General Letters in Mathematics (GLM) *1 (1) (2016) 11-22*

General Letters in Mathematics (GLM)

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Science Reflection

Congruences on Inverse Semigroups using Kernel Normal System

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Abstract

Congruences on inverse semigroups via the (kernel-trace) method introduced by Scheiblich in 1974. In this paper we discuss the congruences on inverse semigroups by using the technique of Kernel normal systems. Congrueneses on inverse semigroup were described in terms of congruences pairs (ker ρ, tr ρ). It is natural to ask if this strategy can be extended to include regular semigroups. Feigenbaum in 1979 has achieved this. However, this approach has not proved to be the best possible for congruences on regular semigroups in general. Whilst it is possible to describe abstractly the trace and kernel of congruence on a regular semigroup, these descriptions are unwieldy. The technique which has proved most useful for studying congruences on arbitrary regular semigroups is that due to Preston of Kernel normal systems.

Indexing terms/Keywords

Inverse semigroup, congruence, Kernel, regular semigroup, Kernel normal systems.

1. Introduction

Congruence on an algebraic is an equivalence relation on the structure which is compatible with all the operations on the structure. The compatibility condition allows corresponding operations to be defined on the equivalence classes of a congruence to obtain a similar structure. All homomorphic images of an algebraic structure arise in this way and consequently, a great deal of effort has been put into studying congruences.

Semigroups provide one of the simplest types of structure for which congruences are not determined by just one equivalence class. In Section 2 we present a number of elementary concepts and propositions on semigroups. In Section 3 we describe the Kernel (or Kernel normal system) of a congruence on an inverse semigroups. The use of the capital K is to distinguish the Kernel of a congruence from its kernel defined in Section 2.

The first characterization of arbitrary congruences was due to Preston [7] who introduced the notion of a Kernel normal system. Later approaches characterized congruences on an inverse semigroup in terms of restrictions to the set of idempotents and the kernels of the congruences, the kernel being the set of elements related to idempotents. This culminated in Petrich's theory of congruence pairs [6]. In [2] Meakin gave an alternative definition for a Kernel normal system on an orthodox semigroup.

We give a full account of Kernel normal systems in Section 4. In particular, we include the result that a congruence on an inverse semigroup is completely determined by a Kernel normal system. The final set of conditions is equivalent to the first set, in fact, a combination of Exercises 6 and 8 on p.63 of [1].

2. Introductory concepts

In this section we shall define some of the basic concepts which will be used throughout the paper.
A relation ρ on a semigroup S is called right compatible if For all $a, b, c, \epsilon S$ $[(a,b)\epsilon\, \rho \Rightarrow (ac, bc)\epsilon\, \rho]$ and ρ is called left compatible if For all $a, b, c\, \epsilon\, S$ $[(a,b)\epsilon\, \rho \Rightarrow (ca, cb)\epsilon\, \rho]$.
A right (left) congruence on a semigroup S is defined to be an equivalence relation ρ on S which is right (left) compatible. A relation ρ on S which is both a right and a left congruence is called a congruence on S.

Proposition 2.1 *An equivalence relation ρ on a semigroup S is a congruence if and only if for any* $a, b, c, d \in S$ $(a, b) \in \rho$, $(c, d) \in \rho \Rightarrow (a\, c, bd) \in \rho$. ■

Given a congruence ρ on a semigroup S, note that

$$x\rho = \{x' \epsilon S : x\, \rho\, x'\}$$

We refer to $x\rho$ as the congruence class of an element $x \in S$. One defines a semigroup structure on the set S/ρ of all congruence classes of S by taking $x\rho . y\rho = (xy)\rho$ for any $x, y \epsilon S$. This operation on S/ρ is well-defined and it is easily seen to be associative and so $(S/\rho, .)$ is a semigroup.
Now the proof of the following fundamental theorem of homomorphisms is straightforward.

Theorem 2.1 *Let ρ be a congruence on a semigroup S. Then S/ρ is a semigroup and the mapping $\rho^{\pm}: S \rightarrow S/\rho$ defined by $x\rho^{\pm} = x\rho$ for all $x \in S$ is a homomorphism with ker $\rho^{\pm} = \rho$.*

Theorem 2.2 *Let $\varphi: S \rightarrow T$ be a semigroup homomorphism. Then*

$$\ker \varphi = \varphi\, o\, \varphi^{-1} = \{(a, b)\, \epsilon\, S \times S : a\, \varphi = b\, \varphi\}$$

is a congruence on S and there is a homomorphism $\varphi' = S/\ker \varphi \rightarrow T$ such that imφ'= im φ and $(\ker \varphi)^{\pm}$ $.\varphi' = \varphi$.

