Abstract In this paper, using “belongingness (∈)” and “quasi-coincidence (q)” of fuzzy points and fuzzy sets, the concept of $(ε, ε ∨ q)$-fuzzy hemirings and $(ε, ε ∨ q)$ -fuzzy hemirings have been introduced and some of its properties have been investigated.

Keywords: Fuzzy set, Hemiring, fuzzy hemiring, $(ε, ε ∨ q)$- fuzzy hemiring, $(ε, ε ∨ q)$ -fuzzy hemiring.
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1 Introduction

After the introduction of fuzzy set by Zadeh [1], a host of researchers applied this impression in different branches of algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. In algebra, fuzzy set was first applied by Rosenfeld [2]; the idea of fuzzy group was defined and its structures were investigated. The $(ε, ε ∨ q)$ -fuzzy group which is generalization of Rosenfeld’s fuzzy group was initiated Bhakat and Das [3] by using the joint notions of “belongingness (ε)” and “quasi-coincidence (q)” of fuzzy points and fuzzy sets, which was instigated by Pu and Liu [4]. Later these structures were studied by some researchers [5-13]. In hemirings structures, Dudek et al. [14] introduced the concept of $(ε, ε ∨ q)$ -fuzzy h-ideals (k-ideal) of a hemirings, and gave its characterizations. Afterward there have been done some works related to $(ε, ε ∨ q)$ -fuzzy h-ideals of a hemiring [15-20]. In this paper, we have introduced $(ε, ε ∨ q)$-fuzzy hemirings and $(ε, ε ∨ q)$ -fuzzy hemirings and some related properties have been described.
2 Preliminaries

Definition 2.1 [21]: A hemiring is a nonempty set $H$ on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

- $[H_1]: (H;+)$ is a commutative monoid with identity element $0$;
- $[H_2]: (H;\cdot)$ is a semigroup;
- $[H_3]:$ Multiplication distributes over addition from either side;
- $[H_4]:$ The element 0 is the absorbing element of the multiplication i.e.,
  \[ 0 \cdot r = 0 = r \cdot 0. \]

Note: We shall omit the symbol “·” writing $ab$ for $a \cdot b (a,b \in H)$.

Definition 2.2 [22]: Suppose $(H;+,-)$ is a hemiring and $\mu$ is a fuzzy set of $H$.

(i) For any $t \in [0,1]$ the set $\mu_t = \{x \in H : \mu(x) \geq t\}$ is called $t$-level set of $\mu$.

(ii) $\text{Supp}(\mu) = \{x \in H : \mu(x) > 0\}$ is called support of $\mu$.

Definition 2.3 [11]: Suppose $(H;+,-)$ is a hemiring and $\mu$ is a fuzzy set of $H$ with support $\{x\}$. Then $\mu$ is called a fuzzy point where,

\[
\mu(y) = \begin{cases} 
  t \in [0,1] & \text{if } y = x \\
  0 & \text{otherwise}
\end{cases}
\]

We denote the fuzzy point with support $\{x\}$ and value $t$ by $x_t$. For a fuzzy point $x_t$ and a fuzzy set $\mu$ of $H$, Pu and Liu [4] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{e, q, e \wedge q, e \vee q\}$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ of $H$, we say that a fuzzy point $x_t$ belongs to $\mu$, denoted by $x_t \in \mu$ if $\mu(x) \geq t$;

(ii) quasi-coincident with $\mu$, denoted by $x_t \mu$ if $\mu(x) + t > 1$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ of $H$, we say that

(iii) $x_t \in e \mu$ if $x_t \in \mu$ or $x_t \mu$;

(iv) $x_t \in e \wedge q \mu$ if $x_t \in \mu$ and $x_t \mu$.

The symbol $x_t \in \bar{\mu}$, $x_t \mu$ and $x_t \in e \wedge q \mu$ mean that $x_t \in \mu$, $x_t \mu$ and $x_t \in e \wedge q \mu$ do not hold respectively.

