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On x-magnetic Surfaces Generated by Trajectory of x-magnetic Curves in Null Cone

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Abstract. In this work, we examine the impact of magnetic fields on the moving particle trajectories by variational approach to the magnetic flow associated with the Killing magnetic field on 2−dimensional lightlike cone $\mathbf{Q}^2 \subset E_1^3$. We give some characterizations for $x-$magnetic curve and $x-$magnetic surface of rotation using the Killing magnetic field of this curve in $\mathbf{Q}^2$ and we give the different types of axes of rotation, then creates three different types of magnetic surfaces of rotation in 2−dimensional lightlike cone $\mathbf{Q}^2 \subset E_1^3$.

Keywords: *Killing vector field, lightlike cone, magnetic curve, magnetic surface, W_x trajectory.*
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1 Introduction

Although we know much about the submanifolds of the pseudo-Riemannian space forms, we have very few papers on submanifolds of the pseudo-Riemannian lightlike cone. A simply connected Riemannian manifold of dimension $n \geq 3$ is conformally flat if and only if it can be isometrically immersed as a hypersurface of the lightlike cone and the Killing vector fields define the Killing magnetic fields. Working on the degenerate submanifolds of Lorentzian manifolds with degenerate metric provides us with meaningful relavance between null submanifolds and spacetime, [1]. Barros and Romeo proved in [2, 3], that if (M, g^*) has constant curvature, then the magnetic curves corresponding to a Killing magnetic field are centerlines of Kirchhoffelastic rods. V.N. Mishra and at al studied different metrics in different ambiant spaces [6, 15, 16, 18, 19, 20]. In [21], the author defined magnetic curves on a Riemannian manifold (M, g^*) according trajectories of charged particles moving on M under the action of a magnetic field F_*. In [4], Bozkurt et al. invastigated the magnetic flow associated with the Killing magnetic field in a three-dimensional oriented Riemann manifold (M^3, g^*). In [12], the author studied cone curves and investigate the notations of the cone curvature function and also gave some examples of cone curves in Minkowski space. In [5], they characterized the Killing vector fields on a Walker manifold M_f^3 aiming to obtain the corresponding Killing magnetic curves and characterized the normal magnetic curves corresponding to some Killing vector fields on M_f^3, obtaining their explicit expressions for certain functions f. In [7], the authors investigated magnetic curves corresponding to the Killing magnetic field W in the 3−dimensional Minkowski space. In [8], they determined all magnetic curves corresponding to the Killing magnetic fields on the 3−dimensional Euclidean space. In [13], the author gave the representation formulas of the curves in $\mathbf{Q}^2$ and $\mathbf{Q}^3$ and defined the functions of the cone curves

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2 Preliminaries

Let E_1^3 be the 3−dimensional pseudo-Euclidean space with the

$$g^*(V,W) = \langle V,W\rangle = v_1w_1 + v_2w_2 - v_3w_3$$

for all $V = (v_1, v_2, v_3)$, $W = (w_1, w_2, w_3) \in E_1^3$. E_1^3 is a flat pseudo-Riemannian manifold of signature $(2,1)$.

Let M be a submanifold of E_1^3. If the pseudo-Riemannian metric g^* of E_1^3 induces a pseudo-Riemannian metric $\widetilde{g}$ (respectively, a Riemannian metric, a degenerate quadratic form) on M, then M is called a timelike (respectively, spacelike, degenerate) submanifold of E_1^3. The lightlike cone is defined by

$$\mathbf{Q}^2 = \left\{x \in E_1^3 : g^*(x,x) = 0\right\}$$

Let E_1^3 be 3−dimensional Minkowski space and $\mathbf{Q}^2$ be the lightlike cone in E_1^3. A vector $W \neq 0$ in E_1^3 is called spacelike, timelike or lightlike, if $\langle W,W\rangle > 0$, $\langle W,W\rangle < 0$ or $\langle W,W\rangle = 0$, respectively. A frame field $\{x,\alpha,y\}$ on E_1^3 is called an asymptotic orthonormal frame field, if

$$\langle x,x\rangle = \langle y,y\rangle = \langle x,\alpha\rangle = \langle y,\alpha\rangle = 0, \langle x,y\rangle = \langle \alpha,\alpha\rangle = 1.$$

We assume that curve $x : I \to \mathbf{Q}^2 \subset E_1^3$ is a regular curve in $\mathbf{Q}^2$ for $t \in I$. In the following, we always assume that the curve is regular.

