Distributions of Gompertz Spacings and Their Moments

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Abstract. This paper considers the distributions of spacings between successive order statistics corresponding to a random sample from a two-parameter Gompertz distribution. Closed-form formulas for the probability density functions of the spacings and their moments are given in terms of the integro-exponential function and the Meijer G-function. Then a recursive identity for the moments of the Gompertz spacings is also derived.

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1 Introduction

The Gompertz probability distribution, introduced by Gompertz [9], is a lifetime continuous probability distribution. It has been recently used in many fields of sciences including actuarial, biology, demography, gerontology, and computer science. Computer scientists use the Gompertz distribution to model the failure rates of computer codes (Dohi [7]). In Marketing Science, the Gompertz model is used as an individual-level simulation for customer lifetime value modeling (Bemmaor [3]). In network theory, particularly the ErdősRényi model, the walk length of a random self-avoiding walk (SAW) is distributed according to the Gompertz distribution (Tishby [18]).

In Marshall and Olkin [12], the authors provide a comprehensive review of the history and theory of the Gompertz probability distribution. Golubev [8] emphasizes the practical importance of this probability distribution. Detailed information about the Gompertz distribution, its mathematical and statistical properties, and its applications can be found in Johnson et al. [10] and Dey et al. [6].

Spacings are very important in mathematical statistics and some areas of applied probability. They can be used in statistical inference and goodness-of-fit (see for example Lockhart et al. [11], Stephens [16]). They can also be used in characterizing probability distributions (see Alsamullah [1], Pyke [15]).

The Gompertz probability distribution has a somewhat mathematically complicated probability density function and distribution function, because these two functions involve the double exponential Pollard and Valkovics [14]. This makes the task of computing the distributions of spacings tedious and complicated.

Besides the introduction, this paper is organized as follows. In Section 2, the Gompertz distribution is introduced. Then the distributions of spacings associated with a random sample of size $n$ from the Gompertz distribution are discussed. In Section 3, closed-form formulas of the moments of spacings are derived and expressed explicitly in terms of the integro-exponential function and the Meijer G-functions. In Section 4, a brief conclusion to the results obtained in this paper.
2 Distributions of Spacings

The random variable $X$ is said to follow a Gompertz distribution with scale parameter $\lambda > 0$ and frailty parameter $\xi > 0$, written for short as $X \sim GM(\lambda, \xi)$, if the probability density function (pdf) of $X$ is given by

$$f_X(x) = \lambda e^{\lambda x + (1-e^{\lambda x})\xi}, \quad x \geq 0. \quad (1)$$

The cumulative distribution function (cdf) of $X \sim GM(\lambda, \xi)$ at $t \geq 0$ is given by

$$F_X(t) = 1 - e^{(1-e^{\lambda t})\xi}. \quad (2)$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution $GM(\lambda, \xi)$ with scale parameter $\lambda > 0$ and frailty parameter $\xi > 0$, respectively. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics, where $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$. Let $D_{i:n} = X_{i+1:n} - X_{i:n}$ be the distance between $X_{i+1:n}$ and $X_{i:n}$ for $i = 1, 2, 3, \ldots, n-1$. By convention, we set $D_{0:n} = X_{1:n}$ (Casella and Berger [4]).

The random variables $D_{0:n}, D_{1:n}, \ldots, D_{n-1:n}$ are called the spacings between successive order statistics (Pyke [5] and Casella and Berger [4]).

Recall that the joint pdf of the $i$th and $j$th order statistics $X_{i:n}$ and $X_{j:n}, 1 \leq i < j \leq n$, is given by DasGupta [3].

$$f_{i,j}(x, y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f(x) f(y) F^{i-1}(x) \times [F(y) - F(x)]^{j-1-1} [1 - F(y)]^{n-j}, \quad (3)$$

for $-\infty < x < y < \infty$.

