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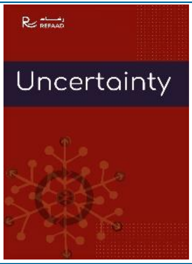

Complex Bipolar Multi-Fuzzy Sets

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Abstract

Convey human being's information to a mathematical formula and vice versa by a proper tool, is considered a first step to solve the problem of assigning a membership degree (suitable value) of an object to a set by decision-makers. Different information with its periodic circumstances highlighted the need for an extra variable to be built into the mathematical tools. This extra variable adds a significant role to represent a special type of information that has the same data but with a different meaning in different time/levels/phases. However, the idea of extending the range of membership degree to unit disk in the complex plane solved this problem. In this research, we choose bipolar multi-fuzzy information to be generalized to the complex realm using the Cartesian form " $x + iy$". Therefore, the formal definition of complex bipolar multi-fuzzy sets (CBMFS) is introduced with a range of membership lies in the unit square. The advantage of CBMFS is that the real and imaginary parts of CBMFSs can represent bipolar multi-fuzzy information. Also, some basic operations and numerical examples on CBMFS are presented and studied its properties.
Moreover, the relations of CBMFS with each of bipolar fuzzy sets (BFS) and complex bipolar fuzzy sets (CBFS), respectively, are illustrated. Finally, two types of distances and δ −equality under CBMFS are presented. Also, some illustrative examples and properties of complex bipolar multi-fuzzy distance and $\delta -$equality are obtained.

Keywords: Bipolar fuzzy sets, Fuzzy multi sets, Complex fuzzy sets, Complex bipolar multi-fuzzy sets.
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1 Introduction

The set of complex numbers is considered a foundation concept in the field of mathematics and as a solution for mathematicians to solve merits problems that cannot be solved in real numbers. Comparability, in terms of membership functions, expanding the range of the membership function from {0, 1} to the interval [0, 1] introduced the novel concept of fuzzy set theory. Thus, the extension of intuitionistic fuzzy set, bipolar fuzzy set, bipolar multi-fuzzy sets, and other uncertainty sets to the field of complex numbers is essential and have a priority to get numerous benefits from the properties and benefits of complex numbers. This may be presented by extending the range of membership values from [0, 1] to the unit square in the complex plane with some inheritance constraints. The benefits of complex numbers can be employed in expressing information that occurs periodically. Several information has the same values, but these values may have different meaning at

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different times. The complex fuzzy set1.1.[22], complex intuitionistic fuzzy set1.1.[2], and bipolar complex fuzzy sets 1.1.[4] deal with uncertainty of periodic nature. The complex bipolar multi-fuzzy set (CBMFS) is used to represent the bipolar multi-fuzzy information that happens periodically (i.e. CBMFS deals with several values of positive and negative memberships of uncertainty of periodic nature). However, the difficulty that lies in presenting optimal theory has the ability to present bipolar multi-fuzzy information (uncertainty semantics) and periodicity semantics, simultaneously, without any limitations and weaknesses. The present work solves the weakness and limitations of bipolar multi-fuzzy information and generalizes a proper mathematical tool to transform human knowledge into mathematical formula and vice versa with respect to two semantic expansions (positive and negative multi-fuzzy uncertainty and periodicity semantics).

1.1. Literature Review

Vagueness, ambiguity, and uncertainty of information have been wildly proceeding to be formalized a in logical way. In 1965, the concept of fuzzy sets (FS) was introduced by Zadeh, which is a class of objects with graded membership functions. The fuzzy set generalizes the classical set (crisp set) by extending the membership function from {0, 1} to the interval [0, 1]. Fuzzy sets came to convey uncertainty information to mathematical form. So, the fuzzy sets can be utilized in an extended range of the membership function in which information is incomplete or inaccurate. Many scholars followed, applied, generalized, and solved many problems in multiple attributes and criteria decision-making (MADM and MCDM), engineering, medical science, artificial intelligence, and pattern recognition by using the concept of fuzzy sets1.1.[13],1.1.[10] Fuzzy sets solves many problems in engineering, medical science, etc.

Different generalization of fuzzy sets came in the last decades to convey and describe different cases and types of information. For example: the concept of intuitionistic fuzzy sets (IFS) introduced by Atanassov [8], he added a non-membership function to represent the falsity values of information besides the truth values in fuzzy sets. But the fuzzy sets and intuitionistic fuzzy sets have limitations, which it cannot process the indeterminate and inconsistent information. Some of these limitations were solved by introducing the concept of a Neutrosophic set 1.1.[26]. Also, Sebastian and Ramakrishnan 1.1.[23] generalized fuzzy sets to multi-fuzzy sets in terms of multidimensional membership functions and multi-level fuzziness. On the other hand, Zhang introduced the concept of bipolar fuzzy sets (BFSs) 1.1.[34], the BFSs have the ability to represent human decision-making based on double-sided "positive and negative side" described by the range of membership function lies in [-1, 0] X [0, 1]. For example: female and male, friendship and enemy, hate and love, the Chinese Yin and Yang system, etc. Also, a bipolar multi-fuzzy set (BMFS) was introduced by Y. AL-Qudah and N. Hassan.

