resp. soft D, soft B) -closed set that contains H_E.*

Lemma 2.1. *We have the following three properties for a soft subset H_E of $(X,\tau,E,\preceq)$:*

(i) $(H_E^{dcl})^c=(H_E^c)^{io}$.

(ii) $(H_E^{icl})^c=(H_E^c)^{do}$.

(iii) $(H_E^{bcl})^c=(H_E^c)^{bo}$.

Proof. **(i)** $(H_E^{dcl})^c=\{\widetilde{\bigcup}F_E:F_E$ is a soft D-closed set containing $H_E\}^c$
$=\widetilde{\bigcap}\{F_E^c:F_E^c$ is a soft I-open set contained in $H_E^c\}=(H_E^c)^{io}$.
By analogy with **(i)**, one can prove **(ii)** and **(iii)**. □

Theorem 2.2. *The following five properties of a soft map $f_\phi:(X,\tau,E\preceq_1)\to(Y,\theta,F,\preceq_2)$ are equivalent:*

(i) *f_ϕ is soft I-continuous;*

(ii) *$f_\phi^{-1}(L_F)$ is a soft D-closed subset of $\widetilde{X}$ for each soft closed subset L_F of $\widetilde{Y}$;*

(iii) *$(f_\phi^{-1}(M_F))^{dcl}\widetilde{\subseteq}f_\phi^{-1}(cl(M_F))$ for every $M_F\widetilde{\subseteq}\widetilde{Y}$;*

(iv) *$f_\phi(N_E^{dcl})\widetilde{\subseteq}cl(f_\phi(N_E))$ for every $N_E\widetilde{\subseteq}\widetilde{X}$;*

(v) *$f_\phi^{-1}(int(M_F))\widetilde{\subseteq}(f_\phi^{-1}(M_F))^{io}$ for every $M_F\widetilde{\subseteq}\widetilde{Y}$.*

Proof. **(i) ⇒ (ii)** : Consider L_F is a soft closed subset of $\widetilde{Y}$. By hypothesis, $f_\phi^{-1}(L_F^c)$ is a soft I-open subset of $\widetilde{X}$ and by the fact that $f_\phi^{-1}(L_F^c)=(f_\phi^{-1}(L_F))^c$, we obtain $f_\phi^{-1}(L_F)$ is soft D-closed as required.
(ii) ⇒ (iii) : It follows from statement **(ii)** that $f_\phi^{-1}(cl(M_E))$ is a soft D-closed subset of $\widetilde{X}$ for every $M_F\widetilde{\subseteq}\widetilde{Y}$. So $(f_\phi^{-1}(M_F))^{dcl}\widetilde{\subseteq}(f_\phi^{-1}(cl(M_F))^{dcl}=f_\phi^{-1}(cl(M_F))$.
(iii) ⇒ (iv) : From the fact that $N_E^{dcl}\widetilde{\subseteq}(f_\phi^{-1}(f_\phi(N_E))^{dcl}$ and from **(iii)**, we have $(f_\phi^{-1}(f_\phi(N_E))^{dcl}\ \widetilde{\subseteq}f_\phi^{-1}(cl(f_\phi(N_E)))$. This implies that $f_\phi(N_E^{dcl})\widetilde{\subseteq}cl(f_\phi(N_E))$.
(iv) ⇒ (v) : For any soft subset M_F of $\widetilde{Y}$, we obtain from Lemma (2.1) that $f_\phi(\widetilde{X}-(f_\phi^{-1}(N_E))^{io})=f_\phi(((f_\phi^{-1}(N_E))^c)^{dcl})$. It follows from statement **(iv)**, that $f_\phi(((f_\phi^{-1}(N_E))^c)^{dcl})\ \widetilde{\subseteq}cl(f_\phi(f_\phi^{-1}(N_E))^c)=cl(f_\phi(f_\phi^{-1}(N_E^c)))\widetilde{\subseteq}cl(\widetilde{Y}-N_E)=\widetilde{Y}-int(N_E)$. Therefore $(\widetilde{X}-(f_\phi^{-1}(N_E))^{io})\widetilde{\subseteq}f_\phi^{-1}(\widetilde{Y}-int(N_E))=\widetilde{X}-f_\phi^{-1}(int(N_E))$. Thus $f_\phi^{-1}(int(N_E))\widetilde{\subseteq}(f_\phi^{-1}(N_E))^{io}$.
(v) ⇒ (i): Consider M_F is a soft open subset of $\widetilde{Y}$. Then $f_\phi^{-1}(M_F)=f_\phi^{-1}(int(M_F))\ \widetilde{\subseteq}(f_\phi^{-1}(M_F))^{io}$. So $(f_\phi^{-1}(M_F))^{io}=f_\phi^{-1}(M_F)$ and this means that $f_\phi^{-1}(M_F)$ is a soft I-open subset of $\widetilde{X}$. Hence the desired result is proved. □