Definition 2.1 *Equivalence relations $\mathcal{L}$ and R on S defined by the rule that $a\, \mathcal{L}\, b$ $(a\, \mathcal{R}\, b)$ if and only if a and b generate the same principal left (right) ideal. D is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$. These relations called Green's relations [2].*

Lemma 2.1 *Let a and b be elements of a semigroup S. Then $a \mathcal{L} b$ if and only if there exist x, y in S^1 such that $xa = b$ and $yb = a$. Dually, $a \mathcal{R} b$ if and only if there exist u, v in S^1 such that $au = b$ and $bv = a$.*

Proposition 2.2 *The relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

Definition 2.2 *An element a of a semigroup S is called **regular** if there exists an element x in S such that $ax\,a = a$. A semigroup S is called **regular** if all its elements are regular.*

Definition 2.3 *A semigroup S is called an **inverse** semigroup if every a in S possesses a unique inverse, that is, there exists a unique element a^{-1} in S such that*

$$aa^{-1}a = a \text{ and } a^{-1}a\,a^{-1} = a.$$

Such a semigroup is certainly regular and $|V(a)| = 1$ for all $a \in S$. In this case we write $V(a) = \{a^{-1}\}$, so that a^{-1} is the unique inverse of a.

Theorem 2.3 *Let S be a semigroup. The following statements are equivalent:*

- *S is an inverse semigroup;*
- *S is regular and its idempotents commute;*
- *Each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains a unique idempotent;*
- *Each principal left ideal and each principal right ideal of S contains a unique idempotent generator.*

The proof of the following proposition follows from the definition.

Proposition 2.3 *Let S be an inverse semigroup with semilattice of idempotent E. Then we have*

a) $(a^{-1})^{-1} = a$ *for every* $a \in S$.
b) $e^{-1} = e$ *for every* $e \in E$.
c) $(ab)^{-1} = b^{-1}a^{-1}$ *for every* $a, b \in S$.
d) $aea^{-1} \in E$, $a^{-1}ea \in E$ *for every* $a \in S$ *and* $e \in E$.
e) $a \mathcal{R} b$ *if and only if* $a\,a^{-1} = bb^{-1}$, $a \mathcal{L} b$ *if and only if* $a^{-1}a = b^{-1}b$.
f) *If* $e, f \in E$, *then* $e\, D\, f \in S$ *if and only if there exists* a *in S such that* $a\,a^{-1} = e$ *and* $a^{-1}a = f$.

3. Kernel normal systems

Let ρ be a congruence on a regular semigroup S. The Kernel normal system associated with ρ, or more briefly, Kernel of ρ (Ker ρ) is defined by

$$\text{Ker}\,\rho = \{e\rho : e \in E(S)\}.$$

One also sees Ker ρ defined as

$$\text{Ker}\,\rho = \{\rho : a\rho \in E(S/\rho)\}.$$

Since S is regular, these are the same, by [3]. The reader should be careful not to confuse the concepts of ker ρ and Ker ρ. By definition we have

$$\ker \rho = \cup \{A : A \in \text{Ker}\,\rho\}.$$

The set $A = \{A_i : i \in I\}$ of subsets of a regular semigroup S is defined to be a Kernal normal system of S if A satisfies

C_1) $A_i \cap A_j = \varphi$ if $i \neq j$

C_2) each A_i contains an idempotent of S and each idempotent of S is contained in some A_i

C_3) $xA_iy \cap A_j \neq \varphi$ implies $xA_iy \subseteq A_j$ for each $x, y \in S^1$ and $i, j \in I$.

We aim to show that the set of Kernel normal systems of S is exactly the set of Kernels of congruences on S, and further, every congruence on S determined by its Kernel.

The Kernel normal systems can be defined also for inverse semigroups as we will show in the next section.

Lemma 3.1 *Let ρ be a congruence on a regular semigroup S. Then the Kernel of ρ is a Kernel normal system of S.*

Proof Let $A = \{A_i = e\rho : i \in I\}$ be the Kernel of ρ. We have to show that A satisfies conditions C_1, C_2 and C_3. Condition C_1 is clear since the A_i's are congruence classes. Condition C_2 is immediate. To show condition C_3, suppose that $xA_iy \cap A_j \neq \varphi$, for some $x, y \in S^1$ and $i, j \in I$. Then there exist $a \in A_i, b \in A_j$ with $xay = b$. Let $c \in A_i$ and consider xcy. Since A_i and A_j are congruence classes, we have $xcy\ \rho\ xay = b$, so that $xcy \in A_j$. Thus $xA_iy \subseteq A_j$. ■

Theorem 3.1 *Two congruences σ and ρ on a regular semigroup S coincide if and only if they have the same Kernel.*

Proof Let ρ and σ be two congruences on a regular semigroup S having the same Kernel. We have to show that $\rho = \sigma$. Let $(x, y) \in \sigma$. Since S is regular, there exist a, b in S such that

$xax = x,\quad ax\,a = a,\quad yby = y\quad$ and $\quad byb = b,$

so that xa, by are idempotents of S. Since σ is right compatible, $(xa, ya) \in \sigma$, and from left compatibility we get $(bx, by) \in \sigma$ and because ρ and σ have the same Kernel, and xa, by are idempotents,

$(xa, ya) \in \rho$ and $(bx, by) \in \rho$.

