General Letters in Mathematics Vol. 2, No. 2, April 2017, pp.99-104
e-ISSN 2519-9277, p-ISSN 2519-9269
Available online at http:// www.refaad.com

Duality Between Differentiability and Minimization in Banach Spaces

Nourddin Saidou *

National Institute of Applied Sciences, Euromed University of Fez
Route de Meknès - Morocco.
n.saidou@insa.ueuromed.org

Abstract. In this paper we will establish a new relationship between differentiability and optimization of convex functions. We consider convex lower semi continuous functions in Banach spaces and we characterize respectively the relation between Fréchet (Gâteaux) differentiability and *strong minimum (*weak minimum) in the dual.

Keywords: *Convex functions, strongly exposed, denting points, strong, weak minimum, differentiability, rotundity.*

1 Introduction

Several works had treated the geometrical properties of convex subsets in Banach spaces. Pionners in the convex analysis like Asplund [2], Collier [9], Bourgain [8], Li-Xin Cheng, Min Li [10], Bourgin [11], R. D. EHuff[3], Phelps [1] had studied the Radon Nikodym properties in duality with Asplund spaces. Very important aspects of points like extrem, exposed and denting points had been used to characterize subsets in Radon Nikodym Spaces. In this work, we will develop our last results about characterizations of geometrical properties of convex lower semi continuous functions. On one hand, we will examine the duality between differentiability of lower proper convex semi-continuous functions and on the other hand the existence of the *strong and *weak minimum of its conjugates. Our procedure is to work in epigraph of convex functions and epigraph of its conjugates. We introduce our study with the extremality such strongly exposed points, exposed points and denting points. In the same way to give the relation between all this points in the dual space and the Fréchet, Gâteau differentiability.

Our paper published in 2003 see [5] deals with the geometrical characterization like extremality in epigraph of convex proper lower semi continuous. We will use some results and corollary in order to prove our main results.

Let X a Banach space and $X*$ its dual. For each function f defined in X with values in $]-\infty,+\infty]$, we define:

$$dom f = \{x \in X / f(x) < +\infty\},$$

and

$$epi f = \{(x,r) \in X \times R / f(x) \leq r, x \in dom f\}.$$

A function is said to be proper if it has some finite values.

The conjugate of f called f^* defined in X^* with values in $]-\infty,+\infty]$ is.

$$f^*(x') = sup\{< x', x > -f(x), x \in X\}.$$

*Corresponding author.Nourddin Saidou [1]n.saidou@insa.ueuromed.org
2010 Mathematics Subject Classification: Primary: 46B20, Secondary: 46B22.

It is a convex proper lower semi continuous function in the topology (X^*, X) respect to the scalary product $<,>$. The subdifferential of f at x_0 is

$$\partial f(x_0) = \{x' \in X^*,\ < x', x_0 > -f(x_0) \geq < x', x > -f(x)\, \forall x \in X\}.$$

The r-subdifferential at x_0 is

$$\partial_r f(x_0) = \{x' \in X^*,\ < x', x_0 > -f(x_0) \geq < x', x > -f(x) - r\, \forall\, x \in X\}.$$

f is said rotund at x_0 if and only if there exists $x' \in X^*$ such that for all $\epsilon > 0$ there exists $r > 0$ such that $f(x) - f(x_0) - r \leq < x', x > - < x', x_0 >$ implies $\|x - x_0\| \leq \epsilon$.

f is said to be weakly rotund at x_0 if and only if for all $\epsilon > 0$ there exists $0 < r < \epsilon$ such that, for each $x' \in \partial_r f(x_0)$, $f^*(x') - r \leq < x', x > -f(x)$ implies that $\|x - x_0\| \leq \epsilon$. Let C be a nonempty closed convex subset of X. We define the indicator function as

$$\delta_C(x) = 0 \ \ if \ \ x \in C \ and \ \delta_C(x) = +\infty \ else.$$

Its conjugate in X^* is

$$\delta_C^*(x') = sup_{x \in C}(< x', x >) \ for\, all\, x' \ \in X^*.$$

f has a strong minimum at x_0 if
i) $f(x_0) \leq f(x)$ for all $x \in X$ and,
ii) If $f(x_n) \to f(x_0)$ then, $x_n \to x_0$ for all $(x_n) \in X$.

