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$\mathcal{I}_\lambda$ -Statistical Convergence of Order α in Topological Groups

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Abstract

The purpose of this paper is to introduce and study $\mathcal{I}_\lambda$ - statistical convergence of order α in topological groups and we shall also present some inclusion theorems.

Keywords: *ideal convergence, ideal statistical convergence, statistical convergence, topological groups.*

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1 Introduction

We begin this section by giving some definitions and preliminaries:

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [7] (see also Schoenberg [26]) as follows : If $\mathbf{N}$ denotes the set of natural numbers and $K \subset \mathbf{N}$ then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \quad \text{and} \quad \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbf{N} : |x_k - L| \geq \epsilon\}$ has natural density zero. In this case, we write $st - \lim_k x(k) = L$ and we denote the set of all statistical convergent sequences by st. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [8] and Šalát [16]. Di Maio and Kočinac [13] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence. The notion has also been defined and studied in different steps, for example, in the locally convex space [12]; in intuitionistic fuzzy normed spaces[14]. Quite recently, Das and Savas [5] introduced the ideas of $\mathcal{I}_\tau$-convergence, $\mathcal{I}_\tau$-boundedness and $\mathcal{I}_\tau$-Cauchy condition of nets in a locally solid Riesz space. Also,[17] $\lambda-$ statistical convergence in random 2-normed space was studied by Savas. Quite recently, Savas [24] studied $\mathcal{I}$-lacunary statistical convergence of order α in topological groups. Further, in [25]$\mathcal{I}_\lambda$ -statistically convergent sequences in topological groups was presented.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1 \tag{1}$$

The collection of such sequence λ will be denoted by Δ.

Also, in [15] a new type of convergence called λ- statistical convergence was introduced as follows: A sequence $x = (x_k)$ of real numbers is said to be λ-statistically convergent to L (or, S_λ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0$$

where $I_n = [n - \lambda_n + 1, n]$ and $|A|$ denotes the cardinality of $A \subset \mathbf{N}$. In [15] the relation between λ- statistical convergence and statistical convergence was established among other things.

In[14], Mohiuddine and Lohani extended the idea of λ- statistical convergence with respect to the intuitionistic fuzzy normed space.

P. Kostyrko et al. [10] introduced the concept of $\mathcal{I}$-convergence of sequence in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [4, 11, 17, 18, 19, 20, 21].

The purpose of this paper is to study $\mathcal{I}_\lambda$- statistical convergence of order α in topological groups and to give some important inclusion theorems.

2 Definitions and Notations

The following definitions and notions will be needed.

Definition 2.1 *[10]. A family $\mathcal{I} \subset 2^{\boldsymbol{N}}$ is said to be an ideal of $\boldsymbol{N}$ if the following conditions hold:*
(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$,

Definition 2.2 *A non-empty family $F \subset 2^{\boldsymbol{N}}$ is said to be an filter of $\boldsymbol{N}$ if the following conditions hold:*
(a) $\phi \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F$, $A \subset B$ implies $B \in F$,

If $\mathcal{I}$ is a proper ideal of $\mathbf{N}$ (i.e., $\mathbf{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbf{N} : \exists\, A \in \mathcal{I} : M = \mathbf{N} \setminus A\}$ is a filter of $\mathbf{N}$. It is called the filter associated with the ideal.

Definition 2.3 *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \boldsymbol{N}$.*

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbf{N}$.

Definition 2.4 *(See [10]) Let $\mathcal{I} \subset 2^{\boldsymbol{N}}$ be a proper admissible ideal in $\boldsymbol{N}$. The sequence $x = (x_k)$ of elements of $\boldsymbol{R}$ is said to be $\mathcal{I}$-convergent to $L \in \boldsymbol{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \boldsymbol{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$.*

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k. \tag{2}$$

A sequence $x = (x_k)$ is said to be $\mathcal{I} - [V, \lambda] -$ summable to L, if

$$\mathcal{I} - lim_n t_n(x) = L \tag{3}$$

i.e. for any $\delta > 0$,

$$\{n \in N : |t_n(x) - L| \geq \delta\} \in \mathcal{I}. \tag{4}$$

If $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}$. $\mathcal{I} - [V, \lambda] -$ summablility becomes $[V, \lambda]$ summability [15].

