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Modified Successive Approximations Method for Solving Fuzzy Second Kind Fredholm Integral Equations with Separable Kernel

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Abstract

This paper presents an analysis, development, and application of a modified method of successive approximations to solve fuzzy second kind Fredholm integral equations with a separable kernel. The fuzziness in the equations is represented utilizing convex normalized triangular fuzzy numbers, which are based on a single and double parametric form of fuzzy numbers. In the single parametric form of the fuzzy number, a fuzzy equation converts to two crisp equations for the solution which increases the computational cost. Therefore, to reduce the computational cost and increase the solution's accuracy, the double parametric form of the fuzzy number will be used in this study. The feasibility of the proposed methods is illustrated through a numerical example. The results obtained using the presented approach demonstrate adequate agreement with relevant theoretical aspects.

Keywords: Fuzzy Fredholm integral equations, double parametric form of fuzzy numbers, method of successive approximations. Numerical solution.

1 Introduction

Integral equations and their specific solutions are of great importance in the realm of scientific inquiry and engineering applications. Integral equations are commonly employed to model a wide range of significant physical phenomena which can be classified into two main categories which are Volterra integral equations and Fredholm integral equations.

Fuzzy set theory is well recognized as a powerful tool in mathematical modeling for addressing uncertain and imprecise problems [[1]-[3]]. Its practical applications have proven to be valuable in various scientific disciplines, offering effective solutions to real-life problems. For example, fuzzy logic has proven to be beneficial in addressing several practical challenges in computer programming, decision-making processes, and communication. [[4]-[6]]. Crisp quantities in the integral equations which are deemed vagueness and uncertain can be replaced by fuzzy quantities to reflect vagueness and uncertainty. This leads to fuzzy integral equations.

The significance of fuzzy integral equations is growing within the field of fuzzy studies. This can be attributed to the rising prominence of high-dimensional systems in data analysis, which involve numerous parameters in their observation [[7]-[14]]. One motivation for investigating equations with fuzzy-valued parameters is the inherent difficulty in measuring or estimating these parameters. Consequently, in order to

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create abstract models that closely resemble real-world processes, the simulation of precise parameters using fuzzy numbers is frequently employed. Among the primary equations involving fuzzy-number-valued parameters, the fuzzy Fredholm integral equation (FFIE) has been extensively analyzed by numerous researchers [[14]-[17]].

Solving FFIE is a challenging task, and analytical solutions are often not available for complex equations. Numerical methods are typically used to approximate the solution of FFIE. Solving FFIE of the second kind is often more challenging than those of the first kind, and may require the use of numerical methods such as the method of successive approximations. The main aim of this study is to analyses, develop, and apply method of successive approximations to solve the fuzzy second kind Fredholm integral equations with separable kernel. Therein lies the main aim. Another aim is the investigating use of double parametric form of fuzzy number. In the single parametric form of fuzzy number, a fuzzy equation converts to two crisp equations for the solution. Here, we have to solve the two crisp equations separately to obtain the lower and upper bounds of the solution which increase the computational cost. Hence to reduce the computational cost and increase the accuracy of the solution, the use of the double parametric form of fuzzy number will be investigated.

2 Fuzzy Fredholm integral equation

Consider the one-dimensional fuzzy second-kind Fredholm integral equations with the initial and boundary conditions

$$\tilde{y}(x;r) = \tilde{f}(x;r) + \tilde{\lambda}\int_a^b k\,(x,t)\tilde{y}(t;r)dt \tag{1}$$

Using the fuzzy set theory based on the Zadeh extension principle, and r-level set approach Eq.(3.1) is rewritten as follows":

$$1)\,\tilde{y}(x) = \begin{cases}\underline{y}(x;r)\\ \overline{y}(x;r)\end{cases} \tag{2}$$

$$2)\,\tilde{f}(x) = \begin{cases}\underline{f}(x;r)\\ \overline{f}(x;r)\end{cases} \tag{3}$$

$$3)\,\tilde{\lambda} = \begin{cases}\underline{\lambda}\\ \overline{\lambda}\end{cases} \tag{4}$$

$$4)\,\tilde{y}(t) = \begin{cases}\underline{y}(t;r)\\ \overline{y}(t;r)\end{cases} \tag{5}$$