Example 2.3 (a): Let $H = \{0,1,2\}$ be a hemiring with the operations addition $\ (\cdot)$ and multiplication $\ (\cdot)$ as follows:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
2 & 2 & 0 & 1
\end{array}
\]

Now, define a fuzzy set $\mu$ of $H$ as follows:

\[
\mu(x) = \begin{cases} 
  0.3 & \text{if } x = 0 \\
  0.6 & \text{if } x = 1 \\
  0.8 & \text{if } x = 2
\end{cases}
\]

Where $\text{Supp}(\mu) = \{0,1,2\}$. Now consider the fuzzy point $0_t$, where $t \in [0,0.3]$ then $0_t \in \mu$. Again we consider the fuzzy point $2_t$, where $t \in [0.2,0.8]$ then $2_t \mu$. 

Definition 2.4 [23]: Let \((H;+,\cdot)\) be a hemiring. A fuzzy set \(\mu\) of \(H\) is said to be \textit{fuzzy hemiring} of \(H\) if it satisfies the following conditions:
\[
[\text{FH}_1]: \mu(x + y) \geq \min\{\mu(x),\mu(y)\}; \\
[\text{FH}_2]: \mu(x \cdot y) \geq \min\{\mu(x),\mu(y)\}; \forall x, y \in H.
\]

Definition 2.5: Let \(\mu_1\) and \(\mu_2\) be two fuzzy hemirings of a hemiring \((H;+,\cdot)\). We define the \textit{sum and product}, denoted by \(\mu_1 + \mu_2\) and \(\mu_1 \cdot \mu_2\) of \(\mu_1\) and \(\mu_2\) by
\[
(\mu_1 + \mu_2)(x) = \max\{\min\{\mu_1(y), \mu_2(z)\}\};
\]
If \(x\) can be expressed as \(x = y + z\) otherwise \((\mu_1 + \mu_2)(x) = 0\).

The conditions
\[
(\mu_1 \cdot \mu_2)(x) = \max\{\min\{\mu_1(y), \mu_2(z)\}\};
\]
If \(x\) can be expressed as \(x = yz\) otherwise \((\mu_1 \cdot \mu_2)(x) = 0\).

3 \((\in,\in \vee q)\)-Fuzzy Hemirings

Definition 3.1: A fuzzy set \(\mu\) of a hemiring \((H;+,\cdot)\) is called \((\in,\in \vee q)\)-\textit{fuzzy hemiring} of \(H\) if \(\forall t, r \in [0,1]\) and \(\forall x, y \in H\), the following conditions hold:
\[
[\text{FH}_1a]: x \in \mu \text{ and } y \in \mu \Rightarrow (x + y)_{\min_{[r,t]}} \in \in \vee q \mu; \\
[\text{FH}_2a]: x \in \mu \text{ and } y \in \mu \Rightarrow (xy)_{\min_{[r,t]}} \in \in \vee q \mu.
\]

Example 3.1 (a): Let \(H = N \cup \{0\}\). Then \(H\) is a hemiring with usual addition and multiplication. Let \(\mu\) be a fuzzy set defined by
\[
\mu(x) = \begin{cases} 
0.9 & \text{if } x \text{ is even} \\
0.8 & \text{if } x \text{ is odd} \\
0.6 & \text{if } x = 0
\end{cases}
\]
Then routine calculation gives that, \(\mu\) is an \((\in,\in \vee q)\) - fuzzy hemiring of \(H\).

Theorem 3.2: The conditions \([\text{FH}_1a]\) and \([\text{FH}_2a]\) in definition 3.1 are equivalent to the following conditions \([\text{FH}_1b]\) and \([\text{FH}_2b]\) respectively, where
\[
[\text{FH}_1b]: \mu(x + y) \geq \min\{\mu(x),\mu(y),0.5\}; \\
[\text{FH}_2b]: \mu(x \cdot y) \geq \min\{\mu(x),\mu(y),0.5\}.
\]

Proof: \([\text{FH}_1a] \Rightarrow [\text{FH}_1b]\): Suppose that \(x, y \in H\), we consider the following cases:

(a) \(\min\{\mu(x),\mu(y)\} < 0.5\),

(b) \(\min\{\mu(x),\mu(y)\} \geq 0.5\).