Using $x'(s) = \alpha(s)$, from an asymptotic orthonormal frame along the curve $x(s)$ and the cone Frenet formulas of $x(s)$ are given by

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ y'(s) &= -\kappa(s)\alpha(s) \end{aligned} \tag{1}$$

where the function $\kappa(s)$ is called cone curvature function of the curve $x(s)$, [12]. The Lorentzian cross-product $\times : E_1^3 \times E_1^3 \to E_1^3$ is defined by the formula

$$V \times W = \begin{bmatrix} i & j & -k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

where $V = (v_1, v_2, v_3)$, $W = (w_1, w_2, w_3) \in E_1^3$. Here i, j, k have usual meaning. We say that this product has similar algebraic properties as the cross product in E^3 . Hence, it is skew-symmetric and $V \times W$ is ortogonal on both V and W. The Lorentz force ψ of a magnetic field F_* on $\mathbf{Q}^2$ is defined to be a skew-symetric operator given by

$$g^*(\psi(V), W) = F_*(V,W)$$

for all $V, W \in \mathbf{Q}^2$. The $\alpha-$magnetic trajectories of F_* are x on $\mathbf{Q}^2$that satisfy the Lorentzian equation

$$\nabla_{x'}x' = \psi(x').$$

Furthermore, the mixed product of the vector fields $V, W, Z \in \mathbf{Q}^2$ is the defined by

$$g^*(V \times W, Z) = dv_{g^*}(V,W,Z),$$

where dv_{g^*} denotes a volume on $\mathbf{Q}^2$. If W is a Killing vector in $\mathbf{Q}^2$ and let $F_{*W} = \imath_W vol_{g^*}$ be the corresponding Killing magnetic field, here the inner product is indicated by $\imath$. Hence the equation Lorentz force of F_{*W} is

$$\psi(X) = W \times X$$

for all $X \in \mathbf{Q}^2$. Corresponding the Lorentz equation can be written as

$$\nabla_{x'}x' = \psi(x') = W \times x'.$$

In Minkowski space E_1^3, consider the Killing vector field $W = a\partial_x + b\partial_y + c\partial_z$, with $a, b, c \in \mathbb{R}$, the magnetic trajectories $x : I \to \mathbf{Q}^2 \subset E_1^3$ determined by W are solutions of the Lorentz equation

$$x'' = W \times x',$$

[21].

Definition 2.1. *A one-parameter group of diffeomorphisms of a manifold M is a smooth map $\psi : M \times \mathbb{R} \to M$, such that $\psi_t(x) = \psi(x,t)$, where*

$\psi_t : M \to M$ *is a diffeomorephism,*

$\psi_0 = id.$

$\psi_{s+t} = \psi_s o \psi_t.$

This group is accociated with a vector field V given by $\frac{d}{dt}\psi_t(x) = W(x)$, and the group of diffeomorphisms is called the flow of W. If a one-parameter group of isometries is generated by a vector field W, then this vector field is called a Killing vector field, [14].

3 x-magnetic Curves in the Null Cone $\mathbf{Q}^2 \subset \mathbf{E}_1^3$

In this section, we give some characterizations for $x-$magnetic curve and surface of rotation using the Killing magnetic field of this curve in $\mathbf{Q}^2 \subset E_1^3$.

Definition 3.1. *Let $x : I \to \mathbf{Q}^2 \subset E_1^3$ be a spacelike curve in $\mathbf{Q}^2$ and F_{*W} be a magnetic field on $\mathbf{Q}^2 \subset E_1^3$. We call the curve x a $x-$magnetic curve if its W_x vector field satisfies the Lorentz force equation*

$$\nabla_{x'} x = \nabla_\alpha x = \psi^x(x) = W_x \times x.$$

Theorem 3.2. *Let $x(s)$ be a unit speed spacelike $x-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{x, \alpha, y\}$. The Lorentz force in the Frenet frame are given as follows*

$$\psi^x = \begin{bmatrix} 0 & 1 & 0 \\ w_3 & 0 & -1 \\ 0 & -w_3 & 0 \end{bmatrix} \tag{2}$$

where w_3 is a function defined by $w_3 = g^*(\psi^x(\alpha), y)$.