The Eq.(2) to Eq.(5) is substituted in the Eq. (1) to get the following:

$$\begin{cases}\underline{y}(x;r) = \underline{f}(x;r) + \underline{\lambda}\int_a^b k\,(x,t)\underline{y}(t;r)dt\\ \overline{y}(x;r) = \overline{f}(x;r) + \overline{\lambda}\int_a^b k\,(x,t)\overline{y}(t;r)dt\end{cases} \tag{6}$$

Eq. (6) represents the general formula of the fuzzy second kind Fredholm integral equation in the single parametric form of a fuzzy number.

As per the singular parametric form, the Eq. (1) can be written as follows:

$$\left[\underline{y}(x,r),\overline{y}(x,r)\right] = \left[\underline{f}(x,r),\overline{f}(x,r)\right] + [\underline{\lambda},\overline{\lambda}]\int_a^b [k(x,t)]\left[\underline{y}(t,r),\overline{y}(t,r)\right]dt$$

By using the double parametric form from definition in [15], the Eq. (2) can be written as:

$$\beta\left[\overline{y}(x;r) - \underline{y}(x;r)\right] + \underline{y}(x;r)$$

$$= \beta\left[\overline{f}(x;r) - \underline{f}(x;r)\right] + \underline{f}(x;r) + [\beta[\overline{\lambda} - \underline{\lambda}] + \underline{\lambda}]\int_a^b [k(x,t)]\beta\left[\overline{y}(t;r) - \underline{y}(t;r)\right] + \overline{y}(t;r)dt$$

Note that:

- $\tilde{y}(x;r,\beta) = \beta\left[\overline{y}(x;r) - \underline{y}(x;r)\right] + \underline{y}(x;r)$
- $\tilde{f}(x;r,\beta) = \beta\left[\overline{f}(x;r) - \underline{f}(x;r)\right] + \underline{f}(x;r)$
- $\tilde{\lambda} = \beta[\overline{\lambda} - \underline{\lambda}] + \underline{\lambda}$
- $\tilde{y}(t;r,\beta) = \beta\left[\overline{y}(t;r) - \underline{y}(t;r)\right] + \overline{y}(t;r)$

By substituting these equations in Eq. (1) to obtain:

$$\tilde{y}(x;r,\beta) = \tilde{f}(x;r,\beta) + \tilde{\lambda}\int_a^b k\,(x,t)\tilde{y}(t;r,\beta)\,dt \tag{7}$$

This is the general formula of the fuzzy Fredholm integral equation in the double parametric form of a fuzzy number.

To obtain the lower and upper bounds of the solutions, we respectively, assume $\beta = 0$ and $\beta = 1$, which may be presented as $\tilde{y}(x, r, 0) = \underline{y}(x, r)$ and $\tilde{y}(x, r, 1) = \overline{y}(x, r)$.

3 Method of successive approximation"

In this section, the successive approximation method is developed, reformatted, and applied to solve the fuzzy Fredholm integral equation with a separable kernel in both single and double parametric forms of fuzzy numbers.

3.1 Solving fuzzy Fredholm integral equation in single parametric form

In this section, we shall study an iterative scheme based on the same principle for linear integral equations of the second order. Throughout our discussion, we shall assume that the functions $f(x)$ and $K(x, t)$ involved in an integral equation are functions.