f is said to have a weak minimum at x_0 if and only if there exists $(x'_n)_n \in X^*$ such that

$$|f^*(x'_n) - (x'_n - f)(x_0)| \to 0,$$

and

$$if \ \ |f^*(x'_n) - (x'_n - f)(x_n)| \to 0 \ then, \ \ x_n \to x_0 \ \forall\, (x_n) \in X.$$

We say that $(x'_n)_n$ is a minimizing subsequence of f. It is easy to verify that if $f - x'$ has a strong minimum at x_0 then f has a weak minimum at x_0.

Let A a nonempty closed convex subset in X. For $r > 0$ and $x' \in X^*$, we define a slice defined by x' as:

$$T(A, x', r) = \{x \in X/ \ \delta_A^*(x') - r \leq < x', x >\}.$$

An element e is said to be exposed by $x' \in X^*$ if and only if:

$$< x', e >< < x', x > \ \ \forall\, x \neq e \ and \ \delta_C^*(x') =< x', e > .$$

An element e is said strongly exposed by $x' \in X^*$ if and only if one of the equivalents statement below holds:
1) $\delta_C^*(x') =< x'.e >$ and $\forall\, x_n \in X \quad < x', x_n > \to < x', >$ implies that $x_n \to e$ respect to the norm in X.
2) There exists $x' \in X^*$ such that for all $\epsilon > 0$ $T(A, x', r)$ contains e and has diameter less than ϵ. We recall that the diameter of a subset K is $diam(K) = sup_{x,y}\|x - y\|$.

An element e is said a denting point of A if and only if for each $\epsilon > 0$, there exists a slice which contains e and its diameter is small than ϵ.

We denote by $s - exp(A)$ the set of strongly exposed points and $dent(A)$ the set of denting points of A in the duality (X, X^*). Similarly, we denote by $*s - exp(A)$ the set of *strongly exposed points and *dent(A) the set of *denting points of A in the duality (X^*, X).

Let $x_0 \in X$. The function f is said to be strictly convex at x_0 if and only if, for all $t \in]0, 1[$:
$f(tx + (1-t)y) < tf(x) + (1-t)f(y)\ \ \forall x, y$ such that $x_0 = tx + (1-t)y$.

It is said to be Gâteaux differentiable at x_0 if,

$$f'(x_0).x = lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists for all $x \in X$ and $f'(x_0) \in X^*$.

It is said to be Fréchet differentiable at x_0 if: there exists $f'(x_0) \in X^*$ such that:

$$lim_{x\to 0}\frac{f(x_0+x)-f(x_0)-f'(x_0).x}{\|x\|}=0.$$

Now, we can announce some preliminary results that characterize the relation between strict convexity, rotundity, subdifferentiability and extremality (exposed and denting points).

Lemma 1.1. *Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$. If $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in the $epif$, then $x' \in \partial f(x_0)$.*

Proof. $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $epif$ then,

$$< (x', -1), (y, \lambda) > \leq < (x', -1), (x_0, f(x_0)) > \ \ \forall\ (y, \lambda) \in epif.$$

In particular, $(x, f(x)) \in epif$ then,

$$< (x', -1), (x, f(x)) > \leq < (x', -1), (x_0, f(x_0)) > \ \ \forall x \in X.$$

Hence for all $x \in X$:

$$< x, x_0 > -f(x_0) \geq < x', x > -f(x).$$

Then $x' \in \partial f(x_0)$.

Lemma 1.2. *If f is rotund at x_0 then $\partial f(x_0)$ is noempty.*

Proof. If f is rotund at x_0 then there exists $x' \in X^*$, and for all $\epsilon > 0$ there exists $r > 0$ such that $f(x) - f(x_0) - r \leq < x', x > - < x', x_0 >$ implies that $\|x - x_0\| \leq \epsilon$.