Proof 3.3. *Let $x(s)$ be a unit speed spacelike $x-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{x, \alpha, y\}$. From the definition of the magnetic curve and equation* (2.1)*, we know that*

$$\psi^x(x) = \overrightarrow{\alpha}.$$

Furthermore, since $\psi^x(\alpha) \in Span\{x, \alpha, y\}$, we can write

$$\psi^x(\alpha) = A_1 \overrightarrow{x} + B_1 \overrightarrow{\alpha} + C_1 \overrightarrow{y}.$$

By using the following equalities

$$\begin{aligned} A_1 &= g^*(\psi^x(\alpha), y) = w_3; B_1 = g^*(\psi^x(\alpha), \alpha) = 0 \\ C_1 &= g^*(\psi^x(\alpha), x) = -g^*(\psi^x(x), \alpha) = -g^*(\alpha, \alpha) = -1, \end{aligned}$$

we get

$$\psi^x(\alpha) = w_3 \overrightarrow{x} - \overrightarrow{y}.$$

Similarly, we can easily obtain

$$\psi^x(y) = -w_3 \overrightarrow{\alpha}.$$

Theorem 3.4. *Let $x(s)$ be a unit speed spacelike $x-$magnetic curve in the $\mathbf{Q}^2 \subset E_1^3$. The curve x is then a $x-$magnetic trajectory of a magnetic vector field W_x if and only if the vector field W_x can be written along the curve x as the following*

$$W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}. \tag{3}$$

Proof 3.5. *Let $x(s)$ be a unit speed spacelike $x-$magnetic curve with $x-$magnetic trajectory of a magnetic field W_x. Using theorem 1 and definition 1, we can easily obtain that*

$$W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}.$$

Conversely, we assume that equation (3.2) *holds. Then we get $\psi^x(x) = W_x \times x$. Therefore the curve x is a $x-$magnetic trajectory of the magnetic vector field W_x.*

Theorem 3.6. *Let x be a $x-$magnetic trajectory according to the Killing vector field $W_x = \overrightarrow{x} + \overrightarrow{y}$ in $\mathbf{Q}^2 \subset E_1^3$. Then the curve x can be expressed as follows*

$$x_x(s) = x(0) + cW_x, \tag{4}$$

where $c \in \mathbb{R}_0$.

Proof 3.7. *We prove the theorem according to the W_x. Since W_x and $x(0)$ are linearly independent and W_x is spacelike. We consider $W_x, W_x \times x(0)$ and W^* be linearly independent and satisfy*

$$\langle W_x, W^* \rangle = 0, \langle W_x, W_x \times x(0) \rangle = 0, \langle W^*, W_x \times x(0) \rangle = 0.$$

So, we can take

$$W^* = 2x(0) - \langle W_x, x(0) \rangle W_x.$$

We can write

$$\begin{aligned} x(s) &= x(0) + \lambda_x(s)W_x + \mu_x(s)W_x \times x(0) + \rho_x(s)W^*, \\ x'(s) &= x'(0) + \lambda_x'(s)W_x + \mu_x'(s)W_x \times x(0) + \rho_x'(s)W^*, \end{aligned}$$

where $\lambda_x(s), \mu_x(s), \rho_x(s), \lambda_x'(s), \mu_x'(s), \rho_x'(s)$ functions satisfied the following

$$\lambda_x(0) = 0, \mu_x(0) = 0, \rho_x(0) = 0, \lambda_x'(0) = 0, \mu_x'(0) = 0, \rho_x'(0) = 0, \tag{5}$$

where $s = 0$. The Lorentz equation $x'(s) = W_x \times x(s)$ can be written as

$$x'(0) + \lambda_x'(s)W_x + \mu_x'(s)W_x \times x(0) + \rho_x'(s)W^* = W_x \times x(0) - \mu_x(s)W^* - \rho_x(s)W_x \times x(0),$$

since $x'(0) = W_x \times x(0)$ for $s = 0$, we have

$$0 = \lambda_x'(s)W_x + (\mu_x'(s) + \rho_x(s))W_x \times x_x(0) + (\mu_x(s) + \rho_x'(s))W^*,$$

which is equivalent to

$$\lambda_x'(s) = 0, \mu_x'(s) + \rho_x(s) = 0, \mu_x(s) + \rho_x'(s) = 0.$$

Solving the previous differential equations and using the initial conditions (3.4)*, we get*

$$\lambda_x(s) = c, \mu_x(s) = 0, \rho_x(s) = 0,$$

where $c \in \mathbb{R}_0$. Hence the curve x is written as $x_x(s) = x(0) + cW_x$.