Consider the fuzzy Fredholm integral equation that is defined in the Eq.(6)

$$\begin{cases} \underline{y}(x; r) = \underline{f}(x; r) + \underline{\lambda} \int_a^b k(x, t)\underline{y}(t; r)dt \\ \overline{y}(x; r) = \overline{f}(x; r) + \overline{\lambda} \int_a^b k(x, t)\overline{y}(t; r)dt \end{cases} \tag{8}$$

Then, the iterated kernels $k_n(x, t), n = 1,2,3, \dots$ are defined as follows:

$$k(x, t) = k_1(x, t) \tag{9}$$

$$k_n(x, t) = \int_a^b k(x, z)k_{n-1}(z, t)dz \quad n = 1,2,3 \dots \tag{10}$$

Now, to find the Resolvent Kernel based on the iterated kernel, we will follow the following steps:

Let $k_n(x, t)$ be nth iterated kernel and let $\tilde{R}(x, t; \lambda)$ be the resolvent kernel of (1). Then we have

$$\tilde{R}(x, t; \lambda) = \begin{cases} \underline{R}(x, t; \lambda) = \sum_{n=1}^{\infty} \underline{\lambda}^{n-1} k_n(x, t) \\ \overline{R}(x, t; \lambda) = \sum_{n=1}^{\infty} \overline{\lambda}^{n-1} k_n(x, t) \end{cases} \tag{11}$$

Suppose the sum of infinite series (2) exists and so $\tilde{R}(x, t, \lambda)$ can be obtained in the closed form. Then, the required solution of (1) is given by:

$$\tilde{y}(x; r) = \tilde{f}(x; r) + \tilde{\lambda} \int_a^b \tilde{R}(x, t; \lambda)\tilde{f}(t; r)\, dt \tag{12}$$

3.2 Solving fuzzy Fredholm integral equation in double parametric form

In this section, we shall study an iterative scheme based on the same principle for linear integral equations of the second order. Throughout our discussion, we shall assume that the functions $f(x)$ and $K(x, t)$ involved in an integral equation are functions.

Consider the fuzzy Fredholm integral equation that is defined in the Eq. (1)

$$\beta\left[\overline{y}(x; r) - \underline{y}(x; r)\right] + \underline{y}(x; r) = \beta\left[\overline{f}(x; r) - \underline{f}(x; r)\right] + \underline{f}(x; r) + [\beta[\overline{\lambda} - \underline{\lambda}] + \underline{\lambda}] \int_a^b [k(x, t)]\beta\left[\overline{y}(t; r) - \underline{y}(t; r)\right] + \overline{y}(t; r)dt \tag{13}$$

Then, the iterated kernels $k_n(x, t), n = 1,2,3, \dots$ are defined as follows:

$$k(x, t) = k_1(x, t)$$

Now, to find the Resolvent Kernel based on the iterated kernel, we will follow the following steps:

Let $k_n(x, t)$ be nth iterated kernel and let $\tilde{R}(x, t; \lambda, \beta)$ be the resolvent kernel of (1). Then we have

$$\tilde{R}(x, t; \lambda, \beta) = \sum_{n=1}^{\infty} \tilde{\lambda}^{n-1} k_n(x, t)$$

Suppose the sum of infinite series (2) exists and so $\tilde{R}(x, t; \lambda, \beta)$ can be obtained in the closed form. Then, the required solution of (1) is given by:

$$\tilde{y}(x; r, \beta) = \tilde{f}(x; r, \beta) + \tilde{\lambda} \int_a^b \tilde{R}(x, t; \lambda, \beta)\tilde{f}(t; r, \beta)\, dt$$

4 Numerical Example

In this section, we present examples of fuzzy Fredholm integral equations.