Theorem 2.3. *The following five properties of a soft map $f_\phi:(X,\tau,E\preceq_1)\to(Y,\theta,F,\preceq_2)$ are equivalent:*

(i) *f_ϕ is soft D-continuous (resp. soft B-continuous);*

(ii) *$f_\phi^{-1}(L_F)$ is a soft I-closed (resp. soft B-closed) subset of $\widetilde{X}$ for each soft closed subset L_F of $\widetilde{Y}$;*

(iii) $(f_\phi^{-1}(M_F))^{icl}\widetilde{\subseteq}f_\phi^{-1}(cl(M_F))$(*resp.* $(f_\phi^{-1}(M_F))^{bcl}\widetilde{\subseteq}f_\phi^{-1}(cl(M_F))$ *for every* $M_F\widetilde{\subseteq}\widetilde{Y}$;

(iv) $f_\phi(N_E^{icl})\widetilde{\subseteq}cl(f_\phi(N_E))$(*resp.* $f_\phi(N_E^{bcl})\widetilde{\subseteq}cl(f_\phi(N_E))$ *for every* $N_E\widetilde{\subseteq}\widetilde{X}$;

(v) $f_\phi^{-1}(int(M_F))\widetilde{\subseteq}(f_\phi^{-1}(M_F))^{do}$(*resp.* $f_\phi^{-1}(int(M_F))\widetilde{\subseteq}(f_\phi^{-1}(M_F))^{bo}$ *for every* $M_F\widetilde{\subseteq}\widetilde{Y}$.

Proof. The proof is similar to that of Theorem (2.2). □

Theorem 2.4. *If a soft map* $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ *is soft I (resp. soft D, soft B) -continuous, then a map* $g : (X, \tau_e, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ *is I (resp. D, B) -continuous.*

Proof. Let U be an open subset of $(Y, \theta_{\phi(e)}, \preceq_2)$. Then there exists a soft open subset G_F of $(Y, \theta, F, \preceq_2)$ such that $G(\phi(e)) = U$. Since g_ϕ is a soft I (resp. soft D, soft B) -continuous map, then $g_\phi^{-1}(G_F)$ is a soft I (resp. soft D, soft B) -open set. From Definition (1.7), it follows that a soft subset $g_\phi^{-1}(G_F) = (g_\phi^{-1}(G))_E$ of $(X, \tau, E, \preceq_1)$ is given by $g_\phi^{-1}(G)(e) = g^{-1}(G(\phi(e)))$ for each $e \in E$. This implies a subset $g^{-1}(G(\phi(e))) = g^{-1}(U)$ of $(X, \tau_e, \preceq_1)$ is I (resp. D, B) -open. Hence a map g is I (resp. D, B) -continuous. □

Theorem 2.5. *Let* $\tau^\star$ *be an extended soft topology on* X. *Then a soft map* $g_\phi : (X, \tau^\star, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ *is soft I (resp. soft D, soft B) -continuous if and only if a map* $g : (X, \tau_e^\star, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ *is I (resp. D, B) -continuous.*

Proof. The proof of the necessary condition is given above.
Sufficiency: Let G_F be a soft open subset of $(Y, \theta, F, \preceq_2)$. Then from Definition (1.7), it follows that a soft subset $g_\phi^{-1}(G_F) = (g_\phi^{-1}(G))_E$ of $(X, \tau^\star, E, \preceq_1)$ is given by $g_\phi^{-1}(G)(e) = g^{-1}(G(\phi(e)))$ for each $e \in E$. Since a map g is I (resp. D, B) -continuous, then a subset $g^{-1}(G(\phi(e)))$ of $(X, \tau_e^\star, \preceq_1)$ is I (resp. D, B) -open. By hypothesis, $\tau^\star$ is an extended soft topology on X, $g_\phi^{-1}(G_F)$ is a soft open subset of $(X, \tau^\star, E, \preceq_1)$. Hence a soft map g_ϕ is soft I (resp. soft D, soft B) -continuous. □

Theorem 2.6. *Let an injective soft map* $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ *be soft* B*-continuous. If* $(Y, \theta, F, \preceq_2)$ *is a p-soft* T_i*-space, then* $(X, \tau, E, \preceq_1)$ *is p-soft* T_i*-ordered for* $i = 0, 1, 2$.