The idea of extending the range of uncertainty set from real interval [0, 1] to unit disc in the complex plane has been widely used in different fields and applications. This extension allows a wider range to convey the special type of information that carries two semantics (uncertainty and periodicity) simultaneously. The first paper proposed this idea was written by 1.1.[22]. Then many researchers incorporated the properties of complex fuzzy sets and other uncertainty sets to get a new mathematical tool with special and unique properties 1.1.[23]. 1.1.[4] have used bipolar fuzzy sets in complex geometry by extending the range of bipolar fuzzy sets to the realm of complex numbers, this extension studies and introduces intensely a new mathematical structure called a bipolar complex fuzzy set (BCFS). The range of values are extended to $[0,1]e^{i\alpha[0,1]} and\ [-1,0]e^{i\alpha[-1,0]}$ for both positive and negative membership functions, respectively, as a replacement for [-1, 0] X [0, 1] as in the bipolar fuzzy set. The main benefit of BCFS is that the amplitude and phase terms of BCFSs can convey bipolar fuzzy information. Moreover, the formal definition of BCF distance measure with an application has been introduced. Some basic mathematical operations on BCFS have been proposed and studied its properties with arithmetical examples by 1.1.[4]. On the other side, 1.1.[5] introduced the concept of complex multi-fuzzy sets (CMFSs) as a generalization of the concept of multi-fuzzy sets by adding the phase term to the definition of multi-fuzzy sets 1.1.[23]. In other words, they extend the range of the multi-membership function from the interval [0, 1] to unit disk in the complex plane. The novelty of CMFSs lies in the ability of complex multi-membership functions to achieve more range of values while handling the uncertainty of data that is periodic in nature. The basic operations on CMFSs, namely complement, union, intersection, product and cartesian product were studied along with accompanying examples. Also, 1.1.[5] introduced the intuitive definition of the distance measure between two complex multi-fuzzy sets which are used to define δ-equalities of complex multi-fuzzy sets. 1.1.[6] studied the truth, indeterminate, and false information all together in the form of positive and negative structures. They applied the extending idea for each range of membership degrees into complex numbers, to assist decision-makers in making suitable decisions. They use the polar form to represent the proposed information. Our approach uses the Cartesian form to avoid the limitation of representing the negative membership degree to the amplitude terms. Because the amplitude term represents the distance in the polar form, which is contrary to the basic idea of amplitude term in the complex numbers.

In this article, we present the notion of the complex bipolar multi-fuzzy set which is a generalization of the bipolar multi-fuzzy set to the complex realm by using cartesian coordinate $x + iy$, in contrast to Alkouri approach which uses polar coordinate $(r.e^{i\alpha\theta})$. This generalization conveys the properties of bipolar fuzzy sets, multi-fuzzy sets, and complex fuzzy sets simultaneously. Also, we introduce main definition of complex bipolar multi-fuzzy sets (CBMFSs), formal definition of distance measure and δ −equalities under CBMFSs. Finally, we represent some basic mathematical operations on CBMFSs and study its properties with concrete examples.

2 Preliminaries

In this chapter, we state the basic concepts and theorems that are necessary in this article.

Definition 2.1. 1.1.[32] A fuzzy set A in a universe of discourse U is characterised by a membership function $\mu_A(x)$ that takes values in the interval [0,1].

Definition 2.2. 1.1.[22] A complex fuzzy set A, defined on a universe of discourse U, is characterised by a membership function $\mu_A(x)$, that assigns to any element $x \in U$ a complex-valued grade of membership in A. By definition, the values of $\mu_A(x)$ may receive all lying within the unit circle in the complex plane, thus $\mu_A(x) = r_A(x).e^{iw_A(x)}$ where $i = \sqrt{-1}$, each of $r_A(x)$ and $w_A(x)$ is real-valued, and $r_A(x) \in [0,1]$. The CFS A may be represented as the set of ordered pairs $A = \{(x, \mu_A(x)): x \in U\}$.

Definition 2.3. 1.1.[23] Let X be a nonempty set, N the set of all natural numbers and $\{Li : i \in N\}$ a family of complete lattices. A multi-fuzzy set A in X is a set of ordered sequences: $A = \{\langle x, \mu_1(x), \mu_2(x), \dots, \mu_i(x), \dots\rangle : x \in X\}, where\ \mu_i \in L_i^x, for\ i \in N$

Remark 2.1. 1.1.[23] The function $\langle \mu_A = \mu_1, \mu_2, \dots\rangle$ is called a multi-membership function of multi-fuzzy set A. If the sequences of the membership functions have only $k - terms$ (finite number of terms), k is called the dimension of A. Let $L_i = [0,1]$ (for $i = 1,2,\dots,k$), then the set of all multi-fuzzy sets in X of dimension k is denoted by $M^kFS(X)$.

Definition 2.4. 1.1.[22] A complex fuzzy complement of A is defined as follows:

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)): x \in U\} = \{(x, r_{\bar{A}}(x).e^{iw_{\bar{A}}(x)}): x \in U\}$$

where $\mathrm{r}_{\bar{A}}(\mathrm{x}) = 1 - \mathrm{r}_{\mathrm{A}}(\mathrm{x})$ and $\mathrm{w}_{\bar{A}}(\mathrm{x}) = 2\pi - \mathrm{w}_{\mathrm{A}}(\mathrm{x})$

Definition 2.5. 1.1.[14] let k be appositive integer. A bipolar multi-fuzzy set M in U is a set of ordered sequences of the form:

$M = \left\{ x, \left(\left(\mu_{M_1}^+(x), \mu_{M_1}^-(x)\right), \dots, \left(\mu_{M_i}^+(x), \mu_{M_i}^-(x)\right), \dots, \left(\mu_{M_k}^+(x), \mu_{M_k}^-(x)\right)\right) : x \in U \right\}$, where $\mu_{M_i}^+(x): U \to [0,1]$, $\mu_{M_i}^-(x): U \to [-1,0], i = 1,2,3,\dots,k$, every $\mu_{M_i}^+(x)$ denotes for positive information and every $\mu_{M_i}^-(x)$ denotes for negative information.

The function $\mu_M = ((\mu_1^+(x), \mu_1^-(x)), (\mu_2^+(x), \mu_2^-(x)), \dots, (\mu_k^+(x), \mu_k^-(x))$ is called the bipolar multi-membership function of bipolar multi-fuzzy set M, k is called the dimension of M .

Remark 2.2. 1.1.[14] Clearly, a bipolar multi-fuzzy set of dimension 1 is a bipolar fuzzy set proposed.

Definition 2.6. (Yong Yang at el., 2014) let $M \in BM^kF(U)$.

If $M = \{x, (0,0), (0,0), \dots, (0,0): x \epsilon U\}$,then

M is called the null bipolar multi-fuzzy set of dimensions k , denoted by 0_k . If $M = \{x, ((1,-1), (1,-1), \dots, (1,-1): x \in U\}$, then L is called the absolute bipolar multi-fuzzy set of dimensions k, denoted by 1_k.

