



Existence and Regularity Results for the Stokes System with Mixed Boundary Conditions

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Abstract

In this paper we consider the nonlinear boundary value problem governed by a stationary perturbed Stokes system with mixed boundary conditions (Dirichlet-Fourier- maximal monotone graph), in a smooth domain. The existence and regularity of the weak solution to this problem are proved. Our approach is based on the maximal monotone graph by its Yosida regularization and the contraction method of Brezis [3].

Indexing terms/Keywords

A priori inequalities; maximal monotone graph; Stokes; Variational problem.

SUBJECT CLASSIFICATION

MSC[2010]: 35B40, 35B65, 35C20.

1. Introduction

If S is Many research papers have been written recently, both on the existence, uniqueness and regularity of solutions of Stokes system in different domains but with the usual boundary conditions (Dirichlet, Neumann, ...), see for example [1, 3,5,9,...] and the references cited therein. In [2] the regularity of a stationary equation for a non-isothermal Newtonian and incompressible fluids, in a three-dimensional bounded domain is studied. The problem is governed by a coupled system involving a balance of linear momentum and the heat energy with Tresca free boundary friction conditions.

This work is devoted to the study of the existence and regularity for the solution of the following problem. Let Ω be a bounded open subset of \mathbb{R}^n ($n = 2, 3$) of class C^2 . The boundary $\Gamma = \partial\Omega$ is assumed to be composed of three parts: $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and Γ_3 with measure $(\Gamma_1) > 0$. The notation β will stand for a



maximal monotone graph such that $0 \in \beta(0)$. For given body forces $f \in L^2(\Omega)^n$, we look for a solution (u, p) in $H^2(\Omega)^n \times H^1(\Omega)$ of the following problem:

$$\left\{ \begin{array}{ll} -\nu\Delta u + k^2 u + \nabla p = f & \text{in } (\Omega) \\ \operatorname{div}(u) = 0 & \text{in } (\Omega) \\ u = 0 & \text{on } (\Gamma_1) \\ \left. \begin{array}{l} \sigma_\tau(u) + \alpha u = 0, \\ u \cdot n = 0; \end{array} \right\} & \text{on } (\Gamma_2) \\ -\sigma(u) \cdot n + L(u) \in \beta(u) & \text{on } (\Gamma_3) \end{array} \right. \quad (1.1)$$

where p , u , n and ν are, respectively, the pressure, the velocity field, the unit outward normal to Γ and the viscosity. L a first order differential operator with Lipschitzian coefficients, k is a real number to be fixed lateron. We recall that the components of the stress tensor itself are

$$\sigma_{ij}(u) = -p\delta_{ij} + 2\nu\varepsilon_{ij}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1 \leq i, j \leq n).$$

The formulation of boundary conditions, with maximal monotone graphs, involves several types of conditions resulting from physical problems, such as the Dirichlet, Neumann or Signorini conditions (see, [7]), a boundary condition involved in elasticity with friction in a problem of air conditioning (see, [6]).

The paper is organized as follows: In section 2 we give some preliminaries which will be needed below, while in section 3 we introduce a non-decreasing function β_λ which is regularized in the sense of Yosida. Then we obtain a new nonlinear problem whose the fixed point method is not well adapted. We introduce an intermediate problem for which the Banach fixed point theorem holds. Finally, the a priori estimate allows us to pass to the limit when λ tends to zero, we prove our main results of existence and regularity of the solution to our initial problem (1).

2. Preliminary

We In this paragraph we give some results of [10] which will be useful in the next.

Lemma 2.1. *Let L a tangent operator of the first order, for all $\phi \in L^1(\Gamma)^n$ such that $L(\phi) \in L(\phi) \in L^1(\Gamma)^n$ we get:*

$$\left| \int_\Gamma L(\phi) ds \right| \leq c_1 \int_\Gamma |\phi| ds, \quad (2.1)$$

where $c_1 = c_1(\Omega, L)$ is a constant.