When $j = i + 1$, [3] reduces to

$$f_{i,i+1}(x, y) = \frac{n!}{(i-1)!(n-i-1)!} f(x) f(y) F^{i-1}(x) [1 - F(y)]^{n-i-1}. \quad (4)$$

It follows from [4] that the pdf of $D_{i:n}$ at $r > 0$ is then the integral

$$f_{D_{i:n}}(r) = \int_0^\infty f_{i,i+1}(x, x + r) dx$$

$$= \int_0^\infty c(i, n)s(i) e^{x\lambda + (1-e^{\lambda x})\xi + (1-e^{(r+x)\lambda})\xi + (1-e^{(r+z)\lambda})\xi} (n-i-1)! \xi dx,$$

where $c(i, n) = \frac{\lambda^2 e^{\lambda n}}{(i-1)! (n-i-1)!}$ and $s(i) = \left(1 - e^{\xi(1-e^{\lambda r})}\right)^{i-1}$.

**Lemma 2.1.** For $a > 0$,

$$\int_1^{\infty} te^{-at} dt = \frac{(1+a)e^{-a}}{a^2}. \quad \Box$$

**Proof.** By integration by parts,

$$\int_1^{\infty} te^{-at} dt = \left[ -\frac{t e^{-at}}{a} \right]_{1}^{\infty} + \int_1^{\infty} \frac{e^{-at}}{a} dt$$

$$= \frac{e^{-a}}{a} + \frac{e^{-a}}{a^2} = \frac{(1+a)e^{-a}}{a^2}.$$  

**Lemma 2.2.** Let

$$I(r) = \int_0^\infty e^{\lambda x + 2r x + (-j-i) e^{\lambda x} - (n-i)e^{(r+x)\lambda} + n \xi} dx.$$

Then

$$I(r) = \frac{e^{\lambda+(-1-e^{\lambda}) \xi} \left(1 + (i - j + (n - i)e^{\lambda \xi}\right)}{(i-j+(n-i)e^{\lambda \xi})^2 \lambda^2 \xi^2}.$$
Proof. Letting \( x = \lambda^{-1} \ln u \), we can write \( I(r) \) as
\[
I(r) = \int_{1}^{\infty} e^{r \lambda + (n-i) \xi} u e^{((1+e^{\lambda})i + j - e^{\lambda} n) \xi} du.
\]
Then, by Lemma 2.1,
\[
I(r) = \frac{e^{r \lambda + (1-e^{\lambda})(n-i) \xi} (1 + (i - j + (n - i)e^{r \lambda}) \xi)}{(i - j + (n - i)e^{r \lambda})^2 \xi^2}.
\]

**Theorem 2.3.** The probability density function of \( D_{i,n} \) at \( r > 0 \), denoted by \( f_{D_{i,n}}(r) \), is given as
\[
f_{D_{i,n}}(r) = \sum_{j=0}^{i-1} (-1)^{j-1} \binom{i-1}{j} \frac{\lambda ne^{r \lambda + (1-me^{r \lambda}) \xi} (1 + (t + me^{r \lambda}) \xi)}{(i-1)!(m-1)!(t + me^{r \lambda})^2},
\]
where \( t = i - j \) and \( m = n - i \).

Proof. By (5), we see that
\[
f_{D_{i,n}}(r) = \int_{0}^{\infty} c(i,n) s(i)e^{r \lambda + (r + z) \lambda + (1-e^{\lambda}) \xi + (1-e^{(r + z) \lambda}) \xi + (1-e^{(r + z) \lambda})(n-i-1) \xi} dx.
\]
Using the binomial theorem, we write \( s(i) \) as
\[
s(i) = \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} e^{(i-j-1)(1-e^{\lambda})}.
\]
Therefore, \( f_{D_{i,n}}(r) \) can be written as
\[
f_{D_{i,n}}(r) = c(i,n) \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} I(r),
\]
where
\[
I(r) = \int_{0}^{\infty} e^{r \lambda + 2e^{\lambda} - (j + e^{\lambda}(i-j) + e^{(r + z) \lambda}(n-i)-n) \xi} dx.
\]
Then the result follows from both of Lemma 2.1 and Lemma 2.2.