Now $x = xax\ \rho\ yax = ybyax\ \rho\ ybxax = ybx\ \rho\ yby = y$. Then $x\rho y$ that is, $(x, y) \in \rho$ which implies that $\sigma \subseteq \rho$. The argument can be repeated with σ and ρ interchanged giving $\rho \subseteq \sigma$ whence the theorem follows. ■

We now proceed to show that associated with any kernel normal system A there is a congruence ρ_A with Ker $\rho_A = A$. Let A be Kernel normal system of a regular semigroup S. The relation ρ_A is defined as follows

$$\rho_A = \{(a, b) \in S \times S : xay \in A_i \Leftrightarrow xby \in A_i\ \forall x, y \in S^1, i \in I\}.$$

Lemma 3.2 *Let $A = \{A_i : i \in I\}$ be a Kernel normal system of regular semigroup S, and define relation ρ_A as above. Then ρ_A is a congruence on S with Kernel A.*

Proof The relation ρ_A is clearly an equivalence relation. To show ρ_A is a congruence, let $(a,b) \in \rho_A$ and $c \in S$. We have $xcay \in A_i$ for some $i \in I$ if and only if $xcby \in A_i$ for all choices of x and y in S^1, since $xc \in S^1$. Hence $(ca, cb) \in \rho_A$. Similarly, $(ac, bc) \in \rho_A$.

To show the Kernel of ρ_A is A, we have to show that each A_i is a ρ_A-class. Let $i \in I$. By condition C_2, there is an idempotent e in A_i. Let $a \in e\rho_A$. We have to show $a \in A_i$. Now $(a,e) \in \rho_A$; since $e \in A_i$, we have $1e1 \in A_i$ and so $a = 1a1 \in A_i$. Thus $e\rho_A \subseteq A_i$.

Conversely, let $b \in A_i$. We have to show that $e\rho_A b$. Let $x, y \in S^1$, and suppose $xey \in A_j$. We have that $xA_iy \cap A_j \neq \varphi$ so that by condition C_3, $xA_iy \subseteq A_j$ and $xby \in A_j$. Similarly, $xby \in A_j$ implies that $xey \in A_j$. Thus $e\rho_A b$ and $A_i \subseteq e\rho_A$. It follows that

Kerρ_A = A.■

Now we have

Theorem 3.2 *Let A be kernel normal system of a regular semigroup S. Then ρ_A is the unique congruence on S with Ker $\rho_A = A$.*

Proof The existence of ρ_A follows from Lemma 3.2, the uniqueness from Theorem 3.1.■

Definition 3.1 *A regular semigroup in which the idempotents form a subsemigroup that is, a band called an **orthodox** semigroup.*

Theorem 3.3 *If S is regular semigroup, then the following statements are equivalent:*

a. *S is orthodox,*
b. *for every $a, b \in S$, if $a^{-1} \in V(a), b^{-1} \in V(b)$, then $b^{-1}a^{-1} \in V(ab)$,*
c. *if e is idempotent then every inverse of e is idempotents.*

As shown, it is possible to characterize the Kernel (Ker) of a congruence ρ on a regular semigroup S as a set $A = \{A_i =: i \in I\}$ of subsets of S satisfies the conditions C_1, C_2 and C_3 .

In 1954 Preston [6] gave easier conditions in the case of inverse semigroups. Later Meakin [4,5] gave corresponding conditions to extend Preston's theory from inverse semigroups to orthodox semigroups.

Let $A = \{A_i : i \in I\}$ be a set of subsets of an orthodox semigroup. Then Meakin's conditions defined as follows

D_1) $A_i \cap A_j = \varphi$ if $i \neq j$,

D_2) each A_i contains an idempotent and each idempotent of S is contained in some A_i,

D_3) if $a \in A = \cup_{i\in I} A_i$, then $V(a) \subseteq A$,

D_4) for each $a \in S$, for each inverse a^{-1} of a and for each $i \in I$, there exists $j \in I$ such that $aA_i a^{-1} \subseteq A_j$,

D_5) for each pair $i, j \in I$, there exists $k \in I$ such that $A_i A_j \subseteq A_k$,

D_6) if $a, ba \in A_i$ and $b^{-1}a, b^{-1}b \in A_j$ for some $b^{-1} \in V(a)$ and $i, j \in I$, then $b \in A_i$.