4 Surface of Rotation Generated by The Lorentz force ψ^x with $x-$magnetic curve

Theorem 4.1. *Let x be a $x-$magnetic trajectory according to the killing vector field $W_x = \mp\kappa(\overrightarrow{x} + \overrightarrow{y})$ in $\mathbf{Q}^2 \subset E_1^3$. Then the following statements holds i) $\Delta_1^x(s,t)$ surface of rotation generated by the ψ^x can be expressed as follow*

$$\Delta_1^x(s,t) = \begin{pmatrix} (\frac{1+\cosh(\sqrt{2w_3}s)}{2} + \frac{1-\cosh(\sqrt{2w_3}s)}{2w_3})(t+b), \\ \frac{w_3-1}{\sqrt{w_3}} \sin(\sqrt{w_3}s), \\ (\frac{w_3(1-\cosh(\sqrt{2w_3}s))}{2} + \frac{1+\cosh(\sqrt{2w_3}s)}{2})(t+b) \end{pmatrix}$$

and the surface of revolution Δ_1^x *with* $x_1(t) = (t+b, 0, t+b), b \in \mathbb{R}_0$ *and* $-\infty < s < \infty, t \in I \subset \mathbb{R}$.

i_1) *If* $w_3 = cons \tan t$, *the surface of revolution* Δ_1^x *with* $x_1(t)$ *is minimal surface and the Gaussian and mean curvatures are* $K, H = 0$.

i_2) *If* $w_3 \neq cons \tan t$ *for* $-\infty < s < \infty, t \in I \subset \mathbb{R}$ *and for* $s, t = 0$, *the Gaussian and mean curvatures of the surface of revolution* Δ_1^x *with* $x_1(t)$ *are*

$$K = b\sqrt{2}\left(\frac{w_3''(w_3-1) - 2w_3'(\frac{w_3-1}{\sqrt{w_3}})'\sqrt{w_3}}{(2(w_3-1)^2 - w_3'^2b^2)(1-w_3^2) - (w_3w_3'b)^2}\right)$$

$$H = \frac{-\sqrt{2}}{2}\frac{\left(\begin{array}{c} w_3'(w_3-1)(1-w_3^2) \\ +b\left(\begin{array}{c} (2(w_3-1)^2 - w_3'^2b^2) \\ (-2(\frac{w_3-1}{\sqrt{w_3}})'w_3'\sqrt{w_3} + w_3''(w_3-1)) \end{array}\right) \end{array}\right)}{(2(w_3-1)^2 - w_3'^2b^2)(1-w_3^2) - (w_3w_3'b)^2}$$

i_3) *The condition of minimal surface of revolution* Δ_2^x *with* $x_2(t)$ *is*

$$\begin{aligned} &w_3''b(w_3-1)\left(w_3'^2b^2 - 2(w_3-1)^2w_3\right) + bw_3'^2(2(w_3-1)^2 - w_3'^2b^2) \\ &= w_3'w_3(w_3-1)(1-w_3^2), \end{aligned}$$

for $s, t = 0$.

ii) $\Delta_2^x(s,t)$ *surface of rotation generated by the* ψ^x *can be expressed as follow*

$$\Delta_2^x(s,t) = \left(\begin{array}{c} \frac{\sinh(\sqrt{2w_3}s)(t+b)}{2\sqrt{w_3}}, \\ \frac{(3-\cosh(\sqrt{2w_3}s))(t+b)}{2}, \\ \frac{-\sqrt{2w_3}\sinh(\sqrt{2w_3}s)(t+b)}{2} \end{array}\right),$$

where κ *is the curvature of the curve* x *and* $x_2(t) = (t+b, 0, t+b), b \in \mathbb{R}_0$ *and* $-\infty < s < \infty, t \in I \subset \mathbb{R}$.