Case (a): Assume that there exists \(x, y \in H\) such that
\[
\mu(x + y) < \min\{\mu(x),\mu(y),0.5\};
\]
Then for \(t \in [0,0.5]\), we have,
\[
\mu(x + y) < t \leq \min\{\mu(x),\mu(y)\}.
\]
It follows that \(x, y \in \mu\) but \((x + y), \not\in \mu\).
Moreover, $\mu(x + y) + t < 1$ and so $(x + y), q\mu$.

Hence $(x + y), \in \vee q\mu$, a contradiction.

**Case (b):** Assume that there exists $x, y \in H$ such that

$\mu(x + y) < 0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, but $(x + y)_{0.5} \in \mu$.

Also $\mu(x + y) + 0.5 < 1$ and so $(x + y)_{0.5} q\mu$.

Therefore $(x + y)_{0.5} \in \vee q\mu$, a contradiction

Hence $[FH1b]$ holds.

$[FH1b] \Rightarrow [FH1a]$: Let $x_i \in \mu$ and $y_i \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$.

Now we have,

$$\mu(x + y) \geq \min \{\mu(x), \mu(y), 0.5\} \geq \min \{t, r, 0.5\}.$$  

If $\min \{t, r\} > 0.5$ then $\mu(x + y) \geq 0.5$, this implies that

$$\mu(x + y) + \min \{t, r\} > 1.$$  

If $\min \{t, r\} \leq 0.5$ then $\mu(x + y) \geq \min \{t, r\}$.

Therefore $(x + y)_{\min \{t, r\}} \in \vee q\mu$.

Similarly, we can prove that $[FH2a] \Rightarrow [FH2b]$.

**Theorem 3.3:** If $\mu$ is a fuzzy set of a hemiring $(H; +, \cdot)$ satisfying $\mu(x) \geq 0; \forall x \in H$, then $\mu$ is an $(\in, \in \vee q)$-fuzzy hemiring of $H$.

**Proof:** Let $x, y \in H$ and $t, r \in [0, 1]$ be such that $x_i \in \mu$ and $y_i \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$.

Now $\forall x, y \in H$, we have $x + y \in H$ and $xy \in H$, then

$$\mu(x + y) \geq 0.5 \text{ and } \mu(xy) \geq 0.5.$$  

If $\min \{t, r\} \leq 0.5$, then $\mu(x + y) \geq 0.5 \geq \min \{t, r\}$

and $\mu(xy) \geq 0.5 \geq \min \{t, r\}$.

This implies $(x + y)_{\min \{t, r\}} \in \mu$ and $(xy)_{\min \{t, r\}} \in \mu$.

Again if $\min \{t, r\} > 0.5$, then $\mu(x + y) + \min \{t, r\} > 1$.

and $\mu(xy) + \min \{t, r\} > 1$.

This implies that $(x + y)_{\min \{t, r\}} q\mu$ and $(xy)_{\min \{t, r\}} q\mu$.

Therefore, $(x + y)_{\min \{t, r\}} \in \vee q\mu$ and $(xy)_{\min \{t, r\}} \in \vee q\mu$.

Thus $\mu$ is an $(\in, \in \vee q)$-fuzzy hemiring of $H$.

**Theorem 3.4:** Every fuzzy hemiring is an $(\in, \in \vee q)$-fuzzy hemiring.

**Proof:** Let $x, y \in H$ and $t, r \in [0, 1]$ be such that $x_i \in \mu$ and $y_i \in \mu$ then $\mu(x) \geq t$ and $\mu(y) \geq r$. Since $\mu$ is a fuzzy hemiring so we have,

$$\mu(x + y) \geq \min \{\mu(x), \mu(y)\} \text{ and } \mu(xy) \geq \min \{\mu(x), \mu(y)\}.$$  


Now \( \mu(x + y) \geq \min\{ \mu(x), \mu(y) \} \geq \min\{ t, r \} \).