We remark here that in [5,3] Meakin called the set A which satisfies these conditions a Kernel normal system. He also gave extra conditions as follows:

$D_{6'}$) if $a, ba, b^{-1}b \in A_i$ for some $b^{-1} \in V(b)$, then $b \in A_i$,

D_7) if e, $ae \in A_i$ and $a^{-1}e, aa^{-1} \in A_j$ for some elements $e \in E$, $a^{-1} \in V(a)$ and $i, j \in I$, then $a \in A_i$.

It is easy to show that these conditions follow from the previous ons.

For a collection A of subsets satisfying $D_1 \ldots D_6$ we define the relation $\sim$ on $\cup_{i\in I} A_i$ by $a \sim b$ if and only if a, b belong to A_i for some $i \in I$.

We now introduce ρ_A as we did previously in the regular semigroup

$\rho_A = \{(a,b) \in S \times S : \exists\, a^{-1} \text{of}\, a, b^{-1} \text{of}\, b \ni ba^{-1} \sim aa^{-1}, b^{-1}a \sim b^{-1}\}$.

Note that, if A is a set of subsets of an orthodox semigroup S satisfying $D_1 \ldots\ldots D_6$, and if $a, b \in S$, then a necessary and sufficient condition for $(a, b) \in \rho_A$ is that

$ba^{-1} \sim aa^{-1},\ b^{-1}a \sim b^{-1},\ a^{-1}b \sim a^{-1}a,\ \ and\ \ ab^{-1} \sim bb^{-1}$

For any $a^{-1} \in V(a)$ and $b^{-1} \in V(b)$.

Lemma 3.3 *Let S be an orthodox semigroup, ρ be a congruence on S and let $Ker\ \rho = A = \{A_i : i \in I\}$. Put $A = \cup A_i$ and $V(A) = \cup_{a\in A} V(a)$. Then $V(A) \subseteq A$.* ■

Theorem 3.4 *Let S be an orthodox semigroup. Then the system $A = \{A_i : i \in I\}$ of subsets of S satisfies conditions C_1, C_2 and C_3 if and only if it satisfies conditions $D_1 \ldots D_6$.*

Proof Suppose that $A = \{A_i : i \in I\}$ satisfies the conditions C_1, C_2 and C_3. Conditions D_1, D_2 are the same as C_1, C_2. By Theorem 3.2 we have that a congruence ρ exists such that A = Ker ρ, and so condition D_3 follows from Lemma 3.3. To prove D_4, let $a \in S$ and $a^{-1} \in V(a)$. By C_2, there exists $e \in A_i$ for some $e \in E$, and since $aea^{-1} \in E$, we have by C_2, $aea^{-1} \in A_i$ for some $j \in I$. Then $aA_i a^{-1} \cap A_j \neq \varphi$.

Now by C_3, $aA_i a^{-1} \subseteq A_j$. The condition D_5 follows from the fact that each A_i is a congruence class. To prove condition K_6, let $a, ba \in A_i$ and $a^{-1}a, b^{-1}b \in A_j$ for some $b^{-1} \in V(b)$ and $i, j \in I$. Then $a\, \rho\, ba$, and $b^{-1}a\, \rho\, b^{-1}b$, and so as ρ is a congruence, $bb^{-1}a\, \rho\, bb^{-1}ba = ba$. Then $a\, \rho\, ba\, \rho\, bb^{-1}a\, \rho\, bb^{-1}b = b$ as $b^{-1}a\, \rho\, b^{-1}b$. Hence $a\, \rho\, b$, and since $a \in A_i$ we have $b \in A_i$.

Conversely, suppose that A satisfies $D_1,\ldots,D_6$. We need to show that A satisfies condition C_3. Suppose that $xA_i\, y \cap A_j \neq \varphi$, and set $a \in A_i$, be such that $xay \in A_j$. Let $c \in A_i$, we have to show that $xcy \in A_j$. Let $x^{-1} \in V(x), y^{-1} \in V(y)$ and $a^{-1} \in V(a)$. Then $y^{-1}a^{-1}x^{-1} \in V(xay)$ and $y^{-1}c^{-1}x^{-1} \in V(xcy)$. Since yy^{-1} is idempotent, $yy^{-1} \in A_k$ for some k. By D_3, $a^{-1} \in A_h$ for some h and so by D_5, $cyy^{-1}a^{-1}$, $ayy^{-1}a^{-1} \in A_m$ for some $m \in I$. Now by D_4, we have
$xcyy^{-1}a^{-1}x^{-1} \sim xayy^{-1}a^{-1}x^{-1}$.
Similarly,

$$y^{-1}c^{-1}x^{-1}xay \sim y^{-1}c^{-1}x^{-1}xcy.$$

It follows that $xay\ \rho_A\ xcy$.