ii_1) *the Gaussian and mean curvatures are given by*

$$K = -\frac{(t+b)^2w_3^2\cosh^4(\sqrt{2w_3}s)}{2\xi}$$

$$H = -\frac{\begin{array}{c}(t+b)^2\sqrt{w_3}\cosh^2(\sqrt{2w_3}s)(\sinh(2\sqrt{2w_3}s) \\ .(w_3 - 2w_3^2 + 1 - 6w_3\sinh(\sqrt{2w_3}s))\end{array}}{4w_3\xi},$$

where $w_3 = cons \tan t$ *and*

$$\xi = (t+b)^2\left(\begin{array}{c} \left(\begin{array}{c}(\frac{1}{2} - w_3^2)\cosh^2(\sqrt{2w_3}s) \\ +\frac{w_3}{2}\sinh^2(\sqrt{2w_3}s)\end{array}\right) \\ .\left(\frac{1}{4}(\frac{1}{w_3} - 2w_3)\sinh^2(\sqrt{2w_3}s) + \frac{(\cosh(\sqrt{2w_3}s)-3)^2}{4}\right) \\ -\left(\frac{(t+b)\sqrt{2w_3}}{4}\left(\begin{array}{c}(\frac{1}{2w_3} - w_3 + \frac{1}{2})\sinh(2\sqrt{2w_3}s) \\ -3\sinh(\sqrt{2w_3}s)\end{array}\right)^2\right) \end{array}\right)$$

ii_2) *The condition of minimal surface of revolution* Δ_2^x *with* $x_2(t)$ *is*

$$\cosh(\sqrt{2w_3}s) = \frac{3w_3}{1 + w_3 - 2w_3^2}.$$

Proof 4.2. *we look for a one parameter group of Lorentz of transformation which fixed all points on the* $xy-$*plane. This requires the Killing vector field to satisfy* $W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}$. *Hence, we can use a* 3×3 *matrix* ψ^x. *Now, we have the one parameter group of homomorphism* $\psi_s(x, \alpha, y)$ *given by* $\psi_s'(x) = \psi^x\psi_s(x)$. *So, we obtain* $\psi_s(x) = e^{s\psi_s}x$ *and calculating the matrix exponential, we have*

$$\Pi^x(s) = \begin{bmatrix} \frac{1+\cosh(\sqrt{2w_3}s)}{2} & \frac{\sinh(\sqrt{2w_3}s)}{\sqrt{2w_3}} & \frac{1-\cosh(\sqrt{2w_3})s}{2w_3} \\ \sqrt{w_3}\sin(\sqrt{w_3}s) & \frac{3-\cosh(\sqrt{2w_3}s)}{2} & \frac{-\sin(\sqrt{w_3}s)}{\sqrt{w_3}} \\ \frac{w_3(1-\cosh(\sqrt{2w_3}s))}{2} & \frac{-\sqrt{2w_3}\sinh(\sqrt{2w_3}s)}{2} & \frac{1+\cosh(\sqrt{2w_3}s)}{2} \end{bmatrix}$$

where $-\infty < s < \infty$. By the Lorentz force ψ^x rotation for $x-$magnetic curve we mean; rotating a curve using the rotation matrix (3.1)*, the plane of rotation is given by $W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}$. Clearly, any point in $\mathbf{Q}^2$ can be carried to the $xy-$plane by some notation, so we assume that the x curve lies in $xy-$plane. Hence, one of its parametrizations is*

$$x_1(t) = (t+b, 0, t+b), b \in \mathbb{R}_0.$$

Hence, the surface of revolution ${}_{W_x}\Delta_1^x$ around W_x can be parametrized as follow:

$$\Delta_1^x(s,t) = \Pi^x(s) \times \begin{bmatrix} t+b \\ 0 \\ t+b \end{bmatrix}$$

$$\Delta_1^x(s,t) = \begin{pmatrix} (\frac{1+\cosh(\sqrt{2w_3}s)}{2} + \frac{1-\cosh(\sqrt{2w_3}s)}{2w_3})(t+b), \\ \frac{w_3-1}{\sqrt{w_3}}\sin(\sqrt{w_3}s), \\ (\frac{w_3(1-\cosh(\sqrt{2w_3}s))}{2} + \frac{1+\cosh(\sqrt{2w_3}s)}{2})(t+b) \end{pmatrix},$$

where $-\infty < s < \infty, t \in I$. We examined the surface of revolution Δ_1^x for $s \neq 0$. For the gaussian and mean curvatures, we have