Theorem 2.1. For Ω of class $C^{0,1}$ and for any $\gamma > 0$, there exists a constant $c_2(\gamma)$ depending only on γ such that:

$$\|\varphi\|_{L^2(\Gamma)^n}^2 \leq \gamma \|\nabla \varphi\|_{L^2(\Omega)^{n \times n}}^2 + c_2(\gamma) \|\varphi\|_{L^2(\Omega)^n}^2, \forall \varphi \in H^1(\Omega)^n. \quad (2.2)$$

$$\|\nabla \varphi\|_{L^2(\Gamma)^{n \times n}}^2 \leq \gamma \|\varphi\|_{H^2(\Omega)^n}^2 + c_2(\gamma) \|\varphi\|_{H^1(\Omega)^n}^2, \forall \varphi \in H^2(\Omega)^n. \quad (2.3)$$

3. Main Results

To reach the desired goal, let us introduce non-decreasing functions β_λ which are regularized in the sense of Yosida of β defined by: $\beta_\lambda = \lambda^{-1}(I + J_\lambda)$, where $J_\lambda = -(I + \lambda\beta)^{-1}$ is the resolvent of β .

At first time, we consider the following approached problem:

$$\left\{ \begin{array}{ll} -v\Delta u_\lambda + k^2 u_\lambda + \nabla p = f & \text{in } (\Omega) \\ \operatorname{div}(u_\lambda) = 0 & \text{in } (\Omega) \\ u_\lambda = 0 & \text{on } (\Gamma_1) \\ \left. \begin{array}{l} \sigma_\tau(u_\lambda) + \alpha u_\lambda = 0, \\ u_\lambda \cdot n = 0; \end{array} \right\} & \text{on } (\Gamma_2) \\ (Lu_\lambda - \sigma(u_\lambda)n) = \beta_\lambda(u_\lambda) & \text{on } (\Gamma_3) \end{array} \right. \quad (3.1)$$

The nonlinear problem (3.1) is not well adapted to the fixed point method, an other difficulty is to give the priori estimate, so we introduce the following intermediate problem for which the Banach's fixed point theorem holds.

$$\left\{ \begin{array}{ll} -v\Delta v + k^2 v + \nabla p = f & \text{in } (\Omega) \\ \operatorname{div}(v) = 0 & \text{in } (\Omega) \\ v = 0 & \text{on } (\Gamma_1) \\ \left. \begin{array}{l} \sigma_\tau(v) + \alpha v = 0, \\ v \cdot n = 0; \end{array} \right\} & \text{on } (\Gamma_2) \\ (Lv - \sigma(v)n) - \frac{v}{\lambda} = \frac{J_\lambda(u)}{\lambda} & \text{on } (\Gamma_3) \end{array} \right. \quad (3.2)$$

where $u \in H^{\frac{1}{2}}(\Gamma_3)^n$.

It is clear that each fixed point of (3.2) (i.e. a solution v is found as $v|_{\Gamma_3} = u$) gives a solution of (3.1).



3.1. Study of the intermediate problem (3.2)

To get a weak formulation, we introduce

$$V_{\text{div}} = \left\{ \varphi \in H^1(\Omega)^n : \varphi_{/\Gamma_1} = 0, \varphi \cdot n = 0 \text{ on } \Gamma_2 \text{ and } \text{div}(\varphi) = 0 \right\};$$

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q dx = 0 \right\}.$$

Theorem 3.1. *There exists a unique $v \in V_{\text{div}}$ and $p \in L_0^2(\Omega)$ (up to an additive constant) solution to problem (3.2).*

Proof. A formal application of Green's formula, using (3.2) leads to the weak formulation:

Find $v \in V_{\text{div}}$ such that: $a(v, \phi) = l(\phi), \forall \phi \in V_{\text{div}}$

where

$$a(v, \varphi) = 2\nu \int_{\Omega} \varepsilon(v) \varepsilon(\varphi) dx + k^2 \int_{\Omega} v \cdot \varphi dx + \alpha \int_{\Gamma_2} \omega \cdot \varphi ds - \int_{\Gamma_3} Lv \cdot \varphi ds + \frac{1}{\lambda} \int_{\Gamma_3} v \cdot \varphi ds,$$

$$l(\varphi) = \int_{\Omega} f \cdot \varphi dx - \frac{1}{\lambda} \int_{\Gamma_3} J_{\lambda}(u) \cdot \varphi ds.$$

Using (2.1)-(2.3) and the Korn's inequality (with its constant c_3). Choosing γ and k such that:

$$\gamma < \frac{2\nu c_3}{c_1}, \quad k \geq \sqrt{c_1 c_2(\gamma)} \quad \text{and} \quad \alpha = \min\left((2\nu c_3 - c_1 \gamma), (k^2 - c_1 c_2(\gamma))\right) \quad (3.3)$$

we obtain that the bilinear form $\tilde{a}(\cdot, \cdot)$ is continuous and coercive.