Example (4.1): (Single Parametric Form)

Consider the fuzzy Fredholm integral equation:

$$\tilde{y}(x; r) = (0.75 + 0.25r, 1.25 - 0.25r)\, \tilde{x} + \tilde{\lambda} \int_0^1 x\, t\, \tilde{y}(t; r)\, dt$$

Solution

$$\begin{cases} \underline{y}(x;r) = (0.75+0.25r)\,\underline{x} + \underline{\lambda}\int_0^1 x\,t\,\underline{y}(t;r)\,dt \\ \overline{y}(x;r) = (1.25-0.25r)\,\overline{x} + \overline{\lambda}\int_0^1 x\,t\,\overline{y}(t;r)\,dt \end{cases}$$

$$\underline{y}(x;r) = (0.75+0.25r)\underline{x} + \underline{\lambda}\int_0^1 x\,t\,\underline{y}(t;r)\,dt \tag{14}$$

In general, we can find the resolvent kernel for the above kernel for both lower and upper case as follows:

$$k(x,t) = k_1(x,t) = x\,t$$

By using the formula

$$k_n(x,t) = \int_a^b k(x,z)k_{n-1}(z,t)dz$$

$$k_2(x,t) = \int_0^1 (x\,z)\,(z\,t)\,dz$$

$$= x\,t\int_0^1 z^2\,dz$$

$$= \frac{1}{3}x\,t$$

and

$$k_3(x,t) = \int_0^1 \frac{1}{3}(x\,z)\,(zt)\,dz$$

$$= \frac{1}{3}xt\int_0^1 z^2\,dz$$

$$= \left(\frac{1}{3}\right)^2 x\,t$$

$$\vdots$$

$$\vdots$$

$$k_n(x,t) = \left(\frac{1}{3}\right)^{n-1} x\,t$$

Now based on the iterated kernel in the above equation, we can find the resolvent kernel as follows:

$$\underline{R}(x,t;\lambda) = \sum_{n=1}^{\infty} \underline{\lambda}^{n-1}\,k_n(x,t) = \sum_{n=1}^{\infty} \underline{\lambda}^{n-1}\left(\frac{1}{3}\right)^{n-1} x\,t$$

$$= xt\sum_{n=1}^{\infty}\left(\frac{\lambda}{3}\right)^{n-1}$$

Then, using the geometric series the above equation will obtained as follows:

$$\underline{R}(x,t;\lambda) = xt\,\frac{1}{1-\left(\frac{\lambda}{3}\right)} = \frac{3xt}{3-\lambda}\quad,\qquad \left|\frac{\lambda}{3}\right| < 1$$

Thus, the fuzzy lower approximate $\underline{y}(x;r)$ is:

$$\underline{y}(x;r) = (0.75+0.25r)\,\underline{x} + \underline{\lambda}\int_0^1 \frac{3xt}{3-\lambda}\,(0.75+0.25r)\,\underline{t}\,\,dt$$

$$\underline{y}(x;r) = (0.75+0.25r)\,\underline{x} + (0.75+0.25r)\frac{3x\,\lambda}{3-\lambda}\int_0^1 t^2\,\,dt$$

$$\underline{y}(x;r) = (0.75+0.25r)\,x + \left(\frac{1}{3}\right)(0.75+0.25r)\frac{3x\,\lambda}{3-\lambda}$$

$$\overline{y}(x;r) = (1.25-0.25r)\,\overline{x} + \overline{\lambda}\int_0^1 x\,t\,\overline{y}(t)\,dt \tag{15}$$

We have

$$\underline{R}(x,t;\lambda) = \frac{3xt}{3-\lambda}\quad,\qquad \left|\frac{\lambda}{3}\right| < 1$$

So, the fuzzy upper approximate $\overline{y}(x;r)$:

$$\overline{y}(x;r) = (1.25 - 0.25r)\,\overline{x} + \overline{\lambda}\int_0^1 \frac{3xt}{3-\lambda}\ (1.25 - 0.25r)\,\overline{t}\,dt$$

$$\overline{y}(x;r) = (1.25 - 0.25r)\,x + \left(\frac{1}{3}\right)(1.25 - 0.25r)\frac{3x\,\lambda}{3-\lambda}$$