Definition 2.7. 1.1.[4] If A and B are $BCFSs$ in a universe of discourse X, where :

$$A = \{\langle x, r_A^+(x).e^{i\alpha w_A^+(x)}, r_A^-(x).e^{i\alpha w_A^-(x)}\rangle : x \in X\}$$

$$B = \{\langle x, r_B^+(x).e^{i\alpha w_B^+(x)}, r_B^-(x).e^{i\alpha w_B^-(x)}\rangle : x \in X\}$$

Then

1. $A \subset B$ if and only if $r_A^+(x) \le r_B^+(x)$, $r_A^-(x) \ge r_B^-(x)$, $w_A^+(x) \le w_B^+(x)$ and $w_A^-(x) \ge w_B^-(x)$ for all $x \epsilon X$.
2. $A = B$ if and only if $r_{A_j}^+(h) = r_{B_j}^+(h)$, $r_A^-(x) = r_B^-(x)$, $w_A^+(x) = w_B^+(x)$ and $w_A^-(x) = w_B^-(x)$ for all $x \epsilon X$.

3 Complex Bipolar Multi-Fuzzy Sets

The main idea is concluded by generalizing the range of membership functions of BMFS from [0, 1] to unit square in the complex plane. This section contains two parts; in section 3.1. the motivation is introduced for our study. Consequently, the formal definition of complex bipolar multi-fuzzy sets is presented and formalized in section 3.2.

3.1 Motivation and expected applications

The novel concept of CBMFS can be divided into two parts, which are pure mathematical motivation and meaning motivation.

The mathematical motivation can be crystallized by applying the historical idea of extending the set {0, 1} to [0, 1], then to complex numbers in different uncertainty sets. This extended idea can be noticed in the uncertainty sets, specifically in the range of membership functions. For instance, the range of membership function of the crisp set was extended from {0, 1} to [0, 1] to create a fuzzy set. After that the range [0, 1] in FS was extended to $\{a: a \in \mathbb{C}: |a| \le 1\}$ (i.e unit disk in $\mathbb{C}$) to create a complex fuzzy set. Also, in an intuitionistic fuzzy set, the range of membership and non-membership was extended from [0, 1] to unit disk in the complex plane to create (CIFS), for both membership and non-membership rang. This idea was widely applied in several uncertainty sets to create CBFS, CMFS, CFSS, and others.

Several uncertainty sets are not yet developed to be in the complex realm. This helps us to discover the present work and introduce the novel concept of CBMFS with its suitable mathematical structure and logical steps to create the formal definitions, basic operations, and properties by satisfying inherit conditions to extend BMFS to the unit square in the complex realm.

Undoubtedly, the reason of diversity and abundance of uncertainty sets refer to differences of abilities and representations to convey human being information or data to a mathematical formula and vice versa. Therefore, the meaning motivation reflects a great affect to choose and create a suitable uncertainty sets to convey human being information or data. The complex realm was added to uncertainty sets to get more range of membership functions. This extra range is covered the ability of representing the phase/ level of information to cover the full meaning of data besides to the represented information itself, (i.e convey uncertainty and periodicity information). So, CBMFS represents the information with properties of representing element/ object that satisfy the property and implicit counter property in k-dimension with respect to its phase/level. In other words, the positive real part represents element/ object that satisfies the property and the positive imaginary part represents the level/phase has been measured to the element/ object that satisfy the property for each dimension. Also, the negative real part represents element/ object that satisfies the implicit counter property and the negative imaginary part represents the level/phase that has been measured to the element/ object which satisfies the implicit counter property for each dimension. As a conclusion, CBMFS has ability to convey BMF information to mathematical formula and vise versa without losing the full meaning of BMF information.

3.2 Formal definition of CBMFS

In this section, a novel concept of complex bipolar multi-fuzzy set is introduced and studied. Some remarks are added to confine representations for some cases of real and imaginary parts. After that, some numerical examples are presented. Also, the inclusion definition between two CBMFSs are presented with illustration example.

Definition 3.2.1. let k be appositive integer. A complex bipolar multi-fuzzy set L in H is a set of ordered sequences of the form:

$L = \left\{ h, \left(\left(\mu_{L_1}^+(h), \mu_{L_1}^-(h)\right), \ldots, \left(\mu_{L_j}^+(h), \mu_{L_j}^-(h)\right), \ldots, \left(\mu_{L_k}^+(h), \mu_{L_k}^-(h)\right) \right) : h \in H \right\}$, where $\mu_{L_j}^+(h): H \to [0,1] + i[0,1]$, $\mu_{L_j}^-(h): H \to [-1,0] + i[-1,0]$. The values of positive information may be denoted by $\mu_{L_j}^+(h) = r_{L_j}^+(h) + iw_{L_j}^+(h)$, and negative information may be denoted by $\mu_{L_j}^-(h) = r_{L_j}^-(h) + iw_{L_j}^-(h)$, for each $j = 1,2,3,\ldots,k$ and take values lies in the unit square in the complex plane, where $i = \sqrt{-1}$, each of $r_{L_j}^+(h)$ and $w_{L_j}^+(h)$ belong to [0,1], while $r_{L_j}^-(h)$ and $w_{L_j}^-(h)$ are belong to [-1,0]. The function $\mu_L = \left(\left(\mu_1^+(h), \mu_1^-(h)\right), \left(\mu_2^+(h), \mu_2^-(h)\right), \ldots, \left(\mu_k^+(h), \mu_k^-(h)\right)\right)$ is called a complex bipolar multi-membership function of complex bipolar multi-fuzzy set L, k is called the dimension of L .

In fact, a complex bipolar multi-fuzzy set can be reduced to be bipolar multi-fuzzy set by letting both positive and negative imaginary parts equal to zero in complex bipolar multi-fuzzy sets. The set of all complex bipolar multi-fuzzy sets of diminution k in H is denoted by $CBM^kF(L)$.

The imaginary part may denote the BMF information, so positive and negative imaginary part values in this demonstration case belong to [0,1] and [-1,0] respectively. Also, they satisfied $BMFS$ constraint.

Remark 3.2.1. In a $BCFS$

$$L = \left\{h, \mu^+_{L_j}(h) = r^+_{L_j}(h) + iw^+_{L_j}(h), \mu^-_{L_j}(h) = r^-_{L_j}(h) + iw^-_{L_j}(h): h \in H\right\}$$

(1) If both $r^+_{L_j}(h)$ and $r^-_{L_j}(h)$ (for all j= 1, 2, …, k) equal to 0, then it's the circumstances that h is observed as neutral having no positive or negative satisfaction for L with effected by any periodicity semantics value of $w^+_{L_j}(h)$ and $w^-_{L_j}(h)$.