3 Moments of Spacings

In this section, Some closed-form formulas for the moments of the Gompertz spacings will be derived and proved. Then a recursive identity for those moments will be proved, as well.

Before proceeding to the main theorem of this section, let us introduce the general integro-exponential function (Bateman and Erdlyi [2]), denoted as \( E_{x}^{(k)}(z) \), that we will use to express the moments of spacings. The general integro-exponential function is given as
\[
E_{x}^{(k)}(z) = \frac{1}{\Gamma(k + 1)} \int_{1}^{\infty} u^{-x} e^{-zu} \log^{k} (u) du.
\]
In other words,
\[
\int_{1}^{\infty} u^{-x} e^{-zu} \log^{k} (u) du = \Gamma(k + 1) E_{x}^{(k)}(z).
\]
Remark 3.1. Note that $E^{(k-1)}_{q+2}(m\xi)$ is the Meijer G-function at $m\xi$. It is given as
\begin{equation}
E^{(k-1)}_{q+2}(m\xi) = G^{k+1,0}_{k,k+1}(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k | m\xi),
\end{equation}
where $a_i = q + 2$ and $b_i = q + 1$, for each $i = 1, \ldots, k$.

Note that $E^{(0)}_{q+2}(m\xi)$ is the exponential integral given as
\begin{equation}
E^{(0)}_{q+2}(m\xi) = \int_1^\infty t^{-q-2} e^{-m\xi t} dt.
\end{equation}
For the definition and other details of the Meijer G-function, see Bateman and Erdlyi [2].

Remark 3.2. The Meijer G-function obeys the following identity (see Styles [17]).
\begin{equation}
z^\rho G^{m,n}_{p,q}(a_p, b_q | z) = G^{m,n}_{p,q}(a_p + \rho, b_q + \rho | z).
\end{equation}

Now, we need the following two lemmas for the proof of the main theorem that gives a closed-form formula for the $k$th moment of the spacings.

Lemma 3.3. For $m \geq 1, \xi > 0, k \geq 1, \text{and } t = i - j \geq 1$, we have
\begin{equation}
I_1 := \int_1^\infty \log^k(u) e^{-m\xi u} du = \Gamma(k+1) \sum_{q=0}^\infty (m/t)^{-q-1} \binom{-1}{q} t^{-1} E^{(k)}_{q+1}(m\xi).
\end{equation}

Proof. Let $v = 1/u$. Then $u = 1/v$ and $du = -dv/v^2$. Then
\begin{equation}
I_1 = \int_0^1 \frac{(-\log(v))^k e^{-m\xi/v}}{tv (\frac{m}{t} + v)} dv.
\end{equation}
Let $a = m/t$. Then $v/a = v(i - j)/(n - i)$. Now, $I_1$ becomes
\begin{equation}
I_1 = \int_0^1 \frac{(-\log(v))^k e^{-m\xi/v}}{tv(a+v)} dv.
\end{equation}
To guarantee that $|v/a| < 1$, we assume that $2i < n$. This guarantees the convergence of the binomial series:
\begin{equation}
(a + v)^{-1} = \sum_{q=0}^\infty \binom{-1}{q} v^q a^{-q-1}.
\end{equation}
Therefore,
\begin{equation}
I_1 = \sum_{q=0}^\infty \int_0^1 a^{-q-1} \binom{-1}{q} (-\log(v))^k v^q e^{-m\xi/v} dv.
\end{equation}
By letting $u = 1/v$, we see that
\begin{equation}
\int_0^1 (-\log(v))^k v^q e^{-m\xi/v} dv = \int_1^\infty \log^k(u) u^{-q-1} e^{-m\xi u} du
= \Gamma(k+1) E^{(k)}_{q+1}(m\xi) \quad \text{(by (9))}.
\end{equation}
Therefore,
\begin{equation}
I_1 = \Gamma(k+1) \sum_{q=0}^\infty a^{-q-1} \binom{-1}{q} t^{-1} E^{(k)}_{q+1}(m\xi).
\end{equation}
Lemma 3.4. For $m \geq 1, \xi > 0, k \geq 1$, and $t = i - j \geq 1$, we have