Recently Savaş and Das [17] defined $\mathcal{I}$- statistical convergence and $\mathcal{I}_\lambda$-statistical convergence as follows:

Definition 2.5 *A sequence $x = (x_k)$ is said to be $\mathcal{I}$-statistically convergent to L or $S(\mathcal{I})$-convergent to L, if for each $\epsilon > 0$ and $\delta > 0$*

$$\{n \in \boldsymbol{N} : \{n \in \boldsymbol{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S(\mathcal{I}))$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted by simply $S(\mathcal{I})$.

Definition 2.6 *A sequence $x = (x_k)$ is said to be $\mathcal{I}_\lambda-$ statistically convergent to L or $S_\lambda(\mathcal{I})$ convergent to L if for any $\epsilon > 0$ and $\delta > 0$*

$$\{n \in \boldsymbol{N} : \frac{1}{\lambda_n}|\{k \in I_n : |x_k - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S_\lambda(\mathcal{I}))$. The class of all $\mathcal{I}_\lambda-$statistically convergent sequences will be denoted by $S_\lambda(\mathcal{I})$.

By X, we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. In [1], a sequence $x = (x_k)$ in X is called to be statistically convergent to an element L of X if for each neighbourhood U of 0,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k - L \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in X is denoted by $st(X)$.

Furthermore we define λ - statistical convergence in topological groups as follows: A sequence $x = (x_k)$ is said to be $S_\lambda-$convergent to L (or λ-statistically convergent to L) if for each neighborhood U of 0,

$$lim_n \frac{1}{\lambda_n} |k \in I_n : x_k - L \notin U\}| = 0$$

. In this case, we write

$$S_\lambda - \lim_{k\to\infty} x_k = L \text{ or } \mathrm{x}_k \to L(S_\lambda) \tag{5}$$

and define

$$S_\lambda(X) = \left\{x = (x_k) : \text{ for Some Ł}, \ S_\lambda - \lim_{k\to\infty} x_k = L\right\}. \tag{6}$$

3 $\mathcal{I}_\lambda$- convergence of order α

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan in [9] and later on statistical convergence of order α and strongly p- Cesàro summability of order α studied by Çolak [2]. Further this concept was generalized by Çolak and Bektas [3] as follows: Let $\lambda = (\lambda_n) \in \Delta$

and $0 < \alpha \le 1$ be given. The sequences $x = (x_k)$ is said to be λ-statistically convergent of order α if there is a complex number L such that

$$\{n \in \mathbf{N} : \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_k - L| \ge \epsilon\}| = 0,$$

where λ_n^α denote the α th power $(\lambda_n)^\alpha$ of λ_n, that is, $\lambda^\alpha = (\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, ..., \lambda_n^\alpha, ...)$.

In this section, we introduce and study $\mathcal{I}_\lambda$- statistical convergence of order α for sequence in topological groups and we shall also present some inclusion theorems.

We now have

Definition 3.1 *A sequence $x = (x_k)$ in X is called to be statistically convergent of order α to L of X if for each neighbourhood U of 0,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \le n : x_k - L \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The set of all statistically convergent of order α sequences in X is denoted by $S^\alpha(X)$.

Also, we define λ - statistical convergence of order α in topological groups as follows:

Definition 3.2 *A sequence $x = (x_k)$ is said to be $S_\lambda-$ convergent of order α to L (or λ- statistically convergent of order α to L) if for each neighborhood U of 0,*

$$lim_n \frac{1}{\lambda_n^\alpha} |k \in I_n : x_k - L \notin U\}| = 0$$

.

In this case, we define

$$S_\lambda^\alpha(X) = \left\{ x = (x_k) : \text{ for some } \mathrm{L}, \mathrm{S}_\lambda^\alpha - \lim_{k\to\infty} x_k = L \right\} \tag{7}$$

If we take $\lambda_n = n$, $S_\lambda^\alpha(X)$ reduce to $S^\alpha(X)$.

