Available online at www.refaad.com
Uncertainty, 1(1)(2024), 32-41
Research Article

Uncertainty

Journal Homepage: http://www.refaad.com/Journal/Index/21

Efficient One-Step Second Derivative Block Numerical Scheme for Solution of first-Order Integro-differential Equations

Mark I. Modebei[a,b,*], Olumide O. Olaiya[a], Olugbade E. Faniyi[c], Ikechukwu J. Otaide[d]

[a] *Mathematics Programme, National Mathematical Centre, Abuja, Nigeria.*
[b] *Department of Mathematics, University of Abuja, Nigeria.*
[c] *Department Of Mathematics, Anchor University, Lagos, Nigeria.*
[d] *Department of Mathematics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria.*

Abstract

In this work, a new class of one-step second derivative block hybrid numerical scheme is developed for the numerical solution of first-order integro-differential equations with the aid of Simpson's $1/3$ quadrature method. The collocation techniques is used for the derivation of this new block method which gives high order of accuracy with very low error constants and its zero stable and convergent. The results from the numerical solution of the examples drawn from literature shows its efficiency in terms of the small errors obtained.

Keywords: Initial value problems, linear multistep method, block method, integro-differential equation.
2010 MSC: 65L05, 65L06, 65L10, 65L1

1. Introduction

An nth order integro-differential equation is of the form

$$y^{(n)} = \sum_{i=0}^{n-1} g_i(x) y^{(i)}(x) + f(x) + \lambda \int_a^b K(x,t) y(t) dt, \quad a \leqslant x, t \leqslant b, \tag{1.1}$$

where $y(x) \in \mathbf{C}(a,b)$ and square integrable on $[a,b]$, is an unknown function to be determined, $f(x)$, $g_i(x)$, $i = 0(1)n-1$ are known continuous functions, $K(x,t)$ is a known kernel, $\lambda \in \mathbf{R}$ is a known parameter. We thus in this work consider (1.1) for $n = 1$, the form of integro-differential equations of the form, (see [1]),

$$y' = y(x) + \lambda \int_a^b K(x,t) y(t) dt, \quad a \leqslant x \leqslant b, \tag{1.2}$$

For simplicity sake, we write (1.2) compactly as

$$\begin{aligned} y' &= f(x, y(x), z(x)), \quad a \leqslant x \leqslant b, \\ z(x) &= \textstyle\int_a^b K(x,t,y(t)) dt. \end{aligned} \tag{1.3}$$

*Corresponding author
Email addresses: gmarc.ify@gmail.com (Mark I. Modebei), ooliaya@gmail.com.com (Olumide O. Olaiya), ofaniyi@aul.edu.ng (Olugbade E. Faniyi), otaideikechukwu@gmail.com (Ikechukwu J. Otaide)

doi:10.31559/uncertainty2024.1.1.5

Received Received: 21 Sep 2024 Revised: 20 Oct 2024 Accepted: 5 Nov 2024

The assumption is that f and K are continuous and thus satisfy the following Lipschitz conditions:

$$\begin{array}{l}|f(x,y_1,z)-f(x,y_2,z)| \leqslant \gamma_1|y_1-y_2| \\ |f(x,y,z_1)-f(x,y,z_2)| \leqslant \gamma_2|z_1-z_2| \\ |K(x,t,y_1)-K(x,t,y_2)| \leqslant \gamma_3|y_1-y_2|\end{array} \tag{1.4}$$

where for small constants γ_i, $i=1,2,3$ and that, (1.2) has a unique solution in $[a,b]$.
It should also be noted that an extension of combination of Fredholm and Volterra known as Fredholm-Volterra integro-differential equations of the form

$$\begin{array}{l}y' = f(x,y(x),z_{fv}(x)), \quad a \leqslant x \leqslant b, \\ z_{fv}(x) = \int_{a_1}^{b} K_f(x,t,y(t))dt + \int_{a_2}^{b(x)} K_v(x,t,y(t))dt + .\end{array} \tag{1.5}$$

is also considered.

Integral equations results from fields of science and having wide range of applications in fluid mechanics, electrical engineering, heat/mass transfer biomechanics, etc [2]. on the other hand, integro-differential equations model many phenomena in sciences. Examples include the model of nonlinear dispersive waves in fluid dynamics, the Wilson-Cowan models describes inhibitory and excitatory neurons interaction which results into system of integro-differential equations, Kirchhoff's second law in alternating current (AC) circuit in electricity, the Kermack-Mckendrick theory of infectious disease transmission in epidemiology, [3]. There are limited approach and technique employed to obtaining the solution of integro-differential equation analytically and thus, development of numerical solution to this kind of models have been on the rise. Some numerical techniques include using the matrix relation to convert IDE to system of algebraic equation as seen in [3], Hermite Collocation Method was applied to obtain approximate of linear Fredholm Integro-Differential Equations in [4], Tau method used to produce approximate polynomial solutions in [5]. A thorough study of Most of this methods mentioned mentioned have overlapping of pieces of solutions nd thus solutions obtained are at prescribed points only. In the same way, they rarely estimate error bound for their methods. All these are taken care of using the block method especially approximating the integral part of the equation using the Simpson rule for integrals. The advantage of using the block methods are that, they are self-starting. They also have the advantage of producing smaller global errors (at the end of the range of integration) than those produced by the step-by-step methods due to the presence of accumulated errors at each step in the step-by-step method. The accuracy is thus improve with the presence of second derivatives in the block scheme. The numerical approach in this work takes care of both Fredholm Integro-Differential Equations, Volterra Integro-Differential Equations and the Fredholm-Volterra Integro-Differential Equations which other methods mentioned above lacked.