(2) If $r^+_{L_j}(x) \neq 0$ and $r^-_{L_j} = 0$, then it's the circumstances that h is observed as having only positive satisfaction for L , with respect to affected value of $w^+_{L_j}(h)$.

(3) If $r^+_{L_j}(h) = 0$ and $r^-_j(h) \neq 0$, then it is the circumstances that h dose not satisfy the property of L , but somewhat satisfies the counter property of L, with respect to affected value of $w^-_{L_j}(h)$.

(4) If $r^+_{L_j}(h) \neq 0$ and $r^-_{L_j}(h) \neq 0$, then the membership function of the property overlaps that of its counter property over some portion of the domain, with respect to affected values of $w^+_{L_j}(h)$ and $w^-_{L_j}(h)$,respectivly.

Remark 3.2.2. If $\frac{1}{2}\sum_{j=1}^{k}(r_j^+(h) + w_j^+(h)) \leq 1$ *and* $\frac{1}{2}\sum_{j=1}^{k}\left(r_j^-(h) + w_j^-(h)\right) \geq -1$, $\forall h \in H$, then L of dimension k is called a normalized $CBMFS$. Otherwise L is non-normalized.

Definition3.2.2. let $L \in CBM^kF(H)$.

If $L = \{h, (0 + i, 0 - i), (0 + i, 0 - i), \ldots, (0 + i, 0 - i): h \epsilon H\}$, then L is called the null complex bipolar multi-fuzzy set of dimensions k, denoted by 0_k.

If $L = \{h, ((1, -1), (1, -1), \ldots, (1, -1): h \in H\}$,(i.e, positive and negative imaginary parts equal to zero), then L is called the real part of complex bipolar multi-fuzzy set of dimensions k, denoted by 1_k.

If $L = \{h, ((i, -i), (i, -i), \ldots, (i, -i): h \in H\}$,(i.e, positive and negative real parts equal to zero), then L is called the imaginary part of complex bipolar multi-fuzzy set of dimensions k, denoted by i_k.

Definition 3.2.3. let k be a positive integer and let A and B be two complex bipolar multi-fuzzy sets in a universe of discourse H of dimension k given as follows:

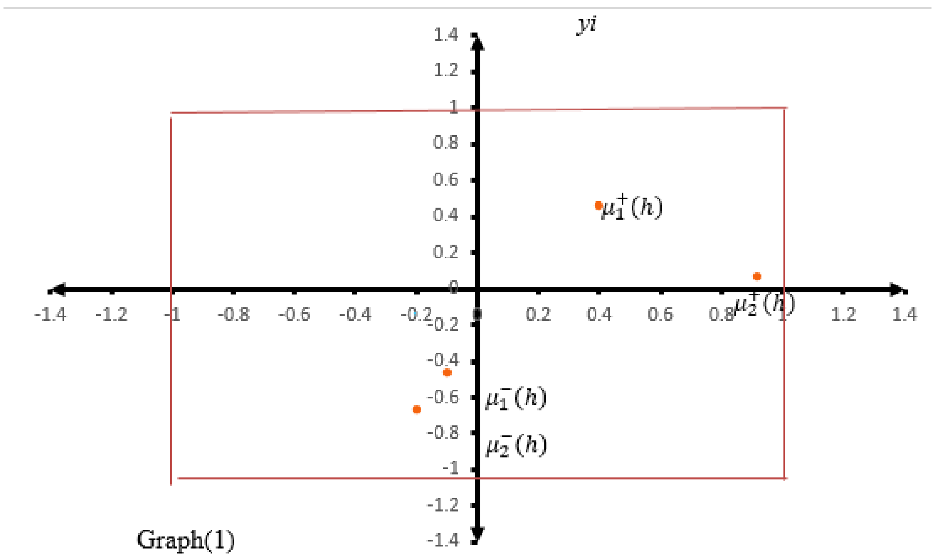
$A = \{h, (r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)): h \in H, j = 1,2,3, \ldots, k\}$

$B = \{h, (r^+_{B_j}(h) + iw^+_{B_j}(h), r^-_{B_j}(h) + iw^-_{B_j}(h)): h \in H, j = 1,2,3, \ldots, k\}$

Then:

1. $A \subset B$ if and only if $r^+_{A_j}(h) \leq r^+_{B_j}(h)$, $r^-_{A_j}(h) \geq r^-_{B_j}(h)$, $w^+_{A_j} \leq w^+_{B_j}$ and $w^-_{A_j} \geq w^-_{B_j}$ for all $h \epsilon H$ and $j = 1,2,3, \ldots, k$.
2. $A = B$ if and only if $r^+_{A_j}(h) = r^+_{B_j}(h)$, $r^-_{A_j}(h) = r^-_{B_j}(h)$, $w^+_{A_j}(h) = w^+_{B_j}(h)$ and $w^-_{A_j}(h) = w^-_{B_j}(h)$ for all $h \epsilon H$ and $j = 1,2,3, \ldots, k$.

Example 3.2.1. let $L = \left\{h, \langle\left(0.4 + i0.5, \left(-0.1 + i(-0.4)\right)\right)\rangle, \langle\left(0.9 + i0.1, \left(-0.2 + i(-0.6)\right)\right)\rangle\right\}$. then the geometric graph of a CBMFS is given by Graph (1)



Graph(1)

4 Basic operations on CBMFS

In this section, the basic operations; complement, union, and intersection are presented and formalized based on the CBMFS. Also, some numerical examples are presented separately for all operations. Finally, relations and properties combined these operations are principally proved.