And \( \mu(xy) \geq \min\{ \mu(x), \mu(y) \} \geq \min\{ t, r \} \).

If \( \min\{ t, r \} \leq 0.5 \) then \( (x + y)_{\min\{t,r\}} \in \mu \) and \( (xy)_{\min\{t,r\}} \in \mu \).

Or if \( \min\{ t, r \} > 0.5 \) then \( (x + y)_{\min\{t,r\}} q \mu \) and \( (xy)_{\min\{t,r\}} q \mu \).

Hence consequently \( (x + y)_{\min\{t,r\}} \in \vee q \mu \) and \( (xy)_{\min\{t,r\}} \in \vee q \mu \).

This implies that \( \mu \) is an \((\varepsilon, \in \lor q)\)-fuzzy hemiring of \( H \).

**Theorem 3.5:** Any \((\varepsilon, \in \lor q)\)-fuzzy hemiring \( \mu \) of a hemiring \((H;+, \cdot)\) satisfying \( \mu(x) < 0.5 \), is an ordinary fuzzy hemiring.

**Proof:** Let \( x, y \in H \) and \( t, r \in [0,1] \) be such that \( x, y \in \mu \) then \( \mu(x) \geq t \) and \( \mu(y) \geq r \). Since \( \mu \) is an \((\varepsilon, \in \lor q)\) fuzzy hemiring of \( H \), so we have,

\[ \mu(x + y) \geq \min\{ \mu(x), \mu(y), 0.5 \} \]

and \( \mu(xy) \geq \min\{ \mu(x), \mu(y), 0.5 \} \).

Now \( \mu(x + y) \geq \min\{ \mu(x), \mu(y), 0.5 \} = \min\{ t, r, 0.5 \} = \min\{ t, r \} \).

And \( \mu(xy) \geq \min\{ \mu(x), \mu(y), 0.5 \} = \min\{ t, r, 0.5 \} = \min\{ t, r \} \).

This implies that \( (x + y)_{\min\{t,r\}} \in \mu \) and \( (xy)_{\min\{t,r\}} \in \mu \).

Thus \( \mu \) is an ordinary fuzzy hemiring.

**Theorem 3.6:** A fuzzy set \( \mu \) of a hemiring \((H;+, \cdot)\) is an \((\varepsilon, \in \lor q)\)-fuzzy hemiring of \( H \) iff for any \( t \in [0,0.5], \mu(t) \neq \emptyset \) is a sub-hemiring of \( H \).

**Proof:** Let \( x, y \in \mu \) then \( \mu(x) \geq t \) and \( \mu(y) \geq t \).

We consider \( \mu \) is an \((\varepsilon, \in \lor q)\)-fuzzy hemiring of \( H \) and \( t \in [0,0.5] \). Then we have,

\[ \mu(x + y) \geq \min\{ \mu(x), \mu(y), 0.5 \} \]

\[ \Rightarrow \mu(x + y) \geq \min\{ t, t, 0.5 \} \]

\[ \Rightarrow \mu(x + y) \geq t \]

\[ \therefore x + y \in \mu. \]

And \( \mu(xy) \geq \min\{ \mu(x), \mu(y), 0.5 \} \)

\[ \Rightarrow \mu(xy) \geq \min\{ t, r, 0.5 \} \]

\[ \Rightarrow \mu(xy) \geq t \]

\[ \therefore xy \in \mu. \]

Hence \( \mu \) is a sub-hemiring of \( H \).

Conversely, suppose \( \mu \) is a sub-hemiring of \( H \). Then \( \forall x, y \in \mu \), we have

\[ x + y \in \mu \]

Now, \( x + y \in \mu \)

\[ \Rightarrow \mu(x + y) \geq t \]
\[ \Rightarrow \mu(x + y) \geq \min\{t, t, 0.5\} = \min\{\mu(x), \mu(y), 0.5\} \]

And 
\[ xy \in \mu, \Rightarrow \mu(xy) \geq t \Rightarrow \mu(xy) \geq \min\{t, t, 0.5\} = \min\{\mu(x), \mu(y), 0.5\}. \]

Hence \( \mu \) is a \((\varepsilon, \varepsilon \land q)\) fuzzy hemiring of \( H \).