So the fuzzy solution of Eq. (14) is :

$$\begin{cases}\underline{y}(x;r)\\ \overline{y}(x;r)\end{cases} = \begin{cases}(0.75 + 0.25r)\,x + \left(\frac{1}{3}\right)(0.75\ + 0.25r)\dfrac{3x\,\lambda}{3-\lambda}\\ (1.25 - 0.25r)\,x + \left(\frac{1}{3}\right)(1.25 - 0.25r)\dfrac{3x\,\lambda}{3-\lambda}\end{cases}$$

Table 1: fuzzy lower and upper solution of FFIE at different values of x for all $r\ \in [0,1]$

	x	Lower	Upper		Lower	Upper
$r=0$	0.1	0.09	0.15	$r=0.3$	0.099	0.141
	0.2	0.18	0.3		0.198	0.282
	0.3	0.27	0.45		0.297	0.423
	0.4	0.36	0.6		0.396	0.564
	0.5	0.45	0.75		0.495	0.705
	0.6	0.54	0.9		0.594	0.846
	0.7	0.63	1.05		0.693	0.987
	0.8	0.72	1.2		0.792	1.128
	0.9	0.81	1.35		0.891	1.269
	1	0.9	1.5		0.99	1.41
	X	Lower	Upper		Lower	Upper
$r=0.7$	0.1	0.111	0.129	$r=1$	0.12	0.12
	0.2	0.222	0.258		0.24	0.24
	0.3	0.333	0.387		0.36	0.36
	0.4	0.444	0.516		0.48	0.48
	0.5	0.555	0.645		0.6	0.6
	0.6	0.666	0.774		0.72	0.72
	0.7	0.777	0.903		0.84	0.84
	0.8	0.888	1.032		0.96	0.96
	0.9	0.999	1.161		1.08	1.08
	1	1.11	1.29		1.2	1.2

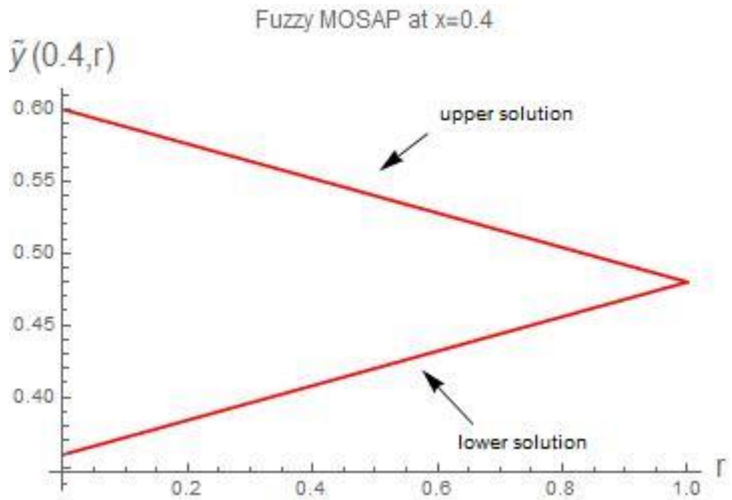
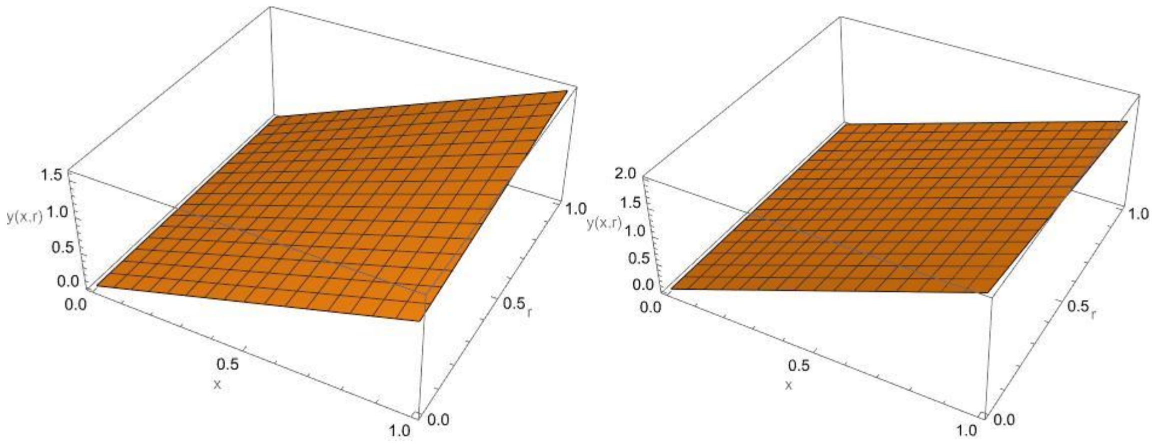