Let $x \neq x_0$. There exists $\epsilon > 0$ such that $\|x - x_0\| > \epsilon$, then there exists $r > 0$ such that:

$$< x', x > -f(x) + r \leq < x', x_0 > -f(x_0).$$

We conclude that $x' \in \partial f(x_0)$. Now we prove that the rotundity is a geometrical property stronger than strict convexity.

Lemma 1.3. *If f is rotund on x_0 then it is strictly convex at x_0.*

Proof. Suppose that f is rotund at $x_0 = tx_1 + (1-t)x_2$ such that $t \in]0, 1[$ and $x_1 \neq x_2$. We suppose that $f(x_0) = tf(x_1) + (1-t)f(x_2)$. $x_0 \neq x_i$ (for $i = 1, 2$), then there exists $\epsilon > 0$ such that $\|x_i - x_0\| > \epsilon$.
Since f is rotund at x_0, there exists $r > 0$ such that:

$$< x', x_i > -f(x_i) + r \leq < x', x_0 > -f(x_0), i = 1, 2.$$

We multiply the inequality (for $i = 1$) by t and the inequality (for $i = 2$) by $(1-t)$ and we add each other to obtain

$$< x', x_0 > -f(x_0) + r \leq < x', x_0 > -f(x_0).$$

So $r \leq 0$, absurd.

Remark 1.1. *It is easy to verify these inclusions:*
1) $s - exp(epif) \subset exp(epif)$.
2) $s - exp(epif) \subset dent(epif)$.

Now, we will be able to demonstrate two important propositions. The first one characterizes the relation between rotundity, strong exposed points and strong minimum of f regularized by x'. The second proposition characterize the relation between weak rotundity, denting points and the weak minimum of f in X.

Proposition 1.1. *Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$. The statements below are equivalents:*
1) There exists $x' \in \partial f(x_0)$ such that $f - x'$ has a strong minimum at x_0.
2) $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in epif.
3) f is rotund at x_0.

Proof. We demonstrate implications: 1) implies 2), 2) implies 3) and 3) implies 1). In the first one, suppose that $x' \in \partial f(x_0)$ such that $f - x'$ has a strong minimum at x_0. let $\epsilon > 0$ and $r \leq min(\frac{\epsilon}{\|x'\|}, \epsilon)$. The definition of strong minimum implies that, there exists $\delta > 0$ and if $|f(x) - f(x_0) - (< x', x > - < x', x_0 >)| \leq \delta$ then $\|x - x_0\| \leq r$ for all $x \in X$. Hence, it is easy to see that $T(epif, (x', -1), \alpha)$ has a diameter less than ϵ where $\alpha \leq min(\frac{\epsilon}{2}, \delta)$. Consequently $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $epif$: (see the second inclusion in Remark 1.1).
2) implies 3). We suppose that $(x_0, f(x_0))$ is strongly exposed by $(x', -1)$ in $epif$. Let $\epsilon > 0$ and α such that the diameter of $T(epif, (x', -1), \alpha)$ small than ϵ. Each $(x, f(x))$ such that $< x', x > -f(x) \geq < x', x_0 > -f(x_0) - \alpha$ is in the $T(epif, (x', -1), \alpha)$, also $(x_0, f(x_0))$ is in the same slice, so we can confirm that

$$\|x - x_0\| \leq \epsilon,$$

which prove the rotundity of f at x_0.

For the implication 3) implies 1), we can remark that $x' \in \partial f(x_0)$ and the second condition is evident.

Proposition 1.2. *Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$. Statements below are equivalents:*
1) f has a weak minimum at x_0.
2) $(x_0, f(x_0))$ is a denting point in $epif$.
3) f is weakly rotund at x_0.

Proof. We start with the equivalence 3) equivalent 2). Suppose that f is weak rotund at x_0. Let $\epsilon > 0$, since f is continuous at x_0, then there exists $\delta > 0$ such that $|f(x) - f(x_0)| \leq \epsilon$ when $\|x - x_0\| \leq \delta$.