Proof. We prove the theorem in the case of $i = 2$ and the others follow similar lines.
For $a \not\preceq_1 b$ in X, there exist $x, y \in Y$ such that $g(a) = x, g(b) = y$ and $x \neq y$. By hypothesis, $(Y, \theta, F, \preceq_2)$ is p-soft T_2, there exist disjoint soft open sets G_F and H_F containing x and y, respectively. Since g_ϕ is soft B-continuous, then $g_\phi^{-1}(G_F)$ is soft I-open and $g_\phi^{-1}(H_F)$ is soft D-open subsets of $\widetilde{X}$. Obviously, $g_\phi^{-1}(G_F)\widetilde{\bigcap}g_\phi^{-1}(H_F) = \widetilde{\emptyset}$. It follows from Proposition (4), that $g_\phi^{-1}(G_F)$ and $g_\phi^{-1}(H_F)$ containing a and b, respectively. Hence $(X, \tau, E, \preceq_1)$ is p-soft T_2-ordered as required. □

Corollary 1. *Let an injective soft map* $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ *be soft* I*-continuous (resp. soft* D*-continuous). If* $(Y, \theta, F, \preceq_2)$ *is a lower (resp. an upper) p-soft* T_1*-space, then* $(X, \tau, E, \preceq_1)$ *is lower (resp. upper) p-soft* T_1*-ordered.*

Proposition 6. *If a bijective soft map* $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ *is soft* B*-continuous such that* θ *is not the indiscrete soft topology, then* $\preceq_1$ *is not linearly ordered.*

3 Soft λ-open and soft λ-closed maps

We begin this section by formulating the concepts of soft λ-open and soft λ-closed maps. Then we supply two examples to show the relationships among them. Finally, we give the equivalent conditions for each one of these soft maps and study the image of some soft separation axioms under these soft maps.

Definition 3.1. *A soft map* $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \tau, F, \preceq_2)$ *is called:*

(i) *Soft I (resp. Soft D, Soft B) -open if the image of every soft open subset of* $\widetilde{X}$ *is a soft I (resp. soft D, soft B) -open subset of* $\widetilde{Y}$.

(ii) *Soft I (resp. Soft D, Soft B) -closed if the image of every soft closed subset of* $\widetilde{X}$ *is a soft I (resp. soft D, soft B) -closed subset of* $\widetilde{Y}$.

Remark 3.1. *From Definition* (3.1)*, we can note the following:*

(i) *Every soft I (soft D, soft B) -open map is soft open.*

(ii) *Every soft I (soft D, soft B) -closed map is soft closed.*

(iii) *Every soft B-open map is soft I-open (soft D-open).*

(iv) *Every soft B-closed map is soft I-closed (soft D-closed).*

In the following two examples, we show that the converse of the three statements in remark above fails.

Example 3.1. *Let the two universe sets $X = \{h, i, j\}$, $Y = \{u, v, w, z\}$ and the two parameters sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$. Consider a map $\phi : A \to B$ is defined as, $\phi(a_m) = b_m$ for $m \in \{1, 2\}$ and a map $f : X \to Y$ is defined as, $f(h) = w$ and $f(i) = f(j) = z$. We define a partial order relation on Y as $\preceq = \triangle \bigcup\{(v, w), (z, u)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_A\}$ and $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, H_B, K_B, L_B, O_B\}$, where $F_A = \{(a_1, \{h, i\}), (a_2, \{i\})\}$, $H_B = \{(b_1, \{w, z\}), (b_2, \{z\})\}$, $K_B = \{(b_1, \{u, v, w\}), (b_2, \{u, v\})\}$, $L_B = \{(b_1, \{w\}), (b_2, \emptyset)\}$ and $O_B = \{(b_1, Y), (b_2, \{u, v, z\})\}$. Then one can readily check that a soft map $f_\phi : S(X_A) \to S(Y_B)$ is soft open and soft closed. On the other hand, $f_\phi(F_A) = H_B$ is neither a soft D-open nor a soft I-open set. Hence f_ϕ is not soft I (soft D, soft B) -open. Also, $f_\phi(F^c_A) = K^c_B$ is neither a soft D-open nor a soft I-closed set. Hence f_ϕ is not soft I (soft D, soft B) -closed.*

Example 3.2. *In Example above, if we replace only the partial order relation by $\preceq = \triangle \bigcup\{(v, w)\}$(resp. $\preceq = \triangle \bigcup\{(z, u)\}$), then the soft map f_ϕ is soft I-open and soft I-closed (resp. soft D-open and soft D-closed), but is not soft B-open and soft B-closed.*