Fig. 1: fuzzy lower and upper solution of FFIE at $x = 0.4$ for all $r\ \in [0,1]$



(a) (b)

Fig 2: (a) the 3D lower solution and 3D upper solution of FFIE for all $x, r \in [0,1]$

Table 1 and Fig .2 shows that the fuzzy numerical solutions obtained using method of successive approximation satisfy the properties of fuzzy number by exhibiting a triangular fuzzy number shape as we can see in Fig. 2.a at $x = 0.4$. And the result obtained for the FFIE are compatible with the fuzzy domain. Fig.2.b show the 3D fuzzy solution for both lower and upper case for all $x, r\epsilon[0,1]$.

(Double Parametric Form)

We well show that the definition in [[15]] is consistence, that is the $\tilde{y}(t,\beta,r) = \beta\left[\overline{y}(t,r) - \underline{y}(t,r)\right] + \underline{y}(t,r)$ is a solution of the integral defined in Eq. (7) which represents the double parametric form of the of the FFIE , using the method of successive approximation. In this we consider the correspondence functions previously considered in Example 1. of the single parametric form, that is, let consider:

$$\tilde{y}(x;r) = (0.75 + 0.25r, 1.25 - 0.25r)\,\tilde{x} + \tilde{\lambda}\int_0^1 x\,t\,\tilde{y}(t;r)\,dt$$

Where, $0 \leq r \leq 1, 0 \leq x \leq 1$, and the kernel

$$k(x,t) = xt, \qquad 0 \leq x,t \leq 1$$

and $a = 0,\ b = 1$.

We have seen in the example 1 that the solution $(\underline{y}(t,r), \overline{y}(t,r))$ of the single parametric form using the method of successive approximation is

$$\begin{cases}\underline{y}(x;r)\\ \overline{y}(x;r)\end{cases} = \begin{cases}(0.75 + 0.25r)\,x + \left(\frac{1}{3}\right)(0.75 + 0.25r)\,\frac{3x\,\lambda}{3-\lambda}\\ (1.25 - 0.25r)\,x + \left(\frac{1}{3}\right)(1.25 - 0.25r)\,\frac{3x\,\lambda}{3-\lambda}\end{cases}$$

Now substituting in $\tilde{y}(t,\beta,r) = \beta\left[\overline{y}(t,r) - \underline{y}(t,r)\right] + \underline{y}(t,r)$ we get (also considering $0 \leq \beta \leq 1$):

$$\tilde{y}(t,\beta,r) = \beta\left[\overline{y}(t,r) - \underline{y}(t,r)\right] + \underline{y}(t,r)$$

$$= \beta\Big[\Big((1.25 - 0.25r)\,x + \left(\tfrac{1}{3}\right)(1.25 - 0.25r)\tfrac{3x\,\lambda}{3-\lambda}\Big) - \Big((0.75 + 0.25r)\,x + \left(\tfrac{1}{3}\right)(0.75 + 0.25r)\tfrac{3x\,\lambda}{3-\lambda}\Big)\Big] + (0.75 + 0.25r)\,x + \left(\tfrac{1}{3}\right)(0.75 + 0.25r)\tfrac{3x\,\lambda}{3-\lambda}$$

$$= x\,\beta\left((1.25 - 0.25r) + \left(\frac{\lambda}{3-\lambda}\right)(1.25 - 0.25r) - (0.75 + 0.25r) - \left(\frac{\lambda}{3-\lambda}\right)(0.75 + 0.25r)\right) + (0.75 + 0.25r)x + \left(\frac{x\,\lambda}{3-\lambda}\right)(0.75 + 0.25r)\Bigg)$$