Now we shall give the definitions of $\mathcal{I}$- statistical convergence and $\mathcal{I}_\lambda-$ statistical convergence of order α in topological groups as follows:

Definition 3.3 *A sequence $x = (x_k)$ is said to be $\mathcal{I}$-statistically convergent of order α to L or $S^\alpha(\mathcal{I})$-convergent to L if for each $\delta > 0$ and each neighbourhood U of 0,*

$$\{n \in \mathbf{N} : \frac{1}{n^\alpha} |\{k \le n : x_k - L \notin U\}| \ge \delta\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S^\alpha(\mathcal{I}))$. The class of all $\mathcal{I}$-statically convergent sequences will be denoted by simply $S^\alpha(\mathcal{I})(X)$.

For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq N : A \text{ is a finitesubset}\}$, $\mathcal{I}$- statistical convergence of order α becomes statistical convergence of order α in topological groups which has not been study till now. Finally for $\mathcal{I} = \mathcal{I}_{fin}$, $\lambda_n = n$ and $\alpha = 1$ it becomes statistical convergence which is studied in [1].

Definition 3.4 *A sequences $x = (x_k)$ is said to be $\mathcal{I}_\lambda$- statistically convergent of order α to L or $S_\lambda^{\mathcal{I}}$-convergent of order α to L if for each $\delta > 0$ and each neighbourhood U of* 0

$$\{n \in \mathbf{N} : \frac{1}{\lambda_n^\alpha} |\{k \in I_n : x_k - L \notin U\}| \geq \delta\} \in \mathcal{I}.$$

In this case, we write

$$S_\lambda^\alpha(\mathcal{I})(X) = \Big\{x = (x_k) : \text{ for some } \mathrm{L}, \mathrm{S}_\lambda^\alpha(\mathcal{I}) - \lim_{k\to\infty} x_k = L\Big\}$$

If we take $\alpha = 1$, we have

$$S_\lambda(\mathcal{I})(X) = \Big\{x = (x_k) : \text{ for some } \mathrm{L}, \mathrm{S}_\lambda(\mathcal{I}) - \lim_{k\to\infty} x_k = L\Big\}$$

For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}_\lambda$- statistical convergence of order α becomes λ- statistical convergence of order α in topological groups which has not been study till now. If $\lambda_n = n$, $\mathcal{I}_\lambda$- statistical convergence of order α becomes statistical convergence of order α in topological groups.

4 Inclusion Theorems

We shall now prove the following theorems.

Theorem 4.1 *Let $0 < \alpha \leq \beta \leq 1$. Then $S_\lambda^\alpha(I)(X) \subset S_\lambda^\beta(I)(X)$.*

Proof: Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{|\{k \in I_n : x_k - L \notin U\}|}{\lambda_n^\beta} \leq \frac{|\{k \in I_n : x_k - L \notin U\}|}{\lambda_n^\alpha}$$

and so for any $\delta > 0$ and any neighbourhood U of 0

$$\{n \in \mathbf{N} : \frac{|\{k \in I_n : x_k - L \notin U\}|}{\lambda_n^\beta} \geq \delta\} \subset \{n \in \mathbf{N} : \frac{|\{k \in I_n : x_k - L \notin U\}|}{\lambda_n^\alpha} \geq \delta\}.$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S_\lambda^\alpha(\mathcal{I})(X) \subset S_\lambda^\beta(\mathcal{I})(X)$.

Corollary 4.2 *If a sequence is I_λ-statistically convergent of order α to L for some $0 < \alpha \leq 1$ then it is $\mathcal{I}_\lambda$-statistically convergent to L i.e. $S_\lambda^\alpha(\mathcal{I}) \subset S_\lambda(\mathcal{I})$.*

Similarly we can show that

Theorem 4.3 *Let $0 < \alpha \leq \beta \leq 1$. Then*
(i) $S^\alpha(\mathcal{I}) \subset S^\beta(\mathcal{I})$.
(ii) In particular $S^\alpha(\mathcal{I}) \subset S(\mathcal{I})$.