$$\begin{aligned} E &= (t+b)^2(1-w_3)^2\sinh^2(\sqrt{2w_3}s)(\frac{1}{4w_3} - \frac{w_3}{2}) \\ &\quad + (1-w_3)^2\cos^2(\sqrt{w_3}s)) \\ F &= \left(\vartheta\frac{(w_3-1)\sqrt{2w_3}}{2}(\frac{1}{w_3^2} - 1)\sinh(\sqrt{2w_3}s)(t+b)\right) \\ G &= \vartheta^2(\frac{1}{w_3^2} - 1); N = 0, M = 0, L = 0; \end{aligned}$$

$$n_{\Lambda_1^x} = \left(-(w_3-1)\cos(\sqrt{w_3}s)\vartheta, 0, -(w_3-1)\cos(\sqrt{w_3}s)\frac{\vartheta}{w_3}\right)$$

where $\frac{(w_3+1)}{2} + \frac{(w_3-1)}{2}\cosh(\sqrt{2w_3}s) = \vartheta$.

Thus, this results in the first fundemental form is $I_{\Delta_1^x} = EG - F^2 \neq 0$, for $s \neq 0$, $K, H = 0$. Hence, we say that the surface of revolution Δ_1^x with $x_1(t)$ is minimal surface.

For $s, t = 0$ and the function $w_3(s)$, we have

$$\begin{aligned} E &= 2(1-w_3)^2 - b^2w_3'^2, \\ F &= -bw_3w_3'; G = (1-w_3^2), \\ N &= b\sqrt{2}\begin{pmatrix} -2w_3'\sqrt{w_3}\left(\frac{w_3-1}{\sqrt{w_3}}\right)' \\ +w_3''(w_3-1) \end{pmatrix}, \\ M &= 0; L = \sqrt{2}(w_3-1)w_3'; \end{aligned}$$

$$n_{\Lambda_1^x} = \left(-\sqrt{2}w_3(w_3-1), -w_3', -\sqrt{2}(w_3-1)\right),$$

using the previous equations, we can write

i_1) If $w_3 = cons\tan t$, since the Gaussian and mean curvatures are $K, H = 0$, the surface of revolution Δ_1^x with $x_1(t)$ is minimal surface.

i_2) If $w_3 \neq cons\tan t$, for $s, t = 0$ the Gaussian and mean curvatures of the surface of revolution Δ_1^x with $x_1(t)$ are

$$K = b\sqrt{2}\left(\frac{w_3''(w_3-1) - 2w_3'(\frac{w_3-1}{\sqrt{w_3}})'\sqrt{w_3}}{(2(w_3-1)^2 - w_3'^2b^2)(1-w_3^2) - (w_3w_3'b)^2}\right)$$

$$H = \frac{-\sqrt{2}}{2}\frac{\begin{pmatrix} w_3'(w_3-1)(1-w_3^2) \\ +b\begin{pmatrix} (2(w_3-1)^2 - w_3'^2b^2) \\ (-2(\frac{w_3-1}{\sqrt{w_3}})'w_3'\sqrt{w_3} + w_3''(w_3-1)) \end{pmatrix} \end{pmatrix}}{(2(w_3-1)^2 - w_3'^2b^2)(1-w_3^2) - (w_3w_3'b)^2}.$$

The condition of minimal surface of revolution Δ_1^x *with* $x_1(t)$ *is*

$$\begin{aligned} &w_3''b(w_3-1)\left(w_3'^2b^2-2(w_3-1)^2w_3\right)+bw_3'^2(2(w_3-1)^2-w_3'^2b^2)\\ &=w_3'w_3(w_3-1)(1-w_3^2). \end{aligned}$$

Let be x *curve as* $x_2(t)=(0,t+b,0), b\in\mathbb{R}_0$. *Hence, using the axis of rotation* $x_2(t)$, *the surface of revolution* ${}_{W_x}\Delta_2^x$ *around* W_x *can be parametrized as follow:*

$$\Delta_2^x(s,t)=\Pi^x(s)\times\begin{bmatrix}0\\ t+b\\ 0\end{bmatrix}$$

$$\Delta_2^x(s,t)=\begin{pmatrix}\frac{\sinh(\sqrt{2w_3}s)(t+b)}{2\sqrt{w_3}},\\ \frac{3-\cosh(\sqrt{2w_3}s)(t+b)}{2},\\ \frac{-(t+b)\sqrt{2w_3}\sinh(\sqrt{2w_3}s)}{2}\end{pmatrix},$$