If we use the weak rotundity for $\epsilon_1 = min(\delta, \frac{\epsilon}{4})$, we verify easily that the diameter of the slice $T(epif, (x', -1), \alpha)$ is less than ϵ when $\alpha \leq \epsilon_1$ and $x' \in \partial f(x_0)$. Inversely, if $(x_0, f(x_0))$ is a denting point in $epif$ then $\forall \epsilon > 0$, there exists $r \leq \epsilon$ such that $(x_0, f(x_0)) \in T(epif, (x', -1), \alpha)$ with a diameter less than ϵ. Let $x' \in \partial_r f(x_0)$ and $< x', x > -f(x) \geq f^*(x') - r$. Then, $(x, f(x)) \in T(epif, (x', -1), r)$, hence $\|x - x_0\| \leq \epsilon$, which prove that f is weakly rotund at x_0.
Now we prove that 1) implies 3).
Suppose that f is weakly rotund at x_0, then, $\forall n \in N$, there exists $x'_n \in \partial_{\frac{1}{n}} f(x_0)$ such that: for all $x \in X$, $(x'_n - f)(x) \geq f^*(x') - \frac{1}{n}$ implies $\|x - x_0\| \leq \epsilon$. Since $x'_n \in \partial_{\frac{1}{n}} f(x_0)$ then we have

$$inf_X(f - x'_n) \leq (f - x'_n)(x_0) \leq inf_X(f - x'_n) + \frac{1}{n},$$

which implies that:

$$|inf_X(f - x'_n) - (f - x'_n)(x_0)| \to 0.$$

Now if $(x'_n)_n$ is a subsequence such that $|inf_X(f - x'_n) - (f - x'_n)(x_n)| \to 0$, then for a rank, we have $f(x_n) - < x', x_n > \leq inf_X(f - x'_n) + \frac{1}{n}$, which it means that $f^*(x'_n) - \frac{1}{n} \leq < x', x_n > -f(x_n)$. Then $\|x_n - x_0\| \leq \frac{1}{n}$.
In the other way, we suppose that f has a weak minimum at x_0. Let $\epsilon > 0$ and $(x'_n)_n$ a subsequence minimizing f. let n such that $\frac{1}{n} \leq \epsilon$ and $x'_n \in \partial_\epsilon f(x_0)$. Let $x \in X$ with $< x'_n, x > -f(x) \geq f^*(x'_n) - \epsilon$.
Since $f^*(x'_n) \geq < x'_n, x > -f(x)$, we have $\|(x'_n - f)(x) - f^*(x'_n)\| \leq \epsilon$.
Hence, $\|x - x_0\| \leq \epsilon$ using the definition of the weak minimum.

Now, in order to introduce to our main results in this paper, we will first announce a proposition see [5] that illustrate the relation between Fréchet (resp. Gâteaux) differentiability of f in the space X and *strongly exposed (resp. *exposed) points in $epif^* \subset X^* \times R$. So we will consider the duality $(X^* \times R, X \times R)$.

Proposition 1.3. *(See [5]) Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$ and $x_0 \in intdomf$. Statements below are equivalents:*
1) f Fréchet (resp.Gâteaux) differentiable at x_0 and $df(x_0) = x'$.
*2) $(x', f^*(x))$ is *strongly exposed (resp. *exposed) point by $(x_0, -1)$ in $epif^*$.*

Proof. For the proof we refer to [5]. In the same way and using the paper [5], we announce an important characterization showing the relationship between Gâteau differentiability and *denting points.

Proposition 1.4. *(See [5]) Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$ and $x_0 \in intdomf$.*
*If f is Gâteaux differentiable at x_0 and $df(x_0) = x'$ then $(x', f^*(x'))$ is a *denting point in $epif^*$.*

Proof. For the proof we refer to our paper see [5].
Now, with all these preliminary results proved as above, we can announce our main results in this work,

Remark 1.2. **strong minimum (resp. *weak minimum) of f^* are used with same definitions above in X^* according to the duality (X^*, X).*

Theorem 1.4. *Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$ and $x_0 \in intdomf$. The following statements are equivalents.*
1) f Fréchet differentiable at x_0 and $df(x_0) = x'$.
*2) $f^* - x_0$ has a *strong minimum in X^* at x'.*
3) f^* is rotund at x'.