Defenition 4.1. Let L be a CBMFS, where L is given as:

$$L = \left\{\left\langle h, \left(r^+_{L_1}(h) + iw^+_{L_1}(h), r^-_{L_1}(h) + iw^-_{L_1}(h)\right), \dots, \left(r^+_{L_j}(h) + iw^+_{L_j}(h), r^-_{L_j}(h) + iw^-_{L_j}(h)\right), \dots, \left(r^+_{L_k}(h) + iw^+_{L_k}(h), r^-_{L_k}(h) + iw^-_{L_k}(h)\right)\right\rangle : h \in H\right\} \textit{ for each } j = 1,2, \dots, k$$

Thus, the complement of L, denoted by L^c, defined as

$$L^c = \left\{\left\langle h, \left(r^+_{L^c_1}(h) + iw^+_{L^c_1}(h), r^-_{L^c_1}(h) + iw^-_{L^c_1}(h)\right), \dots, \left(r^+_{L^c_j}(h) + iw^+_{L^c_j}(h), r^-_{L^c_j}(h) + iw^-_{L^c_j}(h)\right), \dots, \left(r^+_{L^c_k}(h) + iw^+_{L^c_k}(h), r^-_{L^c_k}(h) + iw^-_{L^c_k}(h)\right)\right\rangle : h \in H\right\} \textit{ for each } j = 1,2, \dots, k$$

where, $r^+_{L^c_j}(h) = 1 - r^+_{L_j}(h)$, $r^-_{L^c_j}(h) = -1 - r^-_{L_j}(h)$, $w^+_{L^c_j}(h) = 1 - w^+_{L_j}(h)$, and $w^-_{L^c_j}(h) = -1 - w^-_{L_j}(h)$

Example 4.1. Let L be a $CBMFS$. Consider

$$L = \{\langle h_1, (0.4 + i0.25, -0.3 + i(-0.5)), (0.3 + i0.375, -0.2 + i(-0.625)), (0.2 + i0.5, -0.1 + i(-0.75))\rangle, \langle h_2, (0.5 + i0.2, -0.4 + i(-0.25)), (0.6 + i0.325, -0.3 + i(-0.575)), (0.7 + i0.5, -0.2 + i(-0.525))\rangle \langle h_3, (0.6 + i0.175, -0.5 + i(-0.25)), (0.7 + i0.425, -0.4 + i(-0.675)), (0.8 + i0.325, -0.3 + i(-0.575))\rangle : h_j \in H\} \textit{for each } j = 1,2, \dots, k$$

Thus the complement of (L) may be given as follow:

$$L^c = \{\langle h_1, (0.6 + i0.75, -0.7 + i(-0.5)), (0.7 + i0.625, -0.8 + i(-0.375)), (0.8 + i0.5, -0.9 + i(-0.25))\rangle\}. \langle h_2, (0.5 + i0.8, -0.6 + i(-0.75)), (0.4 + i0.675, -0.7 + i(-0.425)), (0.3 + i0.5, -0.8 + i(-0.475))\rangle \langle h_3, (0.4 + i0.825, -0.5 + i(-0.75)), (0.3 + i0.575, -0.6 + i(-0.325)), (0.2 + i0.675, -0.7 + i(-0.425))\rangle : h_j \in H\} \textit{ for each } j = 1,2, \dots, k$$

Definition 4.2. Let A and B be two $CBMFS$ on the universe of discourse H, with positive and negative complex-valued membership functions such as:

$$A = \left\{\left\langle h, \left(r^+_{A_1}(h) + iw^+_{A_1}(h), r^-_{A_1}(h) + iw^-_{A_1}(h)\right), \dots, \left(r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)\right), \dots, \left(r^+_{A_k}(h) + iw^+_{A_k}(h), r^-_{A_k}(h) + iw^-_{A_k}(h)\right)\right\rangle : h \in H\right\} \textit{for each } j = 1,2, \dots, k \text{ . and}$$

$$B = \left\{\left\langle h, \left(r^+_{B_1}(h) + iw^+_{B_1}(h), r^-_{B_1}(h) + iw^-_{B_1}(h)\right), \dots, \left(r^+_{B_j}(h) + iw^+_{B_j}(h), r^-_{B_j}(h) + iw^-_{B_j}(h)\right), \dots, \left(r^+_{B_k}(h) + iw^+_{B_k}(h), r^-_{B_k}(h) + iw^-_{B_k}(h)\right)\right\rangle : h \in H\right\} \textit{ for each } j = 1,2, \dots, k$$

The $CBMFS$ union of A and B denoted by $A \cup B$, is specified

$$A \cup B = \left\{\left\langle \begin{matrix} h, \left(r^+_{A\cup B_1}(h) + iw^+_{A\cup B_1}(h), r^-_{A\cup B_1}(h) + iw^-_{A\cup B_1}(h)\right), \\ \dots, \left(r^+_{A\cup B_j}(h) + iw^+_{A\cup B_j}(h), r^-_{A\cup B_j}(h) + iw^-_{A\cup B_j}(h)\right), \\ \dots, \left(r^+_{A\cup B_k}(h) + iw^+_{A\cup B_k}(h), r^-_{A\cup B_k}(h) + iw^-_{A\cup B_k}(h)\right) \end{matrix} \right\rangle : h \in H \right\}$$

$\textit{for each } j = 1,2, \dots, k$.

where, $r^+_{A\cup B_j}(h) = max\left[r^+_{A_j}(h), r^+_{B_j}(h)\right]$, $r^-_{A\cup B_j}(h) = min\left[r^-_{A_j}(h), r^-_{B_j}(h)\right]$

$w^+_{A\cup B_j}(h) = max\left[w^+_{A_j}(h), w^+_{B_j}(h)\right]$ and $w^-_{A\cup B_j}(h) = min\left[w^-_{A_j}(h), w^-_{B_j}(h)\right]$

Example 4.2. Let A and B be two $CBMFSs$ in the same common universe H, suppose that $A = \{\langle h, (0.2 + i0.5, -0.8 + i(-0.3)), (0.7 + i0.35, -0.3 + i(-0.5)), (0.1 + i0.75, -0.6 + i(-0.15))\rangle\}$

$B = \{\langle h, (0.1 + i0.15, -0.7 + i(-0.8)), (0.8 + i0.35, -0.6 + i(-0.65)), (0.4 + i0.2, -0.9 + i(-0.35))\rangle\}$

For the $CBMF$ union, we have

$$A \cup B = \left\{\left\langle \begin{matrix} h, (0.2 + i0.5, -0.8 + i(-0.8)), (0.8 + i0.35, -0.6 + i(-0.65)), \\ (0.4 + i0.75, -0.9 + i(-0.35)) \end{matrix} \right\rangle : h \epsilon H\right\}.$$