We now have

Theorem 4.4 $S^\alpha(\mathcal{I})(X) \subset S_\lambda^\alpha(\mathcal{I})(X)$ *if* $\lim_n \inf \frac{\lambda_n^\alpha}{n^\alpha} > 0$

proof: Let us take any neighbourhood U of 0. Then

$$\frac{1}{n^\alpha}|\{k \leq n : x_k - L \notin U\}| \geq \frac{1}{n}|\{k \in I_n : x_k - L \notin U\}| \quad (8)$$

$$\geq \frac{\lambda_n^\alpha}{n^\alpha}\frac{1}{\lambda_n^\alpha}|\{k \in I_n : x_k - L \notin U\}|. \quad (9)$$

If $\lim_{n\to\infty} \inf \frac{\lambda_n^\alpha}{n^\alpha} = a$

then from definition $\left\{n \in N : \frac{\lambda_n^\alpha}{n^\alpha} < \frac{a}{2}\right\}$is finite. For $\delta > 0$ and for each neighbourhood U of 0,

$$\left\{n \in N : \frac{1}{\lambda_n}|\{k \in I_n : x_k - L \notin U\}| \geq \delta\right\} \quad (10)$$

$$\subset \left\{n \in N : \frac{1}{n}^\alpha |\{k \in I_n : x_k - L \notin U\}| \geq \frac{a}{2}\delta\right\} \cup \quad (11)$$

$$\left\{n \in N : \frac{\lambda_n^\alpha}{n^\alpha} < \frac{a}{2}\right\}. \quad (12)$$

The set on the right hand side belongs to I and this completed the proof .
Finally, we prove the reverse inclusion state in Theorem 4.4.

Theorem 4.5 *If $\lambda \in \triangle$ be such that for a particular $\alpha, 0 < \alpha \leq 1$, $\lim_n \frac{n - \lambda_n}{n^\alpha} = 0$ then $S_\lambda^\alpha(\mathcal{I})(X) \subset S(\mathcal{I})^\alpha(X)$.*

proof: Let $\delta > 0$ be given. Since $lim_n \frac{\lambda_n}{n} = 1$, we can choose $m \in N$ such that $|\frac{\lambda_n^\alpha}{n^\alpha} - 1| < \frac{\delta}{2}$, for all $n \geq m$. Let us take any neighbourhood U of 0. Now observe that ,

$$
\begin{aligned}
& \frac{1}{n^\alpha}\left|\{k \leq n : x_k - L \notin U\}\right| = \frac{1}{n}\left|\{k \leq n - \lambda_n : x_k - L \notin U\}\right| & (13)\\
+ \;& \frac{1}{n^\alpha}\left|\{k \in I_n : x_k - L \notin U\}\right| & (14)\\
\leq \;& \frac{n-\lambda_n}{n^\alpha} + \frac{1}{n}\left|\{k \in I_n : x_k - L \notin U\}\right| & (15)\\
\leq \;& 1 - (1 - \frac{\delta}{2}) + \frac{1}{n^\alpha}\left|\{k \in I_n : x_k - L \notin U\}\right| & (16)\\
= \;& \frac{\delta}{2} + \frac{1}{n^\alpha}\left|\{k \in I_n : x_k - L \notin U\}\right|, & (17)
\end{aligned}
$$

for all $n \geq m$. Hence for $\delta > 0$ and for each neighbourhood U of 0,

$$
\begin{aligned}
& \left\{n \in N : \frac{1}{n^\alpha}\left|\{k \leq n : x_k - L \notin U\}\right| \geq \delta\right\} & (18)\\
\subset \;& \left\{n \in N : \frac{1}{n^\alpha}\left|\{k \in I_n : x_k - L \notin U\}\right| \geq \frac{\delta}{2}\right\} \cup \{1,2,3,...,m\}. & (19)
\end{aligned}
$$

If $S_\lambda^{\mathcal{I}} - limx = L$ then the set on the right hand side belongs to $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $x = (x_k)$ is $\mathcal{I}$-statistically convergent to L.

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