Now let $e \in A_j$. We aim to show that if $u \in A_j$ and $u\ \rho_A\ v$, then $v \in A_j$. Now we have that if u^{-1} is any inverse of u, then $u^{-1} \in A_k$ for some $k \in I$, by D_3. Then by D_5, $eu^{-1} \sim uu^{-1}$.
Furthermore, it follows from K_2 and K_5 that each A_i is a subsemigroup and so

$$ue \sim eu \sim e.$$

Since $u\ \rho_A\ v$, we have

$$vu^{-1} \sim uu^{-1} \text{ and } v^{-1}u \sim v^{-1}v.$$

Also

$$u^{-1}v \sim u^{-1}u \text{ and } uv^{-1} \sim vv^{-1}.$$

Now we show that $e\ \rho_A v$. We first prove that $e \sim ev^{-1}ve$ and $vv^{-1} \sim vev^{-1}$. Using D_5 we have

$$e(v^{-1}v)e \sim u(v^{-1}v)e = (u)(u^{-1}u)(v^{-1}v)e \sim u(u^{-1}v)(v^{-1}v)e = u(u^{-1}v)e \sim u(u^{-1}u)e = ue \sim ee = e.$$

Hence $e(v^{-1}v)e \sim e$. Also, using D_5, we have

$$vev^{-1} = v(v^{-1}v)ev^{-1} \sim vv^{-1}uev^{-1} = v[v^{-1}uu^{-1}ue]v^{-1} \sim v[v^{-1}uu^{-1}u]v^{-1} = v[v^{-1}u]v^{-1} \sim vv^{-1}vv^{-1} = vv^{-1}$$

so that

$$vv^{-1} \sim vev^{-1}.$$

Similarly, we can get

$$v^{-1}v \sim v^{-1}ev.$$

Now set $a = uv^{-1}vu^{-1}e$. Using D_5 we have

$$a \sim vv^{-1}vu^{-1}e = vu^{-1}e \sim uu^{-1}e \sim uu^{-1}ue = ue \sim ee = e.$$

Hence $a \sim e$. Now set $b = ve$ and $b^{-1} = ev^{-1}$ so that

$$ba = (veuv^{-1})(vu^{-1})e \sim (vev^{-1})(vu^{-1})e \sim (vv^{-1})(vu^{-1})e = vu^{-1}e \sim uu^{-1}ue = ue \sim e.$$

Hence $ba \sim e$. Also $b^{-1}b = ev^{-1}ve \sim e$. Hence we have $a \sim ba \sim b^{-1}b$, and so by D_6, we have that

$$b = ve \sim e.$$

Again, by putting $a = eu^{-1}vv^{-1}u$ and by D$_5$ we have $a \sim uu^{-1}uv^{-1}u \sim uv^{-1}u \sim ev^{-1}v$. Put $b = ev$ and $b^{-1} = v^{-1}e$. Then $ba = eveu^{-1}vv^{-1}u \sim eeu^{-1}uv^{-1}u \sim uu^{-1}uv^{-1}u = uv^{-1}u \sim uv^{-1}v \sim ev^{-1}v$. Also

$b^{-1}b = v^{-1}eev \sim v^{-1}ev \sim v^{-1}v$ and

$b^{-1}a = v^{-1}eeu^{-1}vv^{-1}u \sim v^{-1}eu^{-1}vv^{-1}u \sim v^{-1}uu^{-1}vv^{-1}v \sim v^{-1}uv^{-1}v \sim v^{-1}vv^{-1}v \sim v^{-1}v.$

Hence we have $a \sim ba$ and $b^{-1}a \sim b^{-1}b$ and so by D$_6$ we have $b = ev \sim ev^{-1}v$, so that $v^{-1}e \epsilon A_k$ for some $k \epsilon I$. Now we have

$$\begin{aligned} v^{-1}e &= (v^{-1}e)(e) \sim (v^{-1}e)(eu) \\ &= (v^{-1}e)(eu)(u^{-1}u) \sim v^{-1}(e(uu^{-1}))v \sim (v^{-1}u)(u^{-1}v) \sim (v^{-1}u)(v^{-1}u) \\ &= v^{-1}u \sim v^{-1}v \end{aligned}$$

and hence

$$v^{-1}e \sim v^{-1}v.$$

Now $e\, \rho_A\, v$. Also we have $e \sim ve \epsilon A_j$ as ϵA_j , and $v^{-1}e \sim v^{-1}v$, so that by D$_7$ we have $v \epsilon A_j$. Now we have proved that if $u \epsilon A_j$ and $u\, \rho_A v$, then $v \epsilon A_j$, and because we have $xay\, \rho_A\, xcy$ and $xcy \epsilon A_j$, we have that $xcy \epsilon A_j$ which implies that $xA_iy \subseteq A_j$ and C$_3$ is satisfied.■

4. Congruences on inverse semigroups

Congruences on inverse semigroups via the (kernel-trace) method introduced by Scheiblich in 1974. In this section we discuss the congruences on inverse semigroups by using the technique of Kernel normal systems.