Definition 4.3. Let A *and* B be two $CBMFSs$ on the universe of discourse H, with positive and negative complex-valued such as:

$$A=\left\{\left\langle h,\left(r_{A_1}^{+}(h)+iw_{A_1}^{+}(h),r_{A_1}^{-}(h)+iw_{A_1}^{-}(h)\right),h,\left(r_{A_1}^{+}(h)+iw_{A_1}^{+}(h),r_{A_1}^{-}(h)+iw_{A_1}^{-}(h)\right),\right.\right.$$

$$\left.\left.\ldots,\left(r_{A_k}^{+}(h)+iw_{A_k}^{+}(h),r_{A_k}^{-}(h)+iw_{A_k}^{-}(h)\right)\right\rangle:h\in H\right\} \text{for each } j=1,2,\ldots,k \text{ . and}$$

$$B=\left\{\left\langle h,\left(r_{B_1}^{+}(h)+iw_{B_1}^{+}(h),r_{B_1}^{-}(h)+iw_{B_1}^{-}(h)\right),\ldots,\left(r_{B_j}^{+}(h)+iw_{B_j}^{+}(h),r_{B_j}^{-}(h)+iw_{B_j}^{-}(h)\right),\ldots,\left(r_{B_k}^{+}(h)+iw_{B_k}^{+}(h),r_{B_k}^{-}(h)+iw_{B_k}^{-}(h)\right)\right\rangle:h\in H\right\} \text{for each } j=1,2,\ldots,k.$$

The $CBMFS$ intersection of A and B, denoted by $A\cap B$, is specified by

$$A\cap B=\left\{\left\langle\begin{matrix}h,\left(r_{A\cap B_1}^{+}(h)+iw_{A\cap B_1}^{+}(h),r_{A\cap B_1}^{-}(h)+iw_{A\cap B_1}^{-}(h)\right),\\ \ldots,\left(r_{A\cap B_j}^{+}(h)+iw_{A\cap B_j}^{+}(h),r_{A\cap B_j}^{-}(h)+iw_{A\cap B_j}^{-}(h)\right),\\ \ldots,\left(r_{A\cap B_k}^{+}(h)+iw_{A\cap B_k}^{+}(h),r_{A\cap B_k}^{-}(h)+iw_{A\cap B_k}^{-}(h)\right)\end{matrix}\right\rangle:h\epsilon H\right\}$$

$for\ each\ j=1,2,\ldots,k.$

where, $r_{A\cap B_j}^{+}(h)=\min\left[r_{A_j}^{+}(h),r_{B_j}^{+}(h)\right]$, $r_{A\cap B_j}^{-}(h)=\max\left[r_{A_j}^{-}(h),r_{B_j}^{-}(h)\right]$,
$w_{A\cap B_j}^{+}(h)=\min\left[w_{A_j}^{+}(h),w_{B_j}^{+}(h)\right]$ and $w_{A\cap B_j}^{-}(h)=\max\left[w_{A_j}^{-}(h),w_{B_j}^{-}(h)\right]$.

Example 4.3. Let $A$and B be tow $CBMFSs$ in the same common universe H, suppose that
$A=\{\langle h,(0.2+i0.5,-0.8+i(-0.3)),(0.7+i0.35,-0.3+i(-0.5)),(0.1+i0.75,-0.6+i(-0.15))\rangle\}$
$B=\{\langle h,(0.1+i0.15,-0.7+i(-0.8)),(0.8+i0.35,-0.6+i(-0.65)),(0.4+i0.2,-0.9+i(-0.35))\rangle\}$.
For the $CBMF$ intersection, we have

$$A\cap B=\left\{\left\langle\begin{matrix}h,(0.1+i0.15,-0.7+i(-0.3)),(0.8+i0.35,-0.3+i(-0.5)),\\ (0.4+i0.2,-0.9+i(-0.35))\end{matrix}\right\rangle:h\epsilon H\right\}.$$

Theorem 4.1. Let A, B and D be any three $CBMFSs$. Then the following are true statements:

i. $(A^c)^c=A$
ii. $A\cup A=A$
iii. $A\cap A=A$
iv. $A\cup B=B\cup A$
v. $A\cap B=B\cap A$
vi. $A\cup(B\cap D)=(A\cup B)\cap(A\cup D)$
vii. $A\cap(B\cup D)=(A\cap B)\cup(A\cap D)$
viii. $A\cup(B\cup D)=(A\cup B)\cup D$
ix. $A\cap(B\cap D)=(A\cap B)\cap D$

Proof:

i) Let A be a CBMFS, given as:

$$A=\left\{\left\langle h,\left(r_{A_1}^{+}(h)iw_{A_1}^{+}(h),r_{A_1}^{-}(h)+iw_{A_1}^{-}(h)\right),\ldots,\left(r_{A_j}^{+}(h)+iw_{A_j}^{+}(h),r_{A_j}^{-}(h)+iw_{A_j}^{-}(h)\right),\ldots,\left(r_{A_k}^{+}(h)+iw_{A_k}^{+}(h),r_{A_k}^{-}(h)+iw_{A_k}^{-}(h)\right)\right\rangle:h\in H\right\}$$

$for\ each\ j=1,2,\ldots,k$,by using definition 1, we get

$$A^c=\left\{\left\langle h,\left(\left(1-r_{A_1}^{+}(h)\right)+i(1-w_{A_1}^{+}(h)),\left(-1-r_{A_1}^{-}(h)\right)+i(-1-w_{A_1}^{-}(h))\right),\ldots,\left(\left(1-r_{A_j}^{+}(h)\right)+i(1-w_{A_j}^{+}(h)),\left(-1-r_{A_j}^{-}(h)\right)+i(-1-w_{A_j}^{-}(h))\right),\ldots,\left(\left(1-r_{A_k}^{+}(h)\right)+i(1-w_{A_k}^{+}(h)),\left(-1-r_{A_k}^{-}(h)\right)+i(-1-w_{A_k}^{-}(h))\right)\right\rangle:h\in H\right\}$$

$for\ each\ j=1,2,\ldots,k$, by using definition 1 again, we get

$$(A^c)^c=\left\{\left\langle h,\left(\begin{matrix}\left[1-\left(1-r_{A_1}^{+}(h)\right)\right]+i[1-(1-w_{A_1}^{+}(h))],\\ \left[-1-\left(-1-r_{A_1}^{-}(h)\right)\right]+i[-1-(-1-w_{A_1}^{-}(h))]\end{matrix}\right),\ldots,\left(\begin{matrix}\left[1-\left(1-r_{A_j}^{+}(h)\right)\right]+i[1-(1-w_{A_j}^{+}(h))],\\ \left[-1-\left(-1-r_{A_j}^{-}(h)\right)\right]+i[-1-(-1-w_{A_j}^{-}(h))]\end{matrix}\right),\ldots,\left(\begin{matrix}\left[1-\left(1-r_{A_k}^{+}(h)\right)\right]+i[1-(1-w_{A_k}^{+}(h))],\\ \left[-1-\left(-1-r_{A_k}^{-}(h)\right)\right]+i[-1-(-1-w_{A_k}^{-}(h))]\end{matrix}\right)\right\rangle:h\in H\right\}=A$$