$$I_2 := \int_1^\infty \frac{\log^k(u) e^{-m \xi u}}{(t + mu)^2} du = \Gamma(k + 1) \sum_{q=0}^\infty (m/t)^{-q-2} \left( -2 \frac{q}{q} \right) t^{-2} E_{q+2}^{(k)}(m \xi).$$

Proof. The proof is similar to the proof of Lemma 3.4. Letting $v = 1/u$, $t = i - j$, $m = n - i$, and $a = m/t$, we get

$$I_2 = \int_0^1 \frac{(-\log(v))^k e^{-m \xi/v}}{t^2(a + v)^2} dv.$$ When $2i < n$, the following binomial series converges

$$(a + v)^{-2} = \sum_{q=0}^\infty \left( -2 \frac{q}{q} \right) v^q a^{-q-2}.$$ Therefore,

$$I_2 = \sum_{q=0}^\infty \int_0^1 a^{-q-2} \left( -2 \frac{q}{q} \right) v^q (-\log(v))^k t^{-2} e^{-m \xi/v} dv.$$ By letting $u = 1/v$, we see that

$$\int_0^1 v^q (-\log(v))^k e^{-m \xi/v} dv = \int_1^\infty u^{-q-2} \log^k(u) e^{-m \xi u} du = \Gamma(k + 1) E_{q+2}^{(k)}(m \xi) \quad \text{(by (9)).}$$ Therefore,

$$I_2 = \Gamma(k + 1) \sum_{q=0}^\infty a^{-q-2} \left( -2 \frac{q}{q} \right) t^{-2} E_{q+2}^{(k)}(m \xi). \quad (14)$$

Theorem 3.5. The $k^{th}$ moment of $D_{i:n}$, denoted by $E(D_{i:n}^k)$, is given as

$$E(D_{i:n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^\infty n!(-1)^{t-1} \left( \frac{i-1}{j} \right) e^{m \xi} \Gamma(k + 1) a^{-q-2} \left( -2 \frac{q}{q} \right) t^{-2} E_{q+2}^{(k)}(m \xi), \quad (15)$$

where $t = i - j, m = n - i$, and $a = m/t$.

Proof. By (14) of Theorem 2.3, the $k^{th}$ moment of $D_{i:n}$ is given as

$$E(D_{i:n}^k) = \sum_{j=0}^{i-1} \int_0^\infty \frac{\log^k(u) e^{-m \xi u} 1 + (t + me^{\lambda t})}{(i-1)!(m-1)!(t + me^{\lambda t})^2} dr.$$ Let $u = e^{\lambda r}$. Then

$$E(D_{i:n}^k) = \sum_{j=0}^{i-1} \frac{e^{m \xi} \int_0^\infty n!(-1)^{t-1} \left( \frac{i-1}{j} \right) e^{m \xi u} 1 + (t + mu)\xi}{(i-1)!(m-1)!(t + mu)^2} du = \sum_{j=0}^{i-1} \frac{e^{m \xi} n!(-1)^{t-1} \left( \frac{i-1}{j} \right)}{\lambda^t(i-1)!(m-1)!} (\xi I_1 + I_2),$$ where
Proof. In terms of the Meijer G-function, by (12), we can rewrite (15) as

\[ E(D_{i,n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} \Gamma(k+1)a^{-q-2}}{\lambda^q j^q(i-1)!(m-1)!} \left( \xi a \right)^{i-1} E^{(k)}_{q+1}(m\xi) + \left( -\frac{2}{q} \right) E_{q+2}^{(k)}(m\xi). \]