In [7], Preston showed that it is possible to characterize the Kernel of a congruence on an inverse semigroup as a set $A = \{A_i : i \epsilon I\}$ of subsets of S satisfying the following conditions:-

I$_1$) each A_i is an inverse subsemigroup of S,

I$_2$) $A_i \cap A_j = \varphi$ if $i \neq j$,

I$_3$) each idempotent in S is contained in some element of A,

I$_4$) for each $a \epsilon S$ and $i \epsilon I, a^{-1}A_i a \subseteq A_j$ for some j, and we can write $j = ia$, so that $a^{-1}A_i a \subseteq A_{ia}$,

I$_5$) if $, ab, bb^{-1} \epsilon A_i$, then $b \epsilon A_i$.

We now introduce the congruence σ_A associated with such a set A of subsets of S as follows

$$\sigma_A = \{(a,b) \epsilon S \times S : aa^{-1}, bb^{-1}, ab^{-1} \epsilon A_i \text{ for some } i \epsilon I\}.$$

Our aim in this section is to show that if A is a collection of subsets of an inverse semigroup S satisfying conditions I$_1$, …I$_5$, then A is a kernel normal system and $\sigma_A = \rho_A$. We begin by proving the following lemma.

Lemma 4.1 *For any i, j in I there exists k in I such that $A_iA_j \subseteq A_k$.*

Proof For each $i \in I$ we denote $E(A_i)$ by E_i and first we prove that given $i, j \in I$ we have $E_i E_j \subseteq E_k$, for some $k \in I$. If $e \in E_i$ and $f \in E_j$, we know that ef is idempotent, so $ef \in E_k$ for some k by I_3. Let $g \in E_i, h \in E_j$. We have to show that $gh \in E_k$. From I_4, we have $eA_je \subseteq A_l$ for some l. But $ef = efe \in A_k$, so $ef \in A_l \cap A_k$ which implies that $A_l = A_k$ by I_1, that is, $eA_j e \subseteq A_k$. Now $eh = ehe \in eA_j e \subseteq A_k$, so $eh \in A_k$ and $eh = heh \in hA_i h \subseteq A_m$ for some m, that is, $eh \in A_k \cap A_m$ which implies that $A_k = A_m$ by I_1 and so $hA_i h \subseteq A_k$. Hence $gh = hgh \in A_k$ and so $E_i E_j \subseteq E_k$.

Now let $a \in A_i$, $b \in A_j$ and put $x = a^{-1}aba^{-1}a$ and $y = a^{-1}ab$. Note that

$$a^{-1}aE_ja^{-1}a \subseteq E_i E_j E_i \subseteq E_k,$$

so that $a^{-1}aA_ja^{-1}a \cap A_k \neq \varphi$. But by I_4, $a^{-1}a A_ja^{-1}a \subseteq A_p$ for some p and so $A_k \cap A_p \neq \varphi$. Hence $p = k$ and

$$a^{-1}aA_j a^{-1}a \subseteq A_k.$$

In particular, $x \in A_k$ and so $xx^{-1} \in A_k$. Put $u = xx^{-1}$. Now

$$uy = xx^{-1}y = a^{-1}aba^{-1}aa^{-1}ab^{-1}a^{-1}aa^{-1}ab = a^{-1}a(ba^{-1}ab^{-1})a^{-1}ab = aa^{-1}aa^{-1}aba^{-1}ab^{-1}b = a^{-1}abb^{-1}ba^{-1}a = a^{-1}aba^{-1}a = x \in A_k$$

.

Also

$$yy^{-1} = a^{-1}abb^{-1}a^{-1}a \in a^{-1}aA_ja^{-1}a$$

and so $yy^{-1} \in A_k$. We now have $u, uy, yy^{-1} \in A_k$ and so by I_5, we get $y \in A_k$.
Using $E_jE_i \subseteq E_k$ and $a^{-1} \in A_i$, a similar argument shows that $bb^{-1}a^{-1} \in A_k$ and so by I_1, we have $abb^{-1} = (bb^{-1}a^{-1})^{-1} \in A_k$. Again by I_1, we have $(abb^{-1})(a^{-1}ab) \in A_k$. But

$$(abb^{-1})(a^{-1}ab) = a(bb^{-1})(a^{-1}a)b = a(a^{-1}a)(bb^{-1})b = ab$$

and so $ab \in A_k$. Thus $A_i A_j \subseteq A_k$ and the lemma is proved.■

The following corollary is an immediate consequence of the lemma.