Theorem 3.1. *The following three properties of a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ are equivalent:*

(i) *f_ϕ is soft I-open;*

(ii) *$int(f^{-1}_\phi(M_F))\widetilde{\subseteq} f^{-1}_\phi(M^{io}_F)$ for every $M_F\widetilde{\subseteq}\widetilde{Y}$;*

(iii) *$f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{io}$ for every $N_E\widetilde{\subseteq}\widetilde{X}$.*

Proof. **(i) ⇒ (ii)**: Given a soft subset M_F of $\widetilde{Y}$, it is obvious that $int(f^{-1}_\phi(M_F))$ is a soft open subset of $\widetilde{X}$. Then, by hypothesis, it follows that $f_\phi(int(f^{-1}_\phi(M_F)))$ is a soft I-open subset of $\widetilde{Y}$. Since $f_\phi(int(f^{-1}_\phi(M_F)))\widetilde{\subseteq} f_\phi(f^{-1}_\phi(M_F))\widetilde{\subseteq} M_F$, then $int(f^{-1}_\phi(M_F))\widetilde{\subseteq} f^{-1}_\phi(M^{io}_F)$.
(ii) ⇒ (iii): Given a soft subset N_E of $\widetilde{X}$, from **(ii)**, we obtain $int(f^{-1}_\phi(f_\phi(N_E)))\widetilde{\subseteq}\ f^{-1}_\phi((f_\phi(N_E))^{io})$. Since $int(N_E)\widetilde{\subseteq} f^{-1}_\phi(f_\phi(int(f^{-1}_\phi(f_\phi(N_E)))))\widetilde{\subseteq} f^{-1}_\phi((f_\phi(N_E))^{io})$, then $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{io}$ as required.
(iii) ⇒ (i): Let G_E be a soft open subset of $\widetilde{X}$. Then $f_\phi(int(G_E)) = f_\phi(G_E)\widetilde{\subseteq}(f_\phi(G_E))^{io}$. Hence f_ϕ is a soft I-open map. □

The following theorem can be proved in a similar manner.

Theorem 3.2. *The following three properties of a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ are equivalent:*

(i) *f_ϕ is soft D-open (resp. soft B-open);*

(ii) *$int(f^{-1}_\phi(M_F))\widetilde{\subseteq} f^{-1}_\phi(M^{do}_F)$(resp. $int(f^{-1}_\phi(M_F))\widetilde{\subseteq} f^{-1}_\phi(M^{bo}_F)$) for every $M_F\widetilde{\subseteq}\widetilde{Y}$;*

(iii) *$f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{do}$(resp. $f_\phi(int(N_E))\widetilde{\subseteq}(f_\phi(N_E))^{bo}$) for every $N_E\widetilde{\subseteq}\widetilde{X}$.*

Theorem 3.3. *The following three statements hold for a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$:*

(i) *f_ϕ is soft I-closed if and only if $(f_\phi(G_E))^{icl}\widetilde{\subseteq} f_\phi(cl(G_E))$ for every $G_E\widetilde{\subseteq}\widetilde{X}$.*

(ii) *f_ϕ is soft D-closed if and only if $(f_\phi(G_E))^{dcl}\widetilde{\subseteq} f_\phi(cl(G_E))$ for every $G_E\widetilde{\subseteq}\widetilde{X}$.*

(iii) *f_ϕ is soft B-closed if and only if $(f_\phi(G_E))^{bcl}\widetilde{\subseteq} f_\phi(cl(G_E))$ for every $G_E\widetilde{\subseteq}\widetilde{X}$.*

Proof. We only give a proof for the first statement and the others follow similar lines.
Necessity: Since f_ϕ is soft I-closed, then $f_\phi(cl(G_E))$ is a soft I-closed subset of $\widetilde{Y}$ and since $f_\phi(G_E)\widetilde{\subseteq}f_\phi(cl(G_E))$, then $(f_\phi(G_E))^{icl}\widetilde{\subseteq}f_\phi(cl(G_E))$.
Sufficiency: Consider H_E is a soft closed subset of $\widetilde{X}$. Then $f_\phi(H_E)\widetilde{\subseteq}(f_\phi(H_E))^{icl}\widetilde{\subseteq} f_\phi(cl(H_E)) = f_\phi(H_E)$. Therefore $f_\phi(H_E) = (f_\phi(H_E))^{icl}$. This means that $f_\phi(H_E)$ is a soft I-closed set. Hence the proof is complete. □

Theorem 3.4. *The following three statements hold for a bijective soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$:*

(i) *f_ϕ is soft I (resp. soft D, soft B) -open if and only if f_ϕ is soft D (resp. soft D, soft B) -closed.*

(ii) *f_ϕ is soft I (resp. soft D, soft B) -open if and only if f_ϕ^{-1} is soft I (resp. soft D, soft B) -continuous.*

(iii) *f_ϕ is soft D (resp. soft I, soft B) -closed if and only if f_ϕ^{-1} is soft I (resp. soft D, soft B) -continuous.*

Proof. For the sake of economy, we only give proofs of cases outside the parenthesis for the three statements in and the cases between parenthesis can be made similarly.