$, for\ each\ j = 1,2, \dots, k$

ii) Similar proof to (i).

iii) Let

$$A = \left\{ \left. \begin{matrix} h, \left(r^+_{A_1}(h) + iw^+_{A_1}(h), r^-_{A_1}(h) + iw^-_{A_1}(h)\right), \\ \dots, \left(r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)\right), \\ \dots, \left(r^+_{A_k}(h) + iw^+_{A_k}(h), r^-_{A_k}(h) + iw^-_{A_k}(h)\right) \end{matrix} \right\rangle : h \in H \right\} for\ each\ j = 1,2, \dots, k,$$

by using definition 2, we get

$$A \cup A = \{h, (max[r^+_{A_1}(h), r^+_{A_1}(h)] + imax[w^+_{A_1}(h), w^+_{A_1}(h)], min[r^-_{A_1}(h), r^-_{B_1}(h)] + imin[w^-_{A_1}(h), w^-_{A_1}(h)]), \dots, \left(max\left[r^+_{A_j}(h), r^+_{A_j}(h)\right] + imax\left[w^+_{A_j}(h), w^+_{A_j}(h)\right], min\left[r^-_{A_j}(h), r^-_{A_j}(h)\right] + imin\left[w^-_{A_j}(h), w^-_{A_j}(h)\right]\right), \dots, (max[r^+_{A_k}(h), r^+_{A_k}(h)] + imax[w^+_{A_k}(h), w^+_{A_k}(h)], min[r^-_{A_k}(h), r^-_{A_k}(h)] + imin[w^-_{A_k}(h), w^-_{A_k}(h)])\}$$

$$= \left\{\left\langle h, \left(r^+_{A_1}(h) + iw^+_{A_1}(h), r^-_{A_1}(h) + iw^-_{A_1}(h)\right), \dots, \left(r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)\right), \dots, \left(r^+_{A_k}(h) + iw^+_{A_k}(h), r^-_{A_k}(h) + iw^-_{A_k}(h)\right)\right\rangle : h \in H\right\} = A, for\ each\ j = 1,2, \dots, k$$

iv) Similar proof to (iii).

v) Let

$$A = \left\{\left\langle h, \left(r^+_{A_1}(h) + iw^+_{A_1}(h), r^-_{A_1}(h) + iw^-_{A_1}(h)\right), \dots, \left(r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)\right), \dots, (r^+_{A_k}(h) + iw^+_{A_k}(h), r^-_{A_k}(h) + iw^-_{A_k}(h))\right\rangle : h \in H\right\} for\ each\ j = 1,2, \dots, k$$ and

$$B = \left\{\left\langle h, \left(r^+_{B_1}(h) + iw^+_{B_1}(h), r^-_{B_1}(h) + iw^-_{B_1}(h)\right), \dots, \left(r^+_{B_j}(h) + iw^+_{B_j}(h), r^-_{B_j}(h) + iw^-_{B_j}(h)\right), \dots, (r^+_{B_k}(h) + iw^+_{B_k}(h), r^-_{B_k}(h) + iw^-_{B_k}(h))\right\rangle : h \in H\right\} for\ each\ j = 1,2, \dots, k,$$

L.H.S by using definition 2, we get

$$A \cup B = \{h, (max[r^+_{A_1}(h), r^+_{B_1}(h)] + imax[w^+_{A_1}(h), w^+_{B_1}(h)], min[r^-_{A_1}(h), r^-_{B_1}(h)] + imin[w^-_{A_1}(h), w^-_{B_1}(h)]), \dots, \left(max\left[r^+_{A_j}(h), r^+_{B_j}(h)\right] + imax\left[w^+_{A_j}(h), w^+_{B_j}(h)\right], min\left[r^-_{A_j}(h), r^-_{B_j}(h)\right] + i\alpha min\left[w^-_{A_j}(h), w^-_{B_j}(h)\right]\right), \dots, (max[r^+_{A_k}(h), r^+_{B_k}(h)] + imax[w^+_{A_k}(h), w^+_{B_k}(h)], min[r^-_{A_k}(h), r^-_{B_k}(h)] + imin[w^-_{A_k}(h), w^-_{B_k}(h)]) : h \in H\}$$

R.H.S by using definition 2, we get

$$B \cup A = \{h, (max[r^+_{B_1}(h), r^+_{A_1}(h)] + imax[w^+_{B_1}(h), w^+_{A_1}(h)], min[r^-_{B_1}(h), r^-_{A_1}(h)] + imin[w^-_{B_1}(h), w^-_{A_1}(h)]), \dots, \left(max\left[r^+_{B_j}(h), r^+_{A_j}(h)\right] + imax\left[w^+_{B_j}(h), w^+_{A_j}(h)\right], min\left[r^-_{B_j}(h), r^-_{A_j}(h)\right] + imin\left[w^-_{B_j}(h), w^-_{A_j}(h)\right]\right), \dots, (max[r^+_{B_k}(h), r^+_{A_k}(h)] + imax[w^+_{B_k}(h), w^+_{A_k}(h)], min[r^-_{B_k}(h), r^-_{A_k}(h)] + imin[w^-_{B_k}(h), w^-_{A_k}(h)]), : h \in H\}$$

$for\ each\ j = 1,2, \dots k$

Thus; R.H.S = L.H.S (i.e $A \cup B = B \cup A$)

vi) Similar proof to (v).

Theorem4.2. (De Morgan's law) Let A and B be $CBMFSs$. Then

i. $(A \cup B)^c = A^c \cap B^c$,

ii. $(A \cap B)^c = A^c \cup B^c$.