The numerical approximation of the integral part in (1.2) will be treated with the 1/3 Simpson's quadrature for multiple segments. Simpsons quadrature can be used to obtain better numerical approximation by dividing the integration interval into a number of segments of equal width as in $h = \frac{b-a}{N}$. Thus, given the integration interval $[a,b]$ with $a = x_0 < x_1 < x_2 < \ldots < x_{n-2} < x_{n-1} < x_n = b$, we can obtain from the 1/3 Simpson's three points the following

$$\begin{array}{rl}\int_a^b f(x)dx \approxeq & (b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6} + (b-a)\frac{f(x_2)+4f(x_3)+f(x_4)}{6} \\ & + \cdots + (b-a)\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\end{array}$$

$$\int_a^b f(x)dx \approxeq (b-a)\frac{f(x_0)+4\sum_{i=1,3,5,\ldots}^{n-1} + 2\sum_{i=2,4,6,\ldots}^{n-2} + f(x_n)}{6n}$$

$$\int_a^b f(x)dx \approxeq \frac{h}{3}\left(f(x_0) + 4\sum_{i=1,3,5,\ldots}^{n-1} + 2\sum_{i=2,4,6,\ldots}^{n-2} + f(x_n)\right) \tag{1.6}$$

When $b = x$, then (1.6) becomes

$$\int_a^x f(x)dx \approxeq \frac{h}{3}\left(f(x_0) + 4\sum_{i=1,3,5,\ldots}^{n-1} + 2\sum_{i=2,4,6,\ldots}^{n-2} + f(x_n)\right) \tag{1.7}$$

In this work, a continuous linear multistep method is derived via a block technique, which is used to formulate continuous block finite difference methods, using power series polynomials as basis functions. Thus, using multistep interpolation and collocation, a continuous finite difference method is derived which are assembled and solved simultaneously to obtain approximate solution of (1.1) at points x_n, $n = 1, \ldots, N-1$

2. Derivation of the method

In this section, we derive a continuous hybrid method which comprise of the main and additional methods. We seek an approximate solution $u(x)$ in the interval $[x_n, x_{n+1}]$, where $n = 0(1)N$. We proceed by considering the approximate solution of the form

$$\Upsilon(x) \approx p(x) = \sum_{k=0}^{d} \nu_k \Gamma_k(x) \tag{2.1}$$

where $\Gamma_k(x)$ with $k = 0, 1, 2, \ldots$ are polynomial basis functions of degree d, $x \in [a, b]$ with the integration interval $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ and ν_k are unknown coefficients. Here, $\Gamma_k(x) = x^k$ and letting the approximate solution $u(x)$ of (1.1) be given as $u(x_{n+k})$ at x_{n+k} with the fact that $f_{n+k} = f(x_{n+k}, u(x_{n+k}), z(x_{n+k})) = p'(x_{n+k})$, $g_{n+k} = f'(x_{n+k}, u(x_{n+k}), z(x_{n+k}))$, $z(x_{n+k}) = \int_a^b K(x_n, t, y(t))dt$, where $x_{n+k} = x_n + kh$, with $h = x_{k+1} - x_k$ is the constant step-size.

Specifically, we derive a one step method with two off-step points clustered at mid point value. Evaluating (2.1) at the points $x = x_n, x_{n+1}$. Similarly, evaluating the first and second derivatives of (2.1) at the points $x = x_{n+k}$, $k = 0, 2/5, 1/2, 3/5, 1$, and $x = x_n, x_{n+1}$, respectively, the following system of equations are obtained. $\Upsilon(x_n) = p(x_n)$, $\Upsilon(x_{n+1}) = p(x_{n+1})$,$\Upsilon'(x_n) = p'(x_n)$, $\Upsilon'(x_{n+\frac{2}{5}}) = p'(x_{n+\frac{2}{5}})$, $\Upsilon'(x_{n+\frac{1}{2}}) = p'(x_{n+\frac{1}{2}})$, $\Upsilon'(x_{n+\frac{3}{5}}) = p'(x_{n+\frac{3}{5}})$, $\Upsilon'(x_{n+1}) = p'(x_{n+1})$, $\Upsilon''(x_n) = p''(x_n)$, $\Upsilon''(x_{n+1}) = p''(x_{n+1})$. Thus, obtaining the constants ν_k with