(i) To prove the necessary condition, let H_E be a soft closed subset of $\widetilde{X}$ and consider f_ϕ is a soft I-open map. Then H_E^c is soft open and $f_\phi(H_E^c)$ is soft I-open. It follows from the bijectiveness of f_ϕ, that $f_\phi(H_E^c) = [f_\phi(H_E)]^c$. This automatically implies that $f_\phi(H_E)$ is soft D-closed. Thus f_ϕ is a soft D-closed map. In a similar manner, we can prove the sufficiency condition.

(ii) Necessity: Let G_E be a soft open subset of $\widetilde{X}$ and consider f_ϕ is a soft I-open map. Then $f_\phi(G_E)$ is soft I-open. It follows from the bijectiveness of f_ϕ, that $f_\phi(G_E) = (f_\phi^{-1})^{-1}(G_E)$. This automatically implies that $(f_\phi^{-1})^{-1}(G_E)$ is soft I-open. Thus f_ϕ^{-1} is a soft I-continuous map. In a similar manner, we can prove the sufficiency condition.

(iii) The proof of this statement comes immediately from **(i)** and **(ii)** above.

□

Theorem 3.5. *If a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) -open and ϕ is an injective map, then a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -open.*

Proof. Let U be an open subset of $(X, \tau_e, \preceq_1)$ and $\phi(e) = f$. Then there exists a soft open subset G_E of $(X, \tau, E, \preceq_1)$ such that $G(e) = U$. Since g_ϕ is a soft I (resp. soft D, soft B) -open map, then $g_\phi(G_E)$ is a soft I (resp. soft D, soft B) -open set. From Definition (1.7), it follows that a soft subset $g_\phi(G_E) = (g_\phi(G))_F$ of $(Y, \theta, F, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{e\in\phi^{-1}(f)} g(G(e))$ for each $f \in F$. This implies a subset $\bigcup_{e\in\phi^{-1}(f)} g(G(e)) = g(U)$ of $(Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -open. Hence a map g is I (resp. D, B) -open. □

Theorem 3.6. *Let $\theta^\star$ be an extended soft topology on Y and ϕ be an injective map. Then a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) -open if and only if a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -open.*

Proof. The proof of the necessary condition is given above.
Sufficiency: Let G_E be a soft open subset of $(X, \tau, E, \preceq_1)$. Then from Definition (1.7), it follows that a soft subset $g_\phi(G_E) = (g_\phi(G))_F$ of $(Y, \theta^\star, F, \preceq_2)$ is given by $g_\phi(G)(f) = \bigcup_{e\in\phi^{-1}(f)} g(G(e))$ for each $f \in F$. Since a map g is I (resp. D, B) -open, then a subset $\bigcup_{e\in\phi^{-1}(f)} g(G(e))$ of $(Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -open. By hypothesis, $\theta^\star$ is an extended soft topology on Y, $g_\phi(G_E)$ is a soft subset of $(Y, \theta^\star, F, \preceq_2)$. Hence a soft map g_ϕ is soft I (resp. soft D, soft B) -open. □

The two theorems above are restated in the case of a soft I (resp. soft D, soft B) -closed map. One can prove them similarly and so the proof will be omitted.

Theorem 3.7. *If a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) -closed and ϕ is an injective map, then a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -closed.*

Theorem 3.8. *Let $\theta^\star$ be an extended soft topology on Y and ϕ is an injective map. Then a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) -closed if and only if a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta^\star_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -closed.*

Theorem 3.9. *Let a bijective soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ be soft B-open (resp. soft B-closed). If $(X, \tau, E, \preceq_1)$ is a p-soft T_i-space, then $(Y, \theta, F, \preceq_2)$ is p-soft T_i-ordered for $i = 0, 1, 2$.*