For the gaussian and mean curvatures, we have

$$\begin{aligned} E&=(t+b)^2\begin{pmatrix}(\frac{1}{2}-w_3^2)\cosh^2(\sqrt{2w_3}s)\\ +\frac{w_3}{2}\sinh^2(\sqrt{2w_3}s)\end{pmatrix}\\ F&=\frac{(t+b)\sqrt{2w_3}}{4}\begin{pmatrix}(\frac{1}{2w_3}-w_3+\frac{1}{2})\sinh(2\sqrt{2w_3}s)\\ -3\sinh(\sqrt{2w_3}s)\end{pmatrix}\\ G&=\frac{1}{4}(\frac{1}{w_3}-2w_3)\sinh^2(\sqrt{2w_3}s)+\frac{(\cosh(\sqrt{2w_3}s)-3)^2}{4}\\ M&=-\frac{\sqrt{2}w_3(t+b)}{2}\cosh^2(\sqrt{2w_3}s), N=0, L=0 \end{aligned}$$

$$n_{\Delta_2^x}=\begin{pmatrix}\frac{w_3(t+b)}{2}(1-3\cosh(\sqrt{2w_3}s)),\\ 0,\\ \frac{-\sqrt{2}(t+b)}{4}(1-3\cosh(\sqrt{2w_3}s))\end{pmatrix}$$

this results in the first fundamental form and the second fundamental form are given as

$$I_{\Delta_2^x}=\xi; II_{\Delta_2^x}=\frac{-w_3^2(t+b)^2}{2}\cosh^4(\sqrt{2w_3}s).$$

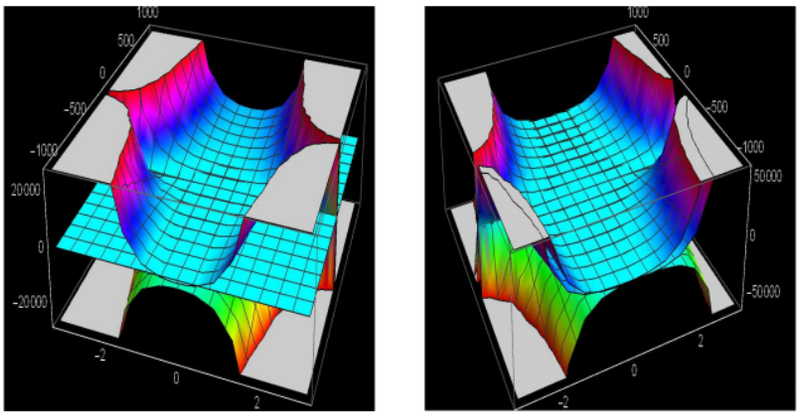
Therefore, the Gaussian and mean curvatures are given by

$$\begin{aligned} K&=\frac{-(t+b)^2w_3^2\cosh^4(\sqrt{2w_3}s)}{2\xi},\\ H&=\frac{-\begin{pmatrix}(t+b)^2\sqrt{w_3}\cosh^2(\sqrt{2w_3}s)(\sinh(2\sqrt{2w_3}s)\\ (w_3-2w_3^2+1-6w_3\sinh(\sqrt{2w_3}s))\end{pmatrix}}{4w_3\xi}, \end{aligned}$$

where

$$\begin{aligned} \xi=(t+b)^2&\begin{pmatrix}(\frac{1}{2}-w_3^2)\cosh^2(\sqrt{2w_3}s)\\ +\frac{w_3}{2}\sinh^2(\sqrt{2w_3}s)\end{pmatrix}\\ &\cdot\left(\frac{1}{4}(\frac{1}{w_3}-2w_3)\sinh^2(\sqrt{2w_3}s)+\frac{(\cosh(\sqrt{2w_3}s)-3)^2}{4}\right)-\\ &\left(\frac{(t+b)\sqrt{2w_3}}{4}\begin{pmatrix}(\frac{1}{2w_3}-w_3+\frac{1}{2})\sinh(2\sqrt{2w_3}s)\\ -3\sinh(\sqrt{2w_3}s)\end{pmatrix}\right)^2. \end{aligned}$$

Hence, we say that the surface of revolution Δ_2^x *with* $x_2(t)$ *is minimal surface for* $\cosh(\sqrt{2w_3}s)=\frac{3w_3}{1+w_3-2w_3^2}$.



(a) The surface of Δ_1^x (b) The surface of Δ_2^x

Figure 1: Graphics of surfaces Δ_1^x and Δ_2^x

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