Proof. The proof is immediately a consequence of Proposition 1.1 and Proposition 1.3 as above.

Example 1.5. *We give an illustration in $X = \mathbb{R}$. we consider $f(x) = e^x$. It is a proper convex continuous function in $\mathbb{R}$. Its conjugate is:*

$$f^*(x^*) = x^* log x^* - x^* \ \forall\, x^* > 0.$$

$$f^*(x^*) = 0 \ if \ x^* = 0.$$

$$f^*(x^*) = +\infty \ else.$$

For example.
1) f is differentiable at 1 and $df(1) = e$. So it is easy to verify that $f^*(x^*) - 1.x^*$ admits a strong minimum at e and it is also easy to verify that f^* is rotund at e.
2) f is differentiable at 3 and $df(3) = e^3$. So it is easy to verify that $f^*(x^*) - 3.x^*$ admits a strong minimum at e^3 and it is also easy to verify that f^* is rotund at e^3.

The second theorem that we will announce below gives us just a sufficient condition instead of equivalence between Gâteau differentiability and *weak properties of f^*(*weak minimum, *weak rotundity).

Theorem 1.6. *Let f a convex proper lower semi continuous (lsc) function in X with values in $]-\infty,+\infty]$ and $x_0 \in intdomf$.*
If f is Gâteaux differentiable at x_0 and $df(x_0) = x'$ then we have the two equivalent statements below.
*1) f^* has a *weak minimum in X^* at x'.*
*2) f^* is *weakly rotund at x'.*

Proof. The proof is immediately a consequence from Proposition 1.2 and Proposition 1.4 as above.

ACKNOWLEDGEMENTS. The author would like to thank the referee for valuable comments.

References

[1] R. R. Phelps. *Convex functions, monotone operators and differentiability*, Lectures Notes in Math., 1364, 1989.

[2] E. Asplund and R.T Rockafellar. *Gradient of convex functions*, Trans. Amer. Math. Soc., 139:443–467, 1969.

[3] R. E. Huff and P. D. Morris. *Geometric Characterizations of the Radon Nikodym property in Banach spaces*, Studia Mathematica, 56:157–164, 1976.

[4] C. Stegall. *The Radon Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc., 364:507–516, 1981.

[5] N. Saidou, A. Bourass and B. Ferrahi. *Extremality and Differentiability*, Bulletin of The Polich Academy of Sciences, 51(1):13–21, 2003.

[6] N. Saidou, A. Bourass and A. Benlaarabi. *Fonctions convexes et propriéé de Radon Nikodym*, Bull. Belg. Math. Soc. Simon Stevin, 10(1):59–64, 2003.

[7] E. Asplund. *Frechet differentiability of convex functions*, Acta Math., 121:31–47, 1968.

[8] J. Bourgain. *On dentability and the Bishop-Phelps property*, Israel J. Math., 28(4):265–271, 1977.

[9] James B. Collier. *The Dual of a space with the Radon Nikodym property*, Pacific Journal of Mathematics, 64(1):103–106, 1976.

[10] Li-Xin Cheng and Min Li. *Extrem points, Exposed Points, Differentiability Points In CL - Space*, Proc. Amer. Math. Soc., 136(7):2445–2451, 2008.

[11] R. D. Bourgin. *Geometric aspects of convex sets with the Radon Nikodym property*, Lecture Notes in Mathematics, 993. Springer-Verlag, Berlin, 1983.

[12] M. Bachir and A. Danilidis. *A dual Characterization of the Radon Nikodym Property*, Bull. Austral. Math. Soc., 62:379–387, 2000.

[13] R. R. Phelps. *Dentability and extreme points in Banach spaces*, J. Functional Analysis. 17:78–90, 1974.