The form l is linear and continuous.

By the Lax-Milgram theorem, there exists a unique solution $v \in V_{\text{div}}$ of $a(v, \phi) = l(\phi), \forall \phi \in V_{\text{div}}$; then the existence of p is obtained by using a duality results of convex optimization.

Therefore, there exists $(v, p) \in V_{\text{div}} \times L_0^2(\Omega)$ solution of the problem (3.2).

Now we establish the solution of a nonlinear problem (3.1).

Theorem 3.2. *Under the assumption of (3.4), there exists a unique $u_{\lambda} \in V_{\text{div}}$, and a unique (up to an additive constant) $p \in L_0^2(\Omega)$, solution to the problem (3.1).*

Proof. We use the Banach fixed point theorem. For this, we introduce the mapping defined by



$$\Lambda : L^2(\Gamma_3)^n \rightarrow L^2(\Gamma_3)^n$$

$$u \rightarrow \Lambda(u) = v_{/\Gamma_3},$$

where v is the solution of (3.2).

The mapping Λ is strictly contracting, then there exists one and only one element $u \in L^2(\Gamma_3)^n$ such that $\Lambda(u) = u = v_{/\Gamma_3}$ and v is solution of (3.2). Finally, we have proved that the solution (u_λ, p) of the problem (3.1) is in $V_{\text{div}} \times L_0^2(\Omega)$.

Theorem 3.3. *If k verify (3.4), the solution (u_λ, p) of the nonlinear problem (3.1) satisfies*

$u_\lambda \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ (up to an additive constant).

4. A priori estimate

Let (u_λ, p) solution of (3.1) we have the:

Theorem 4.1. *Under the assumption of theorem 3.2 there exists a constant C independent of λ such that*

$$\|u_\lambda\|_{H^2(\Omega)^n}^2 + \|\nabla p\|_{L^2(\Omega)^n} \leq C \|f\|_{L^2(\Omega)^n}. \quad (4.1)$$

Theorem 4.2. *If the inequality (3.4) holds, there exists a unique $u_\lambda \in H^2(\Omega)^n$, and a unique (up to an additive constant) $p \in H^1(\Omega)$, solution to the initial problem (1.1).*

Proof. According to (4.1) there exists a sequence λ_j tending to 0 such that

$$\left. \begin{aligned} u_{\lambda_j} &\rightarrow u && \text{weakly in } H^2(\Omega)^n, \\ u_{\lambda_j} &\rightarrow u && \text{strongly in } H^1(\Omega)^n, \\ \text{div}(u_{\lambda_j}) &\rightarrow 0 && \text{strongly in } L^2(\Omega)^n, \\ u &= 0 && \text{on } \Gamma_1, \\ \left. \begin{aligned} \sigma_\tau(u) + \alpha u &= 0, \\ u \cdot n &= 0; \end{aligned} \right\} && \text{on } \Gamma_2. \end{aligned}$$

On the other hand

$$\begin{aligned} L(u_{\lambda_j}) &\rightarrow L(u) && \text{in } L^2(\Gamma_3)^n, \\ \sigma(u_{\lambda_j})\eta &\rightarrow \sigma(u)\eta && \text{in } L^2(\Gamma_3)^n, \end{aligned}$$

Thus $\beta_{\lambda_j}(u_{\lambda_j}) \rightarrow -\sigma(u)\eta + L(u)$ strongly in $L^2(\Gamma_3)^n$.



But since $\beta_{\lambda_j} \subset \beta \circ (-J_{\lambda_j})$, we deduce that

$$u_{\lambda_j} + J_{\lambda_j}(u_{\lambda_j}) = \lambda_j \beta_{\lambda_j}(u_{\lambda_j}) \xrightarrow{\lambda_j \rightarrow 0} 0$$

which is equivalent to $-J_{\lambda_j}(u_{\lambda_j}) \xrightarrow{\lambda_j \rightarrow 0} u$.

Hence the limit u satisfies (1.1) and then we have the regularity.

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