**Definition 3.7:** Let \( \mu \) be any fuzzy hemiring of a hemiring \((H; +, \cdot)\).

The set \( [\mu]_t = \{x \in H : x \in \varepsilon \land q \mu\} \), where \( t \in [0,1] \) is called an \((\varepsilon, \varepsilon \land q)\) -level set of \( \mu \).

**Theorem 3.8:** A fuzzy set \( \mu \) of a hemiring \((H; +, \cdot)\) is an \((\varepsilon, \varepsilon \land q)\) -fuzzy hemiring of \( H \) iff \([\mu]_t\) is a sub-hemiring of \( \forall t \in [0,1] \).

**Proof:** Let \( \mu \) be an \((\varepsilon, \varepsilon \land q)\) fuzzy hemiring of a hemiring \( H \).

Let \( x, y \in [\mu]; \forall t \in [0,1]. \) Then \( \mu(x) \geq t \) or \( \mu(x) + t > 1 \) and \( \mu(y) \geq t \) or \( \mu(y) + t > 1 \). Since \( \mu \) is an \((\varepsilon, \varepsilon \land q)\) -fuzzy hemiring, we have,

**Case-1:** When \( \mu(x) \geq t \) and \( \mu(y) \geq t \).

Then \( \mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\} \)
\[ \Rightarrow \mu(x + y) \geq \min\{t, t, 0.5\}. \]

If \( t > 0.5 \), then \( \mu(x + y) \geq 0.5 \), and so \( \mu(x + y) + t > 1 \). \( (x + y), \mu(q) \).

Or if \( t \leq 0.5 \), then \( \mu(x + y) \geq t \), and so \( (x + y), \mu \).

Thus for \( t \in [0,1] \), we have \( x + y \in [\mu]_t \).

**Case-2:** When \( \mu(x) \geq t \) and \( \mu(y) + t > 1 \).

Then \( \mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\} \)
\[ \Rightarrow \mu(x + y) \geq \min\{t, 1-t, 0.5\}. \]

If \( t > 0.5 \), then \( 1-t < 0.5 \).

This implies \( \mu(x + y) > 1-t \Rightarrow \mu(x + y) + t > 1 \), therefore \( (x + y), \mu(q) \).

Or if \( t \leq 0.5 \), then \( 1-t \geq 0.5 \), then we have \( \mu(x + y) \geq t \).

Therefore \( (x + y), \mu \).

Thus \( x + y \in [\mu]_t ; \forall t \in [0,1] \).

**Case-3:** Similar to case-2.

**Case-4:** When \( \mu(x) + t > 1 \) and \( \mu(y) + t > 1 \).

Then \( \mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\} \)
\[ \Rightarrow \mu(x + y) \geq \min\{1-t, 1-t, 0.5\}. \]

If \( t > 0.5 \) then \( 1-t < 0.5 \) and so \( \mu(x + y) > 1-t \Rightarrow \mu(x + y) + t > 1. \)
This implies \((x + y), q\mu\).

If \(t \leq 0.5\), then \(1 - t \geq 0.5 \geq t\). In this case we have,
\[
\mu(x + y) \geq t \Rightarrow (x + y) \in \mu.
\]
Thus \(x + y \in [\mu]; \forall t \in [0,1]\).

In similar way for each case we can prove that \(xy \in [\mu]; \forall t \in [0,1]\).

Therefore, we have \(x + y \in [\mu]\), and \(xy \in [\mu]\).

Thus \([\mu]\) is a sub-hemiring of \(H\).

Conversely, let \(\mu\) be a fuzzy set and \([\mu]\) is a sub-hemiring of \(H; \forall t \in [0,0.5]\).

If \(\mu(x + y) < t < \min\{\mu(x), \mu(y), 0.5\}\) for some \(t \in [0,0.5]\), then \(x, y \in [\mu]\) and \(x + y \in [\mu]\). Hence \(\mu(x + y) \geq t\) or \(\mu(x + y) + t > 1\) contradiction, similar result holds for \(\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}\).