Table 2: fuzzy lower and upper solution of FFIE at different values of x for all $r \in [0,1]$

		x	$\tilde{y}(t,\beta,r)$			$\tilde{y}(t,\beta,r)$
		0.1	0.102			0.1218
		0.2	0.204			0.2436
		0.3	0.306			0.3654
$r=0$	$\beta=0.2$	0.4	0.408	$r=0.7$	$\beta=0.6$	0.4872
		0.5	0.51			0.609
		0.6	0.612			0.7308
		0.7	0.714			0.8526
		0.8	0.816			0.9744
		0.9	0.918			1.0962
		1	1.02			1.218
		x	$\tilde{y}(t,\beta,r)$			$\tilde{y}(t,\beta,r)$
		0.1	0.1158			0.12
		0.2	0.2316			0.24
		0.3	0.3474			0.36
$r=0.3$	$\beta=0.4$	0.4	0.4632	$r=1$	$\beta=0.8$	0.48
		0.5	0.579			0.6
		0.6	0.8106			0.72
		0.7	0.8106			0.84
		0.8	0.9264			0.96
		0.9	1.0422			1.08
		1	1.158			1.2

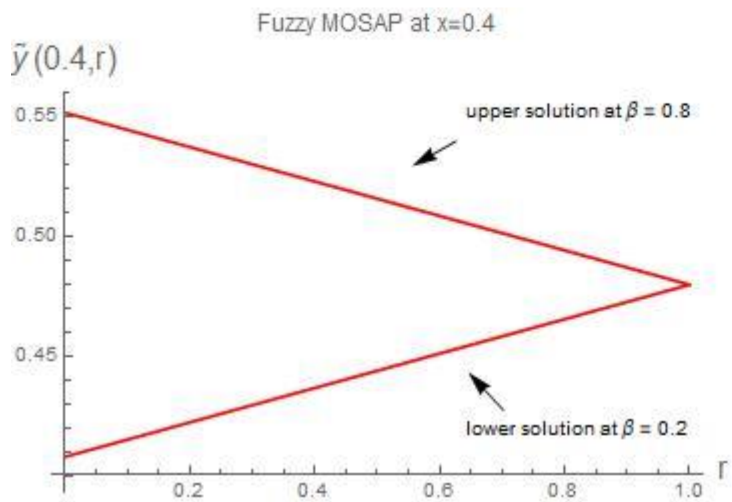


Fig. 3: fuzzy lower and upper solution of FFIE at $x = 0.4$ for all $r \in [0,1]$

Table 2 and Figs.3 shows that the fuzzy numerical solutions obtained using method of successive approximation satisfy the properties of double fuzzy number by exhibiting a triangular fuzzy number shape as we can see in Fig. 3 at $x = 0.4$. And the result obtained for the FFIE are compatible with the fuzzy domain. the double parametric form approach is observed to be simple, generally applicable, and computationally efficient due to the conversion of the governing equation from uncertain to crisp form.

5 Conclusion

In summary, this paper contributes to the field of fuzzy integral equations by proposing and exploring a numerical method for solving fuzzy second-kind Fredholm integral equations with separable kernels. The investigation of the double parametric form of fuzzy numbers adds an innovative dimension to the study, aiming to improve computational efficiency without compromising solution accuracy. The findings and insights presented in this thesis are expected to advance the understanding and application of fuzzy integral equations in scientific and engineering domains.

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