Proof. We prove the theorem in the case of $i = 2$ and the others follow similar lines.
For $x \not\preceq_1 y$ in Y, there exist $a, b \in X$ such that $g^{-1}(x) = a, g^{-1}(y) = b$ and $a \neq b$. By hypothesis, $(X, \tau, E, \preceq_1)$ is p-soft T_2, there exist disjoint soft open sets G_F and H_F containing a and b, respectively. Since g_ϕ is soft B-open, then $g_\phi(G_F)$ is soft I-open and $g_\phi(H_F)$ is soft D-open subsets of $\widetilde{Y}$. The bijective of g_ϕ implies that $g_\phi(G_F)\widetilde{\bigcap} g_\phi(H_F) = \widetilde{\emptyset}$. It follows from Proposition (4), that $g_\phi(G_F)$ and $g_\phi(H_F)$ containing x and y, respectively. Hence $(Y, \theta, F, \preceq_2)$ is p-soft T_2-ordered as required. □

Corollary 2. *Let a bijective soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ be soft I-open or soft D-closed (resp. soft D-open or soft I-closed). If $(X, \tau, E, \preceq_1)$ is a lower (resp. an upper) p-soft T_1-space, then $(Y, \theta, F, \preceq_2)$ is lower (resp. upper) p-soft T_1-ordered.*

Proposition 7. *Consider τ is not the indiscrete soft topology on X. If an injective soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft B-open or soft B-closed, then $\preceq_2$ is not linearly ordered.*

4 Soft λ-homeomorphism maps

In this section, the concepts of soft λ-homeomorphism maps are established and their main properties are discussed. The image and pre image of p-soft T_i and p-soft T_i-ordered under the these soft maps are investigated and some compositions soft maps are constructed.

Definition 4.1. *A bijective soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is called soft I (resp. soft D, soft B) - homeomorphism if it is soft I-continuous and soft I-open (resp. soft D-continuous and soft D-open, soft B-continuous and soft B-open).*

Remark 4.1. *From Definition* (4.1)*, we can note the following:*

(i) *Every soft I (soft D, soft B) -homeomorphism map is soft homeomorphism.*

(ii) *Every soft B-homeomorphism map is soft I-homeomorphism (soft D-homeomorphism).*

To elucidate that the two items of the remark above are not conversely, we give two examples below.

Example 4.1. *Let the two universe sets $X = \{h, i\}$, $Y = \{w, z\}$ and the two parameters sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$. Consider a map $\phi : A \to B$ is defined as, $\phi(a_m) = b_m$ for $m \in \{1, 2\}$ and a map $f : X \to Y$ is defined as, $f(h) = w$ and $f(i) = z$. We define two partial order relations on X and Y, respectively, as $\preceq_1 = \triangle \bigcup \{(h, i)\}$ and $\preceq_2 = \triangle \bigcup \{(z, w)\}$ and we define two soft topologies τ and θ on X and Y, respectively, as $\tau = \{\widetilde{\emptyset}, \widetilde{X}, F_A\}$ and $\theta = \{\widetilde{\emptyset}, \widetilde{Y}, H_B\}$, where $F_A = \{(a_1, X), (a_2, \{h\})\}$ and $H_B = \{(b_1, Y), (b_2, \{w\})\}$. Then one can readily check that a soft map $f_\phi : S(X_A) \to S(Y_B)$ is soft homeomorphism. On the other hand, $f_\phi(F_A) = H_B$ is not a soft D-open set and $f_\phi^{-1}(H_B) = F_A$ is not a soft I-open set. Hence f_ϕ is not soft I (soft D, soft B) -homeomorphism.*

Example 4.2. *In Example above, if we replace only the partial order relation $\preceq_2$ by $\preceq = \triangle \bigcup \{(w, z)\}$, then the soft map f_ϕ is soft D-homeomorphism, but is not soft B-homeomorphism. Also, if we replace only the partial order relation $\preceq_1$ by $\preceq = \triangle \bigcup \{(i, h)\}$, then the soft map f_ϕ is soft I-homeomorphism, but is not soft B-homeomorphism.*

Theorem 4.1. *Consider $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is a bijective soft map and let $(\lambda, \gamma) \in \{(I, dcl), (D, icl), (B, bcl)\}$. Then f_ϕ is soft λ-homeomorphism if and only if $(f_\phi(G_E))^\gamma = f_\phi(cl(G_E)) = cl(f_\phi(G_E)) = f_\phi(G_E^\gamma)$ for every $G_E \widetilde{\subseteq} \widetilde{X}$.*