Therefore \(\mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\}\) and \(\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}\).

Again \(\forall t \in [0.5,1]\).

If \(\mu(x + y) < t < \min\{\mu(x), \mu(y)\}\), then \(x, y \in [\mu]\) and \(x + y \in [\mu]\),
but \((x + y) \in \vee q\mu\), a contradiction.

Thus \(\mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\}\)

In similar way, \(\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}\).

Hence \(\mu\) is an \((\varepsilon, \varepsilon \vee q)\)-fuzzy hemiring of \(H\).

**Definition 3.9:** Let \(\mu_1\) and \(\mu_2\) be two \((\varepsilon, \varepsilon \vee q)\)-fuzzy hemiring of a hemiring \((H; +, \cdot)\). We define \(\mu_1 \cap \mu_2\) by
\[
(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x), 0.5\}.
\]

**Example 3.9 (a):** Let \(H = [0,1]\) be a hemiring with the following two operations:

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
- & 0 & 1 \\
0 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

Define two \((\varepsilon, \varepsilon \vee q)\)-fuzzy hemirings of a hemiring \(H\) by
\[
\mu_1(0) = 0.6, \mu_1(1) = 0.7, \quad \text{and} \quad \mu_2(0) = 0.4, \mu_2(1) = 0.7, \quad \text{respectively.}
\]

Then routine verification gives that, \((\mu_1 \cap \mu_2)(0) = 0.4, (\mu_1 \cap \mu_2)(1) = 0.5\).

**Theorem 3.10:** Let \(\mu_1\) and \(\mu_2\) be two \((\varepsilon, \varepsilon \vee q)\)-fuzzy hemirings of a hemiring \(H\). Then \(\mu_1 \cap \mu_2\) is also \((\varepsilon, \varepsilon \vee q)\)-fuzzy hemiring of \(H\).

**Proof:** Let \(x, y \in \mu_1 \cap \mu_2\). Then \(x, y \in \mu_1\) and \(\mu_2\).

Now,
\[
(\mu_1 \cap \mu_2)(x + y) = \min\{\mu_1(x + y), \mu_2(x + y), 0.5\}
\]
\[
\geq \min\{\min\{\mu_1(x), \mu_1(y), 0.5\}, \min\{\mu_2(x), \mu_2(y), 0.5\}, 0.5\}
\]

\[= \min\{\min\{\mu_1(x), \mu_2(x), 0.5\}, \min\{\mu_1(y), \mu_2(y), 0.5\}, 0.5\}\]
\[= \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y), 0.5\}\]

Similarly, we can prove that \((\mu_1 \cap \mu_2)(xy) \geq \min\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y), 0.5\}\). Thus \(\mu_1 \cap \mu_2\) is a fuzzy hemiring of \(H\).

**Definition 3.11:** Let \(\mu_1\) and \(\mu_2\) be two \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemirings of a hemiring \((H; +, \cdot)\). Then the **product** of \(\mu_1\) and \(\mu_2\) is defined as
\[(\mu_1 \circ \mu_2)(x) = \max\{\min\{(\mu_1(y), \mu_2(z), 0.5\}\} \text{ as } x, y, z \text{ are expressed as } x = yz\text{ otherwise } (\mu_1 \circ \mu_2)(x) = 0.\]

**Example 3.11 (a):** Let \(H = \{0, a, b\}\) be a hemiring with the following two operations:

\[
| & 0 & a & b \\
0 & 0 & a & b \\
a & a & a & b \\
b & b & b & b
\]

Define two \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemirings of a hemiring \(H\) by
\[\mu_1(0) = 0.2, \mu_1(a) = 0.5, \mu_1(b) = 0.6 \text{ and } \mu_2(0) = 0.2, \mu_2(a) = 0.2, \mu_2(b) = 0.6 \text{ respectively. Then routine verification gives that, } (\mu_1 \circ \mu_2)(0) = 0.2, (\mu_1 \circ \mu_2)(a) = 0.2, (\mu_1 \circ \mu_2)(b) = 0.5.\]

**Theorem 3.12:** Let \(\mu_1\) and \(\mu_2\) be two \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemirings of a hemiring \((H; +, \cdot)\). Then their product \(\mu_1 \circ \mu_2\) is also \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemiring of \(H\).