$$\begin{aligned}
&\nu_0 = u_n, \quad \nu_1 = f_n, \quad \nu_2 = \tfrac{g_n}{2},\\
&\nu_3 = -\tfrac{1}{108h^2}\left(1471f_n - 9375f_{n+\frac{2}{5}} + 13824f_{n+\frac{1}{2}} - 6250f_{n+\frac{3}{5}} + 330f_{n+1} + 294hg_n - 36hg_{n+1}\right)\\
&\nu_4 = -\tfrac{1}{144h^3}\left(-6173f_n + 53125f_{n+\frac{2}{5}} - 85248f_{n+\frac{1}{2}} + 40625f_{n+\frac{3}{5}} - 2329f_{n+1} - 930hg_n + 258hg_{n+1}\right)\\
&\nu_5 = -\tfrac{1}{180h^4}\left(10563f_n - 109375f_{n+\frac{2}{5}} + 186624f_{n+\frac{1}{2}} - 93750f_{n+\frac{3}{5}} + 5938f_{n+1} + 1422hg_n - 672hg_{n+1}\right)\\
&\nu_6 = \tfrac{1}{216h^5}25\left(331f_n - 3875f_{n+\frac{2}{5}} + 6912f_{n+\frac{1}{2}} - 3625f_{n+\frac{3}{5}} + 257f_{n+1} + 42hg_n - 30hg_{n+1}\right)\\
&\nu_7 = -\tfrac{1}{126h^6}25\left(49f_n - 625f_{n+\frac{2}{5}} + 1152f_{n+\frac{1}{2}} - 625f_{n+\frac{3}{5}} + 49f_{n+1} + 6hg_n - 6hg_{n+1}\right)
\end{aligned} \tag{2.2}$$

Substituting all ν_k $k = 0(1)7$ into (2.1) and evaluating, we obtain the following continuous formula

$$u(x) = \alpha_0 u_n + h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+\frac{2}{5}} + \beta_3 f_{n+\frac{1}{2}} + \beta_4 f_{n+\frac{3}{5}}) + h^2(\gamma_0 g_n + \gamma_1 g_{n+1}) \tag{2.3}$$

with known coefficients α_0, $\beta_{1,2,3,4}$ and $\gamma_{1,2}$. The block method is obtained by evaluating (2.3) at the points $x = x_{n+k}$, $k = 0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1$ in what follows.

$$
\begin{aligned}
u_{n+\frac{2}{5}} &= u_n + h\left(\frac{69943f_n}{421875} + \frac{661f_{n+\frac{2}{5}}}{945} - \frac{82176f_{n+\frac{1}{2}}}{109375} + \frac{283f_{n+\frac{3}{5}}}{945} - \frac{5657f_{n+1}}{421875}\right) + h^2\left(\frac{8048g_n}{984375} + \frac{1402g_{n+1}}{984375}\right)\\
u_{n+\frac{1}{2}} &= u_n + h\left(\frac{1271f_n}{7680} + \frac{71875f_{n+\frac{2}{5}}}{96768} - \frac{24f_{n+\frac{1}{2}}}{35} + \frac{3125f_{n+\frac{3}{5}}}{10752} - \frac{911f_{n+1}}{69120}\right) + h^2\left(\frac{73g_n}{8960} + \frac{113g_{n+1}}{80640}\right)\\
u_{n+\frac{3}{5}} &= u_n + h\left(\frac{41431f_n}{250000} + \frac{411f_{n+\frac{2}{5}}}{560} - \frac{67824f_{n+\frac{1}{2}}}{109375} + \frac{187f_{n+\frac{3}{5}}}{560} - \frac{3369f_{n+1}}{250000}\right) + h^2\left(\frac{7149g_n}{875000} + \frac{1251g_{n+1}}{875000}\right)\\
u_{n+1} &= u_n + h\left(\frac{329f_n}{2160} + \frac{3125f_{n+\frac{2}{5}}}{3024} - \frac{48f_{n+\frac{1}{2}}}{35} + \frac{3125f_{n+\frac{3}{5}}}{3024} + \frac{329f_{n+1}}{2160}\right) + h^2\left(\frac{17g_n}{2520} - \frac{17g_{n+1}}{2520}\right)
\end{aligned}
\tag{2.4}
$$

This makes a total of 4 formulas which are assembled in a block form for obtaining the numerical solution of (1.1) with the aid of the 1/3 Simpson's quadrature method.