Proof. We make a proof for the theorem in the case of $(\lambda, \gamma) = (I, dcl)$ and the other follow similar line.
Necessity: The property of f_ϕ is a soft I-homeomorphism map implies that $f_\phi(G_E^{dcl}) \widetilde{\subseteq} cl(f_\phi(G_E))$ and $(f_\phi(G_E))^{dcl} \widetilde{\subseteq} f_\phi(cl(G_E))$ for every $G_E \widetilde{\subseteq} \widetilde{X}$. So $f_\phi(cl(G_E)) \widetilde{\subseteq} f_\phi(G_E^{dcl}) \widetilde{\subseteq} cl(f_\phi(G_E)) \widetilde{\subseteq} (f_\phi(G_E))^{dcl}$ and $cl(f_\phi(G_E)) \widetilde{\subseteq} (f_\phi(G_E))^{dcl} \widetilde{\subseteq} f_\phi(cl(G_E)) \widetilde{\subseteq} f_\phi(G_E^{dcl})$. By the preceding two inclusion relations, we obtain the required equality relation.
Sufficiency: The equality relation $(f_\phi(G_E))^{dcl} = f_\phi(cl(G_E)) = cl(f_\phi(G_E)) = f_\phi(G_E^{dcl})$ implies that $f_\phi(G_E^{dcl}) \widetilde{\subseteq} cl(f_\phi(G_E))$ and $(f_\phi(G_E))^{dcl} \widetilde{\subseteq} f_\phi(cl(G_E))$. So f_ϕ is soft I-continuous and soft D-closed map. Hence the desired result is proved. □

Theorem 4.2. *If a bijective soft map $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I-continuous (resp. soft D-continuous, soft B-continuous), Then the following three statements are equivalent:*

(i) *f_ϕ is soft I-homeomorphism (resp. soft D-homeomorphism, soft B-homeomorphism);*

(ii) *f_ϕ^{-1} is soft I-continuous (resp. soft D-continuous, soft B-continuous);*

(iii) *f_ϕ is soft D-closed (resp. soft I-closed, soft B-closed).*

Proof. **(i) ⇒ (ii)** Since f_ϕ is a soft I-homeomorphism (resp. soft D-homeomorphism , soft B-homeomorphism) map, then f_ϕ is soft I-open (resp. soft D-open , soft B-open). It follows from item **(ii)** of Theorem (3.4), that f_ϕ^{-1} is soft I-continuous (resp. soft D-continuous, soft B-continuous).
(ii) ⇒ (iii) The proof follows from item **(iii)** of Theorem (3.4).
(iii) ⇒ (i) It sufficient to prove that f_ϕ is a soft I-open (resp. soft D-open, soft B-open) map. This follows from item **(i)** of Theorem (3.4). □

Theorem 4.3. *If a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft I (resp. soft D, soft B) -homeomorphism, then a map $g : (X, \tau_e, \preceq_1) \to (Y, \theta_{\phi(e)}, \preceq_2)$ is I (resp. D, B) -homeomorphism.*

Proof. The proof is obtained immediately from Theorem (2.4) and Theorem (3.5) □

Theorem 4.4. *Let $\tau^\star$ and $\theta^\star$ be extended soft topologies on X and Y, respectively. Then a soft map $g_\phi : (X, \tau^\star, E, \preceq_1) \to (Y, \theta^\star, F, \preceq_2)$ is soft I (resp. soft D, soft B) -homeomorphism if and only if a map $g : (X, \tau_e^\star, \preceq_1) \to (Y, \theta_{\phi(e)}^\star, \preceq_2)$ is I (resp. D, B) -homeomorphism.*

Proof. The proof is obtained immediately from Theorem (2.5) and Theorem (3.6) □

Theorem 4.5. *Let a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ be soft B-homeomorphism. Then $(X, \tau, E, \preceq_1)$ is a p-soft T_i-space if and only if $(Y, \theta, F, \preceq_2)$ is p-soft T_i-ordered for $i = 0, 1, 2, 3, 4$.*