**Proof:** Trivial

**Theorem 3.13:** Let \(\lambda, \mu\) and \(\nu\) be \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemirings of a hemiring \((H; +, \cdot)\). If \(\lambda \leq \mu\), then \(\lambda \circ \nu \leq \mu \circ \nu\) and \(\nu \circ \lambda \leq \nu \circ \mu\).

**Proof:** Let \(x \in H\). If \(x\) is not expressible as \(x = yz\), then
\[(\lambda \circ \nu)(x) = 0 = (\mu \circ \nu)(x).\]

Otherwise,
\[(\lambda \circ \nu)(x) = \max\{\min\{\lambda(y), \nu(z), 0.5\}\} \leq \max\{\min\{\mu(y), \nu(z), 0.5\}\} = (\mu \circ \nu)(x).\]

Hence \(\lambda \circ \nu \leq \mu \circ \nu\).

Similarly, we can prove that \(\nu \circ \lambda \leq \nu \circ \mu\).

**Theorem 3.14:** Let \(\lambda, \mu\) and \(\nu\) be \((\varepsilon, \varepsilon \lor q)\) fuzzy hemirings of a hemiring \((H; +, \cdot)\). Then \((\lambda \circ \mu) \circ \nu = \lambda \circ (\mu \circ \nu)\).

**Proof:** Trivial

**Definition 3.15:** Let \(\mu_1\) and \(\mu_2\) be two \((\varepsilon, \varepsilon \lor q)\) -fuzzy hemirings of a hemiring \(H\). We define the **sum** of \(\mu_1\) and \(\mu_2\), denoted by \(\mu_1 + \mu_2\) by
\[(\mu_1 + \mu_2)(x) = \max\{\min\{(\mu_1(y), \mu_2(z), 0.5\}\} \text{ as } x, y, z \text{ are expressed as } x = yz\text{ otherwise } (\mu_1 + \mu_2)(x) = 0.\]
Example 3.15 (a): Let $H = \{0, a, b\}$ be a hemiring with the following two operations:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

Define two $(\in, \in \lor \in)$-fuzzy hemirings of a hemiring $H$ by $\mu_i(0) = 0.2, \mu_i(a) = 0.5, \mu_i(b) = 0.6$ and $\mu_2(0) = 0.2, \mu_2(a) = 0.2, \mu_2(b) = 0.6$, respectively. Then routine verification gives that, $(\mu_i + \mu_2)(0) = 0.2, (\mu_i + \mu_2)(a) = 0.2, (\mu_i \circ \mu_2)(b) = 0.5$.

Theorem 3.16: Let $\mu_i$ and $\mu_2$ be two $(\in, \in \lor \in)$-fuzzy hemirings of a hemiring $(H; +, \cdot)$. Then their sum $\mu_i + \mu_2$ is also $(\in, \in \lor \in)$-fuzzy hemiring of $H$.

Proof: Trivial

Theorem 3.17: A fuzzy set $\mu$ of a hemiring $(H; +, \cdot)$ is an $(\in, \in \lor \in)$-fuzzy hemiring of $H$ if and only if $\mu + \mu \leq \mu$ and $\mu \circ \mu \leq \mu$.

Proof: Trivial

4 $(\in, \in \lor \in)$-Fuzzy Hemiring

Definition 4.1: A fuzzy set $\mu$ of a hemiring $(H; +, \cdot)$ is called an $(\in, \in \lor \in)$-fuzzy hemiring of $H$ if $\forall t, r \in [0,1]$ and $\forall x, y \in H$, the following conditions are hold:

$[FH3a]: (x + y)_{\min[t,r]} \in \mu \Rightarrow x_i \in \lor q \mu$ or $y_i \in \lor q \mu$;

$[FH4a]: (xy)_{\min[t,r]} \in \mu \Rightarrow x_i \in \lor q \mu$ or $y_i \in \lor q \mu$.