Note that

\[ \left( -\frac{2}{q} \right) = (q+1) \left( -\frac{1}{q} \right) = (-1)^q (q+1). \]

Therefore,

\[ E(D_{i,n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} \Gamma(k+1)a^{-q-2}}{\lambda^q j^q(i-1)!(m-1)!} \left( m\xi E_{q+1}^{(k)}(m\xi) + (q+1)E_{q+2}^{(k)}(m\xi) \right). \]

We use Equation 2.4 of Milgram [13]:

\[ m\xi E_{q+1}^{(k)}(m\xi) + (q+1)E_{q+2}^{(k)}(m\xi) = E_{q+2}^{(k)}(m\xi). \]

Therefore,

\[ E(D_{i,n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} \Gamma(k+1)a^{-q-2}}{\lambda^q j^q(i-1)!(m-1)!} \left( m\xi E_{q+1}^{(k)}(m\xi) + (q+1)E_{q+2}^{(k)}(m\xi) \right). \]

The following corollary makes the implementation of the moments of spacings relatively fast with Wolfram Mathematica software.

**Corollary 3.6.** In terms of the Meijer G-function, by [13], we can rewrite (15) as

\[ \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} k!(-1)^q(i-j)^q}{\lambda^q j^q(i-1)!(m-1)!} G_{k,k+1}^{(i+1,0)} \left( \begin{array}{c} 0, \ldots, 0 \end{array} \bigg| \begin{array}{c} -q-2, -1, \ldots, -1 \end{array} \right)^{\xi}. \]

Proof.

\[ E(D_{i,n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} \Gamma(k+1)\left( \frac{1}{1} \right) a^{-q-2}E_{q+2}^{(k-1)}(m\xi)}{\lambda^q j^q(i-1)!(m-1)!} \]

\[ = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(1)^{i-1}(-1)^j e^{m\xi} k!(-1)^q(i-j)^q}{\lambda^q j^q(i-1)!(m-1)!} m^{-q-2} E_{q+2}^{(k-1)}(m\xi). \]

By [10],

\[ m^{-q-2} E_{q+2}^{(k-1)}(m\xi) = m^{-q-2} G_{k,k+1}^{(i+1,0)} \left( \begin{array}{c} q+2, \ldots, q+2 \end{array} \bigg| \begin{array}{c} 0, q+1, \ldots, q+1 \end{array} \right)^{\xi}. \]

\[ = G_{k,k+1}^{(i+1,0)} \left( \begin{array}{c} 0, \ldots, 0 \end{array} \bigg| \begin{array}{c} -q-2, -1, \ldots, -1 \end{array} \right)^{\xi}. \]

□
4 Conclusion

It has been shown in this paper that the probability density function of $D_{i:n}$ at $r > 0$, denoted by $f_{D_{i:n}}(r)$, is given by

$$f_{D_{i:n}}(r) = \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} \frac{\lambda n! e^{r\lambda+(1-m)e^{r\lambda}} \xi (1+(t+me^{r\lambda})\xi)}{(i-1)!(m-1)! (t+me^{r\lambda})^2},$$

where $t = i - j$ and $m = n - i$.

Equation (18) has been used to compute the moments of the Gompertz spacings. In other words; the $k$th moment of the $n$th spacing $D_{i:n}$, denoted by $E(D_{i:n}^k)$, is given by

$$E(D_{i:n}^k) = \sum_{j=0}^{i-1} \sum_{q=0}^{\infty} \frac{n!(i-1)!^{i-j} \Gamma(k+1)a^{-q}(-1)^q e^{m\xi} (m\xi)^{k-1} (m-1)!}{\lambda^k (i-1)!(m-1)!},$$

where $t = i - j, m = n - i$, and $a = m/t$.

References