3. Characteristics of the derived method

Associated to the formulas in (2.3), we define the linear difference operator as

$$
\begin{aligned}
\mathcal{L}_k[u(x);\ h] \equiv\ & u(x+kh) - u(x) - \left[\beta_1 u'(x) + \beta_2 u'(x+\tfrac{2}{5}h) + \beta_3 u'(x+\tfrac{1}{2}h) + \beta_4 u'(x+\tfrac{3}{5}h)\right.\\
& \left. - +h^2(\gamma_0 u''(x) + \gamma_1 g(x+h))\right]
\end{aligned}
\tag{3.1}
$$

The local truncation error of each of the formulas in (2.4) is the amount by which the exact solution of the ODEs fails to satisfy the corresponding difference operator (3.1). Expanding $u(x)$ in (3.1), in Taylor series around x, the local truncation errors LTEs of the form

$$
\mathcal{L}[u(x);\ h] = C_{p+1}h^{p+1}u^{(p+1)}(x) + O(h^{p+2}) \tag{3.2}
$$

is obtained, where the constants C_i for $i = 0(1)\rho$ varnishes and $C_{\rho+1} \neq 0$ is called the principal error constant and ρ is the order of the formula. The linear operators of each formula in (2.4) are given as

$$
\begin{aligned}
\mathcal{L}_{\frac{2}{3}} =\ & u(x_n+\tfrac{2}{5}h) - u(x_n) - h\left(\frac{69943u'(x_n)}{421875} + \frac{661u'(x_n+\frac{2}{5}h)}{945} - \frac{82176u'(x_n+\frac{1}{2}h)}{109375} + \frac{283u'(x_n+\frac{3}{5})}{945} - \frac{5657u'(x_n+h)}{421875}\right)\\
& +h^2\left(\frac{8048u''(x_n)}{984375} + \frac{1402u''(x_n+h)}{984375}\right)\\
\mathcal{L}_{\frac{1}{2}} =\ & u(x_n+\tfrac{1}{2}h) - u(x_n) - h\left(\frac{1271u'(x_n)}{7680} + \frac{71875u'(x_n+\frac{2}{5}h)}{96768} - \frac{24u'(x_n+\frac{1}{2}h)}{35} + \frac{3125u'(x_n+\frac{3}{5})}{10752} - \frac{911u'(x_n+h)}{69120}\right)\\
& +h^2\left(\frac{73u''(x_n)}{8960} + \frac{113u''(x_n+h)}{80640}\right)\\
\mathcal{L}_{\frac{3}{5}} =\ & u(x_n+\tfrac{3}{5}h) - u(x_n) - h\left(\frac{41431u'(x_n)}{250000} + \frac{411u'(x_n+\frac{2}{5}h)}{560} - \frac{67824u'(x_n+\frac{1}{2}h)}{109375} + \frac{187u'(x_n+\frac{3}{5}h)}{560} - \frac{3369u'(x_n+h)}{250000}\right)\\
& +h^2\left(\frac{7149u''(x_n)}{875000} + \frac{1251u''(x_n+h)}{875000}\right)\\
\mathcal{L}_1 =\ & u(x_n+h) - u(x_n) - h\left(\frac{329u'(x_n)}{2160} + \frac{3125u'(x_n+\frac{2}{5}h)}{3024} - \frac{48u'(x_n+\frac{1}{2}h)}{35} + \frac{3125u'(x_n+\frac{3}{5}h)}{3024} + \frac{329u'(x_n+h)}{2160}\right) +\\
& h^2\left(\frac{17u''(x_n)}{2520} - \frac{17u''(x_n+h)}{2520}\right)
\end{aligned}
\tag{3.3}
$$

Expanding each formula in (3.3) in Taylor's series about x_n we obtain the following LTEs

$$
\begin{aligned}
\mathcal{C}_{p:\frac{2}{3}} &= -\frac{3u^{(p+1)}(x_n)}{109375000}h^{p+1} + O(h^{p+2})\\
\mathcal{C}_{p:\frac{1}{2}} &= -\frac{u^{(p+1)}(x_n)}{36864000}h^{p+1} + O(h^{p+2})\\
\mathcal{C}_{p:\frac{3}{5}} &= -\frac{3u^{(p+1)}(x_n)}{109375000}h^{p+1} + O(h^{p+2})\\
\mathcal{C}_{p:1} &= \frac{11u^{(p+2)}(x_n)h^(p+2)}{5080320000} + O(h^{p+3})
\end{aligned}
\tag{3.4}
$$

Where $p = 7$. This shows that the method is 7th order. A linear difference operator $\mathcal{L}$ is said to be consistent of order p if it satisfies (3.2) with $p > 0$ for every smooth function u. Thus, a difference operator $\mathcal{L}$ is consistent if the order $p > 0$.