Example 4.1 (a): Let $H = \{0, a\}$ is a hemiring where addition and multiplication are defined as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

Consider the fuzzy sets $\mu(0) = 0.3$ and $\mu(a) = 0.4$. Then by routine calculation gives that, $\mu$ is an $(\in, \in \lor \in)$-fuzzy hemiring.

Theorem 4.2: The conditions $[FH3a]$ and $[FH4a]$ in definition 4.1 are equivalent to the following conditions $[FH3b]$ and $[FH4b]$ respectively.

$[FH3b]: \max\{\mu(x + y), 0.5\} \geq \min\{\mu(x), \mu(y)\};$

$[FH4b]: \max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}.$

Proof: First we prove $[FH3a]$ is equivalent to $[FH3b]$.

$[FH3a] \Rightarrow [FH3b]:$ Suppose $[FH3b]$ does not hold.

Then there exists $x, y \in H$ such that $\max\{\mu(x + y), 0.5\} < \min\{\mu(x), \mu(y)\}$. Then we can choose $t \in [0.5,1]$ such that, $\max\{\mu(x + y), 0.5\} < t \leq \min\{\mu(x), \mu(y)\}$.

This implies $(x + y) \notin \mu$ but $x_i \in \land q$ and $y_i \in \land q$, which is a contradiction.
So \( [FH3b] \) holds.

\[ [FH3b] \Rightarrow [FH3a]: \]

Let \( x, y \in H \) and \( t, r \in [0,1] \) be such that \( (x+y)_{\min\{t,r\}} \notin \mu. \)

Then \( \mu(x+y) < \min\{t,r\} \).

If \( \max\{\mu(x+y),0.5\} = \mu(x+y) \), then,

\[ \min\{\mu(x),\mu(y)\} \leq \mu(x+y) < \min\{t,r\} \Rightarrow \mu(x) < t \text{ or } \mu(y) < t \Rightarrow x, y \notin \bar{\mu} \]

or \( y, y \notin \bar{\mu} \).

If \( \max\{\mu(x+y),0.5\} = 0.5 \), then \( \min\{\mu(x),\mu(y)\} \leq 0.5. \)

Suppose \( x, y \in \mu \) and \( y, y \in \mu \), then \( \mu(x) \geq t \) and \( \mu(y) \geq r \).

Thus \( t \leq \mu(x) < 0.5 \) or \( r \leq \mu(y) < 0.5 \Rightarrow x, y \notin \bar{\mu} \) or \( y, y \notin \bar{\mu} \Rightarrow x, y \notin \bar{\mu} \)

Or \( y, y \notin \bar{\mu}. \) Thus prove \( [FH3a] \).

Similarly we can prove that \( [FH4a] \Leftrightarrow [FH4b] \).

**Theorem 4.3:** A fuzzy set \( \mu \) of a hemiring \((H;+,:)\) is an \((\bar{\varepsilon},\bar{\varepsilon} \lor \bar{q})\)-fuzzy hemiring of \( H \) iff for any \( t \in [0.5,1] \), \( \mu(t \neq \phi) \) is a sub-hemiring of \( H \).

**Proof:** It is immediate consequence of the theorem 3.6.

### 5 Conclusion

Fuzzy algebraic structures play an important role in many branches of applied mathematics and engineering sciences. In order to broad fuzzy algebraic structures in this article we introduce \((\bar{\varepsilon},\bar{\varepsilon} \lor \bar{q})\)-fuzzy hemiring and \((\bar{\varepsilon},\bar{\varepsilon} \lor \bar{q})\)-fuzzy hemiring and some of its structures are investigated. These structures will be helpful to broad the applications of fuzzy hemirings in mathematics and engineering sciences.

### References

7. B. Davvaz, \((\bar{\varepsilon},\bar{\varepsilon} \lor \bar{q})\)-fuzzy subnear-rings and ideals, Soft Computing, 10, (2006), pp. 206-211.