Corollary 4.1 *Let $A = \{A_i : i \in I\}$ be a collection of subsets of an inverse semigroup S and let $A = \cup_{i\in I} A_i$. If A satisfies conditions $I_1,\ldots,I_5$ then A is an inverse subsemigroup of S.*

Theorem 4.1 *Let S be an inverse semigroup and let $A = \{A_i : i \in I\}$ be a collection of subsets of S. Then A is a Kernel normal system if and only if A satisfies the conditions $I_1,\ldots,I_5$.*

Proof Suppose that A satisfies conditions $D_1,\ldots,D_6$. Then by Theorem 3.4, A is Kernel normal system, and by Theorem 3.2, there is a unique congruence ρ with Ker $\rho = A$. Now conditions I_2 and I_4 are the same as conditions D_1 and D_4. Condition I_3 follows trivially from condition D_2.

To prove condition I_1, let $a, b \in A_i$ for some $i \in I$. Also we have $e \in A_i$ for some $e \in E$ by D_2. Then $a \rho e$ and $b \rho e$ which implies that $ab \rho ee = e$ as ρ is a congruence. Hence $ab \in A_i$ and so A_i is a subsemigroup. Also A_i is inverse since if $a \in A_i$ and $e \in A_i$ for $e \in E$, then $a \rho e$ which implies that $a^{-1} \rho e^{-1} = e$ as S is inverse. Hence $a^{-1} \in A_i$ for $i \in I$.

To prove condition I_5, suppose that $a, ab, bb^{-1} \in A_i$. Then $a \rho bb^{-1}$, and $ab \rho bb^{-1}b = b$ as ρ is a congruence. Thus $ab \rho b$ which implies that $b \in A_i$ as $ab \in A_i$.

Conversely, suppose that A satisfies the conditions I_1, …,I_5. We prove that A satisfies conditions D_1, ….D_6 . Conditions K_1 and K_4 are the same as conditions I_2 and I_4. Condition D_2 is immediate from conditions I_1 and I_3, and condition D_3 follows from I_1. Condition D_5 follows from Lemma 4.1.

To prove condition D_6 suppose that a, $ba \in A_i$ and $b^{-1}a, b^{-1}b \in A_j$ for some $i, j \in I$. We have to show that $b \in A_i$. To use I_5, we need that $ab, bb^{-1} \in A_i$. Set $y = bb^{-1}$and $x = ba$ so that $x \in A_i$. By I_3, we have $y \in A_k$ for some k and so $xy \in A_i A_k$. Now $A_i A_k \subseteq A_p$ for some p by Lemma 4.1, and so $xy \in A_p$. Since $a a^{-1} \in A_i$ we also have $yaa^{-1} = aa^{-1}y \in A_i A_k$ and consequently, $A_k A_i \subseteq A_p$. Since $ba \in A_i$, we now have

$$ba = bb^{-1}ba = yba \in A_p.$$

Hence $i = p$ and $xy \in A_i$.

Also $y = y^2 = bb^{-1}bb^{-1} \in bA_j b^{-1}$ and by I_4, $bA_j b^{-1} \subseteq A_r$ for some r. But $y \in A_k$ so that $bA_j b^{-1} \subseteq A_k$. Hence $bb^{-1}ab^{-1} \in A_k$ and so

$$aa^{-1} bb^{-1}ab^{-1} = bb^{-1}aa^{-1}ab^{-1} = bb^{-1}ab^{-1} \in A_k.$$

Now $aa^{-1}bb^{-1}ab^{-1} \in A_i bA_j b^{-} \subseteq A_i A_k \subseteq A_i$ and so $bb^{-1}ab^{-1} \in A_i$. Thus $i = k$ and so $yy^{-1} = y \in A_i$. Thus $x, xy, yy^{-1} \in A_i$. Using I_5 we have $y = bb^{-1} \in A_i$.

Now put $z = baa^{-1}b^{-1}$ and note that $z \in A_i$ since $ba \in A_i$ and A_i is an inverse subsemigroup of S. Also

$$zb = baa^{-1}b^{-1}b = bb^{-1}baa^{-1} = baa^{-1} \in A_i$$

Since $a, ba \in A_i$ and A_i is an inverse subsemigroup of S. We have already seen that $bb^{-1} \in A_i$ so that now we have, $z, zb, bb^{-1} \in A_i$ and hence by I_5, $b \in A_i$. Therefore A is Kernel normal system.■

Preston gave some alternative conditions which are

$I_{4'}$) if $aa^{-1}bb^{-1}, ab^{-1} \in A_i$, then for any $j \in I$, there exists $k \in I$ such that $aA_j a^{-1} \subseteq A_k$ and $aA_j b^{-1} \subseteq A_k$,

$I_{5'}$) if $a, ab^{-1}, bb^{-1} \in A_i$, then $b \in A_i$.

In fact, conditions I_1, I_2, I_3, $I_{4'}$ and $I_{5'}$ are equivalent to conditions I_1,….I_5. To prove this we need to prove the following lemma.