The matrix form of the formulas in (2.4) is given as

$$A_1 U_{\mu+1} = A_0 U_\mu + h(B_0 F_\mu + B_1 F_{\mu+1}) + h^2(C_0 G_\mu + C_1 G_{\mu+1}) \tag{3.5}$$

where

$$\begin{aligned}
&U_\mu = (u_{\mu-\frac{2}{3}}, u_{\mu-\frac{1}{2}}, u_{\mu-\frac{3}{5}}, u_{\mu-1})^T\\
&U_{\mu+1} = (u_{\mu+\frac{2}{3}}, u_{\mu+\frac{1}{2}}, u_{\mu+\frac{3}{5}}, u_{\mu+1})^T\\
&F_\mu = (f_{\mu-\frac{2}{3}}, f_{\mu-\frac{1}{2}}, f_{\mu-\frac{3}{5}}, f_{\mu-1})^T\\
&F_{\mu+1} = (f_{\mu+\frac{2}{3}}, f_{\mu+\frac{1}{2}}, f_{\mu+\frac{3}{5}}, f_{\mu+1})^T G_\mu = (g_{\mu-\frac{2}{3}}, g_{\mu-\frac{1}{2}}, g_{\mu-\frac{3}{5}}, g_{\mu-1})^T\\
&G_{\mu+1} = (g_{\mu+\frac{2}{3}}, g_{\mu+\frac{1}{2}}, g_{\mu+\frac{3}{5}}, g_{\mu+1})^T
\end{aligned}$$

with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{69943}{421875}\\ 0 & 0 & 0 & \frac{1271}{7680}\\ 0 & 0 & 0 & \frac{41421}{250000}\\ 0 & 0 & 0 & \frac{329}{2160} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \frac{661}{945} & \frac{-82176}{109375} & \frac{283}{975} & \frac{-5657}{421875}\\ \frac{71875}{96768} & \frac{-24}{35} & \frac{3125}{10752} & \frac{-911}{69120}\\ \frac{411}{560} & \frac{-67824}{109375} & \frac{187}{560} & \frac{3369}{250000}\\ \frac{3125}{3024} & \frac{-48}{35} & \frac{3125}{3024} & \frac{329}{2160} \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{8048}{984375}\\ 0 & 0 & 0 & \frac{73}{8960}\\ 0 & 0 & 0 & \frac{7149}{875000}\\ 0 & 0 & 0 & \frac{17}{2520} \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1402}{984375}\\ 0 & 0 & 0 & \frac{113}{80640}\\ 0 & 0 & 0 & \frac{1251}{875000}\\ 0 & 0 & 0 & \frac{-17}{2520} \end{pmatrix}$$

The zero-stability refers to stability of the difference system in (3.5) as $h \to 0$. As such, if $h \to 0$ in (3.5), we obtain

$$A_1 U_{\mu+1} - A_0 U_\mu = 0 \tag{3.6}$$

The characteristic equation of (3.6) is given as $\rho(r) = r^3(r-1)$. Solving for the root of $\rho(r) = 0$, we obtain $r = 1, 0, 0, 0$.

A linear multistep method (2.4) which by extension is in (3.5) is said to be 0-stable if the root of the polynomial $p(r) = 0$ satisfies the strict root condition, i.e, if all its roots lie inside the unit circle, $|r| < 1$ and with those on the boundary being simple, see [9].

The conditions for convergence of the method in (2.4) is such that, it must be consistent and 0-stable. Hence the derived method is convergent, see [9].

3.1. Algorithm for solving the IDE

This section underlines the implementation algorithm for solving the integro-differential equations using the above scheme.

INPUT VARIABLES
step 1: Initial and final points of integration interval (x_0, x_N);
step 2: Number of subintervals of integration N;
step 3: Stepsize for the block $h = \frac{x_N - x_0}{N}$;
step 4: Number of steps in the main block: $N - 1$;
OUTPUT
step 5: Approximation of the integral part $z(x)$ of (1.2) with the aid of (1.6);
step 6: Use the block in (2.4) to run the algorithm using stepsize h to obtain the values (x_1, u_1, z_1);
step 7: while $x < x_N$ **do**.
step 8: Use the block to obyain values $(x_{n+1}, u_{n+1}, z_{n+1})$; Setting $x = x + h$, $n = n + 1$;

step 9: end **while**
End

4. Numerical Examples

To illustrate computational efficiency of the derived method, we consider some problems and make some comparison with maximum absolute error $EMax$ defined as $EMax = \max_N |y(x_i) - y_i|$.
Example 1. Consider the nonlinear Volterra-Fredholm integral equation of the form

$$y'(x) = -xy(x) + f(x) + \int_0^x (x+t)y(t)dt + \int_0^1 (x+t)^2 y(t)^2 dt, \quad y(0) = 1$$

where $f(x) = -\frac{1}{30}x^6 - \frac{1}{3}x^4 + x^3 - 2x^2 + \frac{5}{3}x + \frac{3}{4}$.
The exact solution is given as $y(x) = \exp(x)$