Proof. The proof of the theorem in the cases of $i = 0, 1, 2$ follows from Theorem (2.6) and Theorem (3.9).
To prove the theorem in the case $i = 3$, let F_E be a soft closed subset of $\widetilde{X}$ such that $a \notin F_E$. Then $g_\phi(F_E)$ is a soft balancing closed subset of $\widetilde{Y}$ such that $g(a) \notin g_\phi(F_E)$. By hypothesis, $(Y, \theta, F, \preceq_2)$ is p-soft regular ordered, there exist disjoint soft open sets G_F and H_F containing $g(a)$ and $g_\phi(F_E)$, respectively. So $g_\phi^{-1}(G_F)$ and $g_\phi^{-1}(H_F)$ are disjoint soft open sets containing a and F_E, respectively. Thus the necessary condition holds. Conversely, let L_F be an increasing or decreasing soft closed subset of $\widetilde{Y}$ such that $x \notin L_F$. Say, L_F is increasing soft closed. Then $g_\phi^{-1}(L_F)$ is a soft closed set and $g^{-1}(x) \notin g_\phi^{-1}(L_F)$. By hypothesis, $(X, \tau, E, \preceq_1)$ is p-soft regular, there exist disjoint soft open sets G_E and H_E containing $g^{-1}(x)$ and $g_\phi^{-1}(L_F)$, respectively. So $g_\phi(G_F)$ and $g_\phi(H_F)$ are disjoint soft open sets containing x and L_F, respectively, such that $g_\phi(G_F)$ is decreasing and $g_\phi(H_F)$ is increasing. Hence the desired result is proved.
One can made similarly the theorem's proof for $i = 4$. □

Corollary 3. *Let a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ be soft I-homeomorphism (resp. soft D-homeomorphism). Then $(X, \tau, E, \preceq_1)$ is a lower (resp. an upper) p-soft T_1-space if and only if $(Y, \theta, F, \preceq_2)$ is lower (resp. upper) p-soft T_1-ordered.*

Corollary 4. *Let a soft map $g_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ be soft I-homeomorphism (resp. soft D-homeomorphism). Then $(X, \tau, E, \preceq_1)$ is an upper (resp. a lower) p-soft regular if and only if $(Y, \theta, F, \preceq_2)$ is an upper (resp. a lower) p-soft regular ordered.*

Definition 4.2. *The composition of two soft maps $g_\lambda : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ and $f_\phi : (Y, \theta, F, \preceq_2) \to (Z, \upsilon, K, \preceq_3)$ is a soft map $f_\phi \circ g_\lambda : (X, \tau, E, \preceq_1) \to (Z, \upsilon, K, \preceq_3)$ and is given by $(f_\phi \circ g_\lambda)(P_e^x) = f_\phi(g_\lambda(P_e^x))$.*

Proposition 8. *Let $f_\phi : (X, \tau, E, \preceq_1) \to (Y, \theta, F, \preceq_2)$ and $g_\eta : (Y, \theta, F, \preceq_2) \to (Z, \upsilon, K, \preceq_3)$ be two soft maps. Then then following properties hold for $\lambda \in \{I, D, B\}$.*

(i) *If g_η is a soft λ-homeomorphism map and f_ϕ is an increasing soft homeomorphism map, then $g_\eta \circ f_\phi$ is a soft λ-homeomorphism map.*

(ii) *If f_ϕ is a soft λ-continuous map and g_η is a soft continuous map, then $g_\eta \circ f_\phi$ is a soft λ-continuous map.*

(iii) *If f_ϕ is a soft open (resp. soft closed) map and g_η is a soft λ-open (resp. soft λ-open) map, then $g_\eta \circ f_\phi$ is a soft λ-open (resp. soft λ-closed) map.*

(iv) *If $g_\eta \circ f_\phi$ is a soft λ-open map and f_ϕ is surjective soft continuous, then g_η is a soft λ-open map.*

(v) *If $g_\eta \circ f_\phi$ is a soft closed map and g_η is an injective soft λ-continuous map, then f_ϕ is a soft γ-closed map, where $(\lambda, \gamma) \in \{(I, D), (D, I), (B, B)\}$.*

Proposition 9. *Let τ and θ be two soft topologies on X and Y, respectively, such that they do not belong to $\{$discrete soft topology, indiscrete soft topology$\}$. If a soft map $f_\phi : (X, \tau, E \preceq_1) \to (Y, \theta, F, \preceq_2)$ is soft B-homeomorphism, then $\preceq_1$ and $\preceq_2$ are not linearly ordered.*

Conclusion

Al-shami et al. [9] initiated the notion of a soft topological ordered space by equipping a soft partially ordered set with a soft topological space. As a continuation of this, we present in this manuscript, the concepts of soft λ-continuous, soft λ-open, soft λ-closed and soft λ-homeomorphism maps, where $\lambda \in \{I, D, B\}$. To obtain a deeper understanding of these concepts, we give some examples which show the relationships among them and illustrate that they are strictly stronger than their counterparts on topological ordered spaces. Also, we characterize each one of the introduced soft maps and deduce some results which connect them with those maps on topological ordered spaces. Moreover, we probe the behavior of p-soft T_i and p-soft T_i-ordered spaces under these soft maps and study some compositions of the introduced soft maps. In the end, we hope that the concepts presented herein give rise to constitute fundamental background for studying several topics on soft topological ordered spaces.

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