Proof:

i)Let

$$A = \left\{\left\langle h, \left(r^+_{A_1}(h) + iw^+_{A_1}(h), r^-_{A_1}(h) + iw^-_{A_1}(h)\right), \dots, \left(r^+_{A_j}(h) + iw^+_{A_j}(h), r^-_{A_j}(h) + iw^-_{A_j}(h)\right), \dots, \left(r^+_{A_k}(h) + iw^+_{A_k}(h), r^-_{A_k}(h) + iw^-_{A_k}(h)\right)\right\rangle : h \in H\right\}$$ and

$$B = \left\{\left\langle h, \left(r^+_{B_1}(h) + iw^+_{B_1}(h), r^-_{B_1}(h) + iw^-_{B_1}(h)\right), \dots, \left(r^+_{B_j}(h) + iw^+_{B_j}(h), r^-_{B_j}(h) + iw^-_{B_j}(h)\right), \dots, (r^+_{B_k}(h) + iw^+_{B_k}(h), r^-_{B_k}(h) + iw^-_{B_k}(h))\right\rangle : h \in H\right\} for\ each\ j = 1,2, \dots, k,$$

then by using definition 2, we get $A \cup B = \{h, (max[r^+_{A_1}(h), r^+_{B_1}(h)] + imax[w^+_{A_1}(h), w^+_{B_1}(h)], min[r^-_{A_1}(h), r^-_{B_1}(h)] + imin[w^-_{A_1}(h), w^-_{B_1}(h)]),$

$$\dots, \left(max\left[r^+_{A_j}(h), r^+_{B_j}(h)\right] + imax\left[w^+_{A_j}(h), w^+_{B_j}(h)\right], min\left[r^-_{A_j}(h), r^-_{B_j}(h)\right] + imin\left[w^-_{A_j}(h), w^-_{B_j}(h)\right]\right),$$

$$\left.\ldots,\left(max\left[r_{A_k}^{+}(h),r_{B_k}^{+}(h)\right]+imax\left[w_{A_k}^{+}(h),w_{B_k}^{+}(h)\right],min\left[r_{A_k}^{-}(h),r_{B_k}^{-}(h)\right]+imin\left[w_{A_k}^{-}(h),w_{B_k}^{-}(h)\right]\right),\right\}:h\in H$$

Now, by using definition 1, we get $(A\cup B)^c=\{h,(\langle 1-max[r_{A_1}^{+}(h),r_{B_1}^{+}(h)]\rangle+i\langle 1-max[w_{A_1}^{+}(h),w_{B_1}^{+}(h)]\rangle,$
$\langle -1-min[r_{A_1}^{-}(h),r_{B_1}^{-}(h)]\rangle+i\langle -1-min[w_{A_1}^{-}(h),w_{B_1}^{-}(h)]\rangle),$
$\ldots,\left(\langle 1-max\left[r_{A_j}^{+}(h),r_{B_j}^{+}(h)\right]\rangle+i\,\langle 1-max\left[w_{A_j}^{+}(h),w_{B_j}^{+}(h)\right]\rangle,\right.$
$\left.\langle -1-min\left[r_{A_j}^{-}(h),r_{B_j}^{-}(h)\right]\rangle+i\,\langle -1-min\left[w_{A_j}^{-}(h),w_{B_j}^{-}(h)\right]\rangle\right),$

$$\left.\begin{matrix}(\langle 1-max[r_{A_k}^{+}(h),r_{B_k}^{+}(h)]\rangle+i\langle 1-max[w_{A_k}^{+}(h),w_{B_k}^{+}(h)]\rangle,\\ \ldots,\langle -1-min[r_{A_k}^{-}(h),r_{B_k}^{-}(h)]\rangle+i\langle -1-min[w_{A_k}^{-}(h),w_{B_k}^{-}(h)]\rangle):\\ h\in H\end{matrix}\right\}=$$

$$\left\{\begin{matrix} & \begin{pmatrix}min[r_{A^c{}_1}^{+}(h),r_{B^c{}_1}^{+}(h)]+imin[w_{A^c{}_1}^{+}(h),w_{B^c{}_1}^{+}(h)],\\ max[r_{A^c{}_1}^{-}(h),r_{B^c{}_1}^{-}(h)]+imax[w_{A^c{}_1}^{-}(h),w_{B^c{}_1}^{-}(h)]\end{pmatrix}, & \\ h, & \begin{pmatrix}min\left[r_{A^c{}_j}^{+}(h),r_{B^c{}_j}^{+}(h)\right]+imin\left[w_{A^c{}_j}^{+}(h),w_{B^c{}_j}^{+}(h)\right],\\ max\left[r_{A^c{}_j}^{-}(h),r_{B^c{}_j}^{-}(h)\right]+imax\left[w_{A^c{}_j}^{-}(h),w_{B^c{}_j}^{-}(h)\right]\end{pmatrix}, & \\ \ldots & & \\ & \ldots\begin{pmatrix}min[r_{A^c{}_k}^{+}(h),r_{B^c{}_k}^{+}(h)]+imin[w_{A^c{}_k}^{+}(h),w_{B^c{}_k}^{+}(h)],\\ max[r_{A^c{}_k}^{-}(h),r_{B^c{}_k}^{-}(h)]+imax[w_{A^c{}_k}^{-}(h),w_{B^c{}_k}^{-}(h)]\end{pmatrix} & :h\in H\end{matrix}\right\}$$

$=A^c\cap B^c, for\ each\ j=1,2,\ldots,k$

ii) Similar to (i).

5 Conclusion

We have successfully implemented and found a proper mathematical tool to deal with special data and the ability to represent the bipolar multi fuzzy uncertainty into the imaginary parts. This special data has two semantics: bipolar multi fuzzy uncertainty semantic and periodicity semantic with the ability to be represented as bipolar multi fuzzy data. The novelty of CBMFS appears in its ability to consider multiple values of positive and negative membership functions with k dimension in more detail than BMFS, CFS, and CBFS as illustrated in section 3.2. Also, the CBMFS represents the belongingness values that satisfies the property and implicit counter property for any object for both uncertainty and periodicity semantics in k dimension. On the other hand, both CBMFS and CBFS reduce to conventional BMFS and BFS, respectively by omitting the imaginary parts in CBMFS and phase term in CBFS. We should mention that the presented approach has a limitation in representing the truth, indeterminate, and false information simultaneously. This limitation highlighted the need for future research to incorporate our approach to the Neutrosophic sets. Also, as future research, we may combine our approach to Hypersoft sets to avoid the disability of handling data that can be formulated as several trait-valued disjoint sets which blend to various traits (see 1.1.[7]).

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