Table 1: Absolute errors obtained for Example 1:

x	Error (N=5) in [10]	Error (N=5) in our method	Error (N=7) in [10]	Error (N=7) in our method
0.1	2.44×10^{-5}	3.471×10^{-8}	2.44×10^{-7}	1.334×10^{-11}
0.2	1.00×10^{-4}	2.41×10^{-8}	8.53×10^{-7}	2.511×10^{-11}
0.3	1.88×10^{-4}	4.21×10^{-8}	1.64×10^{-6}	3.722×10^{-11}
0.4	2.67×10^{-4}	1.01×10^{-8}	2.42×10^{-6}	1.02×10^{-12}
0.5	3.20×10^{-4}	5.24×10^{-8}	3.02×10^{-6}	4.55×10^{-12}
0.6	3.23×10^{-4}	4.22×10^{-8}	3.25×10^{-6}	3.27×10^{-12}
0.7	2.71×10^{-4}	3.11×10^{-8}	2.95×10^{-6}	2.81×10^{-12}
0.8	1.72×10^{-4}	2.22×10^{-8}	2.05×10^{-6}	2.24×10^{-12}
0.9	5.98×10^{-5}	7.13×10^{-8}	7.90×10^{-7}	2.77×10^{-12}
1.0	0	1.11×10^{-8}	1.00×10^{-9}	4.17×10^{-12}

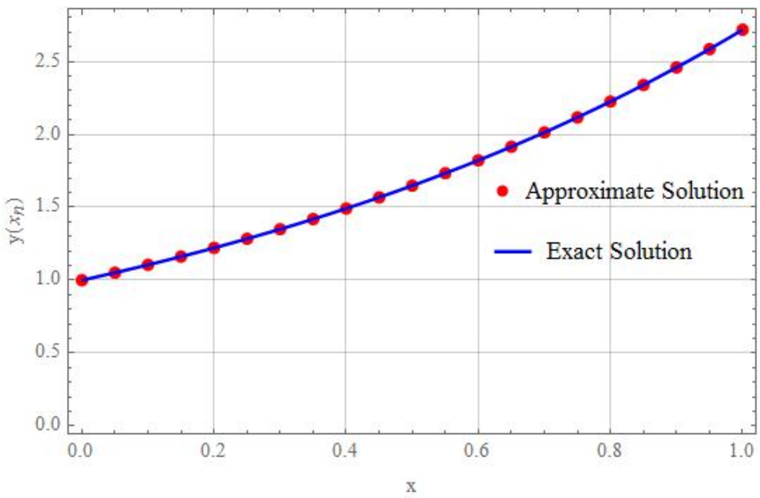

Figure 1: showing exact solution in continuous line and approximate solution in dots

Figure 1 shows the plot of the exact solution (indicated with line) and the approximate solution (with dots). Both formes of solution aligning shows the accuracy of the numerical scheme developed.
Example 2. Consider the nonlinear Volterra integral equation of the form

$$y'(x) = 1 + 2x - y(x) + \int_0^x x(1+2x)e^{t(s-t)}y(t)dt, \quad 0 \leqslant x \leqslant 1, \quad y(0) = 1$$

where the exact solution is $y(x) = \exp x^2$.

Table 2: Absolute errors obtained for Example 2:

x	Tau-err in [11]	Esti-err in [11]	Abs err in new method
N=10			
0.00	0	0	0
0.20	5.72156e-12	5.68889e-12	1.65354e-14
0.40	2.38451e-08	2.33017e-08	1.15652e-10
0.60	3.18608e-06	3.02331e-06	3.00557e-10
0.80	1.04921e-04	9.54437e-05	8.12785e-10
1.00	1.61516e-03	1.38889e-03	5.31712e-09
N=15			
0.00	0	0	0
0.20	1.63300e-16	1.62540e-16	1.00245e-22
0.40	1.08446e-11	1.06522e-11	4.53412e-21
0.60	7.28709e-09	6.99680e-09	2.08274e-21
0.80	7.51118e-07	6.98103e-07	7.00571e-20
1.00	2.78602e-05	2.48016e-05	1.11213e-19
N=20			
0.00	0	0	0
0.20	1.00000e-19	3.60219e-22	2.62364e-26
0.40	4.46000e-17	4.40759e-17	1.45121e-25
0.60	3.39914e-13	3.29740e-13	5.00154e-25
0.80	1.95219e-10	1.84852e-10	2.35451e-24
1.00	2.73127e-08	2.50521e-08	3.22721e-23

Figure 2 shows the plot of the exact solution (as indicated with line) and the approximate solution (with dots). Table 2 here presents the numerical results of the absolute error obtained by the proposed method for $N = 10, 15$ and 20 in comparison with the method presented in [11]. Considering performance in terms of the errors obtained, the proposed method performed better than the method in [11].

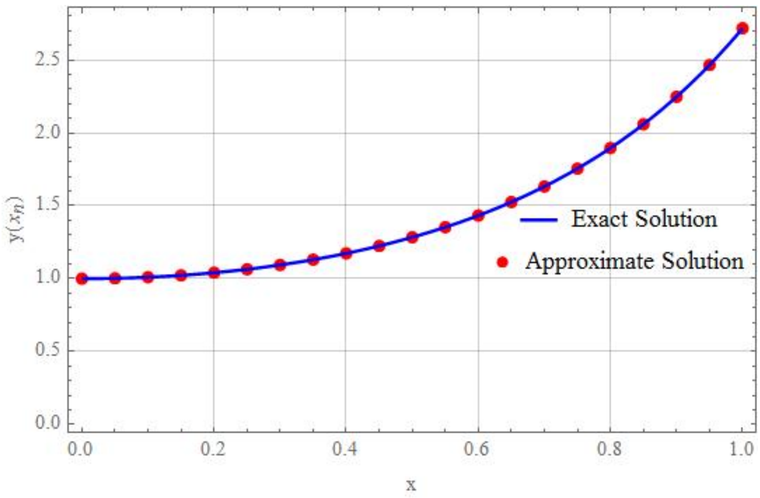

Figure 2: showing exact solution in continuous line and approximate solution in dots