Lemma 4.2 *Let $A = \{A_i : i \in I\}$ be a collection of subsets of an inverse semigroup S satisfying conditions I_1,….,I_5. If $ab^{-1} \in A_i$, then $A_{ia} = A_{ib}$.*

Proof By I_1, A_i is subsemigroup of S so that $ba^{-1} \in A_i$ and also $(ab^{-1})(ba^{-1}) \epsilon A_i$ and $(ba^{-1})(ab^{-1}) \epsilon A_i$. Hence

$$a^{-1}ab^{-1}b = a^{-1}aa^{-1}ab^{-1}b = a^{-1}ab^{-1}ba^{-1}a = a^{-1}(ab^{-1}ba^{-1})a \epsilon A_{ia}.$$

Also

$$a^{-1}ab^{-1}b = b^{-1}ba^{-1}a = b^{-1}bb^{-1}ba^{-1}a = b^{-1}ba^{-1}ab^{-1}b = b^{-1}(ba^{-1}ab^{-1})b \epsilon A_{ib}.$$

Hence $A_{ia} \cap A_{ib} \neq \varphi$ and by I_2, $A_{ia} = A_{ib}$.■

Proposition 4.1 *Let $A = \{A_i : i \epsilon I\}$ be a collection of subsets of inverse semigroup S. Then A satisfies conditions $I_1, \ldots, I_5$ if and only if it satisfies conditions I_1, I_2, I_3, $I_{4'}$, $I_{5'}$.*

Proof Suppose first that A satisfies conditions $I_1, \ldots, I_5$. To show that condition $I_{4'}$ holds, let $i, j \epsilon I$ and let $a, b \epsilon S$ be such that $aa^{-1}, bb^{-1}, ab^{-1}$ are all in A_i. By condition I_4 we have $aA_j a^{-1} \subseteq A_k$ for some $k \epsilon I$. Note that $aa^{-1} \epsilon A_i$ and so

$$a A_j a^{-1} = a A_j a^{-1} aa^{-1} \subseteq A_k A_i$$

so that $A_k \cap A_k A_i \neq \varphi$. It follows from Lemma 4.1 that $A_k A_i \subseteq A_k$.

Since A_j is an inverse subsemigroup of S, it contains an idempotent, say e.Then

$$aeb^{-1} = aa^{-1}a eb^{-1} = (aea^{-1})(ab^{-1}) \varepsilon A_k A_i \subseteq A_k$$

so that $A_k \cap aA_j b \neq \varphi$. By Theorem 3.4, A is a Kernel normal system and hence by definition, A satisfies condition C_3. Thus $a A_j b^{-1} \subseteq A_k$ as required.

We now prove condition $I_{5'}$. Now A satisfies condition $I_1, \ldots, I_5$ and so by Theorem 3.4, A is Kernel normal system and by Theorem3.2, there is a unique congruence ρ with Ker $\rho = A$. Let $a, b \epsilon S$ be such that $a, ab^{-1}, bb^{-1} \epsilon A_i$. We have to show that $b \epsilon A_i$. Now we have $a \rho bb^{-1}$, so that $ab \rho bb^{-1}b = b$ as ρ is a congruence. Also $a \rho ab^{-1}$, so that

$$ab \rho ab^{-1}b = aa^{-1}ab^{-1}b = ab^{-1}ba^{-1}a.$$

But $(ba^{-1})a \rho bb^{-1}$, since A_i is an inverse subsemigroup of S and $ba^{-1}, a \epsilon A_i$. Then we have $ab \rho ab^{-1}ba^{-1}a \rho ab^{-1}bb^{-1} = ab^{-1}$.

Since $ab^{-1} \epsilon A_i$, it follows $ab \epsilon A_i$ which implies that $b \epsilon A_i$ by I_5. Conversely, suppose that A satisfies I_1, I_2, I_3, $I_{4'}$ and $I_{5'}$. We need to show that A satisfies conditions I_4 and I_5. Let $a \epsilon S$ and $i \epsilon I$. Since A_i is an inverse subsemigroup of S, it follows that $a^{-1}a \epsilon A_i$. Using $I_{4'}$ with a^{-1} in place of a and b we deduce that

$$a^{-1}A_i (a^{-1})^{-1} \subseteq A_k$$

for some k. But $(a^{-1})^{-1} = a$ and so I_4 is satisfied.

To show that condition I_5 holds, let $a, b \epsilon S$ be such that $a, ab, bb^{-1} \epsilon A_i$. By I_1, A_i is an inverse subsemigroup and so $(ab)b^{-1} \epsilon A_i$. Therefore, putting $x = ab$ we have $x, xb^{-1}, bb^{-1} \epsilon A_i$. By $I_{5'}$, this gives $b \epsilon A_i$.■

5. Conclusions

Congruences on inverse semigroups by using the technique of kernel normal systems developed in this paper and several characterizations of Kernel normal systems are shown to be equivalent.

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