Example 3. Consider the nonlinear Volterra integral equation of the form

$$y'(x) + \int_0^x y(t)dt = 1, \quad 0 \leqslant x \leqslant 1, \quad y(0) = 0,$$

where the exact solution is $y(x) = \sin x$.

The accuracy of the numerical method developed is shown in Figure 3 which shows the exact solution (as a line) and the approximate solution (with dots). Both solution agree as they are embedded. Table 3 shows comparison of the proposed method and those presented in [11] and [12] for $N = 5, 10$ and 15

Table 3: Comparison absolute errors obtained for Example 3:

x	Tau-err in [11]	Esti-err in [11]	method in [12]	Abs err in new method
N=5				
0.00	0	0	0	0
0.20	2.60e-09	2.54e-09	4.02e-07	1.27e-10
0.40	3.24e-07	3.25e-07	2.06e-07	4.01e-10
0.60	5.53e-06	5.55e-06	3.77e-07	9.42e-10
0.80	4.12e-05	4.16e-05	1.82e-07	4.34e-09
1.00	1.96e-04	1.98e-04	9.67e-06	2.28e-09
N=10				
0.00	0	0	na	0
0.20	1.00e-10	5.33e-13	na	2.00e-15
0.40	1.00e-10	5.28e-12	na	4.27e-15
0.60	1.00e-10	1.05e-10	na	3.42e-15
0.80	2.20e-09	2.18e-09	na	1.21e-14
1.00	2.49e-08	2.51e-08	na	8.11e-13
N=15				
0.00	0	0	na	0
0.20	1.00e-10	5.32e-13	na	1.02e-18
0.40	1.00e-10	4.23e-12	na	2.41e-18
0.60	0	1.41e-11	na	2.24e-16
0.80	0	3.30e-11	na	5.74e-15
1.00	0	6.33e-11	na	3.37e-15

respectively. The new method has been shown to produce smaller error when compared with the exact solution $y = \sin(x)$. This shows that the present method performed favourably when compared to method in [11] and [12].

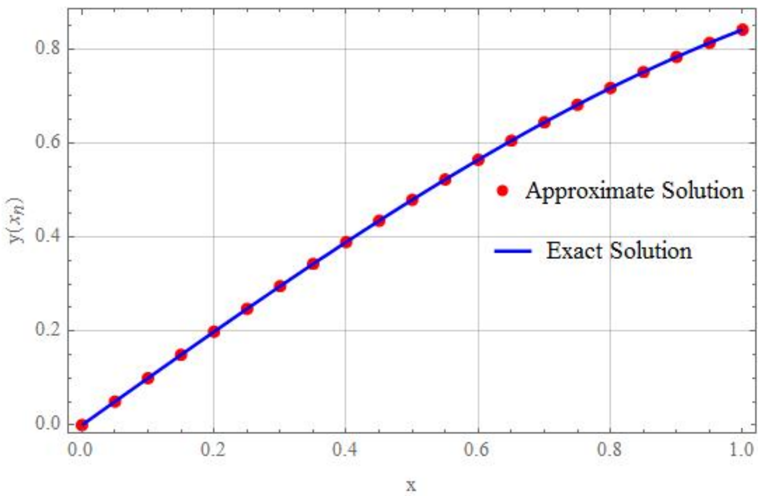

Figure 3: showing exact solution in continuous line and approximate solution in dots

Example 4. Consider the linear Fredholm integro-differential equation discussed in [12].

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, \quad 0 \leqslant x, \quad 0 \leqslant t, \quad y(0) = 0,$$

The exact solution is $y(x) = x \exp^x$.

Table 4 shows numerical results of the present method and the method in [12] at some selected points of x for $N = 10$. The present method produced smaller error as compared to the method in [12]. The proposed method performs better at higher values of N which is a very good approximation of the exact solution $y(x) = x \exp^x$. Figure 4 shows the closeness of both the numerical and exact solutions.

Table 4: Numerical results of the absolute error (for N = 10) for Example 4:

x	Method in [12]	Err in new method
0.1	1.1472e-012	2.4578e-16
0.2	8.6964e-013	5.2466e-16
0.3	na	3.8001e-16
0.4	5.5689e-013	1.2012e-16
0.5	na	2.2473e-16
0.6	4.4853e-014	7.7112e-16
0.7	na	4.1812e-16
0.8	1.0740e-012	2.2719e-16
0.9	5.4293e-011	2.0912e-16
1.0	na	8.5283e-16

na-not available

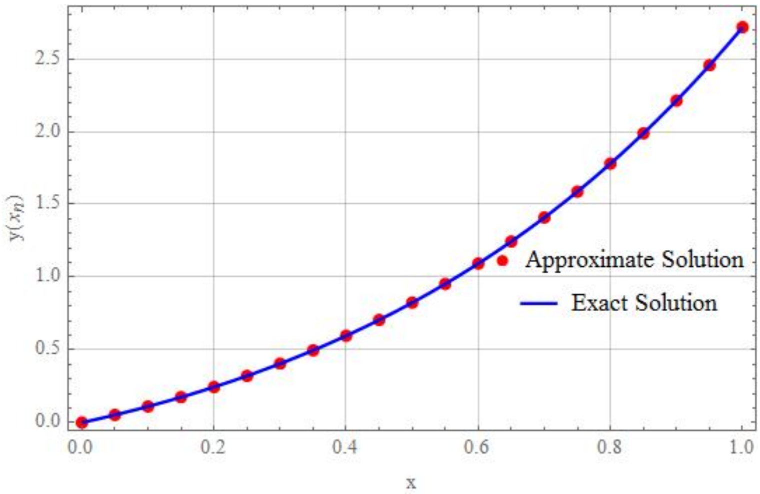

Figure 4: showing exact solution in continuous line and approximate solution in dots

5. Conclusion

A one-step second derivative block hybrid numerical scheme has been derived for the numerical solution of first-order integro-differential equations of the Fredholm and Volterra kind. This numerical scheme is an ensemble of linear multistep formula called a block hybrid method. The development of the block hybrid method is via the collocation techniques and its easy to derive and as well to implement with a simple code written in Wolfram Mathematica. This block method has been used to solve integro-differential equation of the form in (1.1) with $n = 1$. The implementation process was carried out in a step-by-step fashion in order to obtain the final approximation in the given grid interval. The integral part has been approximated with the Simpsons 1/3 quadrature method which shows high degree of accuracy. The accuracy of this method increases with increase in the number of sub-interval N which is evident in the above tables of results. Comparison made of the exact solution, the proposed method and other methods shows that the presented method is very accurate as seen from the small global error obtained.

References

[1] A. M. Wazwaz, *Fredholm Integro-Differential Equations, In: Linear and Nonlinear Integral Equations*. Springer, Berlin, Heidelberg, (2010). https://doi.org/10.1007/978-3-642-21449-3-6 1

[2] A. V. Andreev, *Development of direct numerical integration methods of one-dimensional integro-differential equations in mechanics*, Mech. Solids, **42**, 2, (2007), 209-222. https://doi.org/10.3103/s0025654407020069 1

[3] N. Baykus, M. Sezer, *Solution of High-Order Linear Fredholm Integro-Differential Equations with Piecewise Intervals*, Numerical Methods for Partial Differential Equations, **27**, (2011), 13271339. https://doi.org/10.1002/num.20587 1

[4] N. Akgnll, N. Şahin, M. Sezer, A, *Hermite Collocation Method for the Approximate Solutions of High-Order Linear Fredholm Integro-Differential Equations*, Numerical Methods for Partial Differential Equations. **27**, (2011). 1707–1721 https://doi.org/10.1002/num.20604 1

[5] S. Shahmorad, *Numerical solution of the general form linear FredholmVolterra integro-differential equations by the Tau method with an error estimation*, Applied Mathematics and Computation **167**, (2005), 1418–1429. https://doi.org/10.1016/j.amc.2004.08.045 1

[6] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems, The Initial Value Problem*, first John Wiley, New York, (1991).

[7] C. Huang, *Stability of linear multistep methods for delay integro-differential*, Computers and Mathematics with Applications, **55** (2008) 2830-2838 https://doi.org/10.1016/j.camwa.2007.09.005

[8] CH. Lubich *On the Stability of Linear Multistep Methods for Volterra Convolution Equations*, IMA Journal of Numerical Analysis, **3** (1983), 439–465 https://doi.org/10.1093/imanum/3.4.439

[9] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, (1962). 3

[10] B. Blbl, M. Glsu and M. Sezer, A New Taylor Collocation Method for Nonlinear Fredholm-Volterra Integro-Differential Equations, Numer Methods Partial Differential, Wiley Periodicals, Inc **26** (2010), 1006-1020. https://doi.org/10.1002/num.20470 1

[11] S. Shahmorad *Numerical solution of the general form linear FredholmVolterra integrodifferential equations by the Tau method with an error estimation*, Applied Mathematics and Computation **167**, (2005) 1418-1429 https://doi.org/10.1016/j.amc.2004.08.045 2, 4, 4, 3

[12] . Yzba, N. ahin, A.Yildirim, *A collocation approach for solving high-order linear FredholmVolterra integro-differential equations*, Mathematical and Computer Modelling **55** (2012), 547–563 https://doi.org/10.1016/j.mcm.2011.08.032 4, 3, 4, 4