Characterizations of Interval-Valued Intuitionistic Fuzzy n-Fold Positive Implicative Deal of BCK-Algebras

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Abstract Using the concept of interval-valued intuitionistic fuzzy set, the notion of interval-valued intuitionistic fuzzy n-fold BCK-ideals and interval-valued intuitionistic fuzzy n-fold positive implicative ideas are introduced, in BCK-Algebras and investigate some of its related properties. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideal are investigated.

Keywords: Interval-valued intuitionistic fuzzy n-fold positive implicative ideal, Interval-valued intuitionistic fuzzy n-fold BCK-ideal.

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1 Introduction:

For the general development of BCK-algebras, the ideal theory plays an important role. In [7] 1966 by K. Iseki and Y. Imai introduced a new notion called a BCK-algebras, and then many researchers have investigate various properties of this algebra. Hung and Chen [6] introduced the notion of n-fold implicative ideals, n-fold (weak) commutative ideals, n-fold positive implicative ideals and investigate some of its properties. Jun and Kim [9], introduced the notions of n-fold fuzzy positive implicative ideals in BCK-algebras and investigate some of its related properties. Satyanayana and Durga Prasad [14], introduced the intuitionistic fuzzification of n-fold BCK-ideal and n-fold positive implicative ideal in BCK-algebras and some of its related properties are investigated.

In this paper, we apply the concept of interval-valued intuitionistic fuzzy set to n-fold BCK-ideals and n-fold positive implicative ideals in BCK-algebras and introduced the notion of interval-valued intuitionistic fuzzy n-fold BCK-ideal and interval-valued intuitionistic fuzzy n-fold positive implicative ideal and we gave relations between these notions and investigate some of its related properties. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideals are investigated.
2 Preliminaries:

Definition 2.1. Let X be a set with a binary operation “*” and a constant “0”. Then (X, *, 0) is called BCK-algebra, if it satisfies the following conditions.

(BCK-1) \((x * y) * (z * y) = (x * z) * y = 0\),
(BCK-2) \((x * (x * y)) * y = 0\),
(BCK-3) \(x * x = 0\),
(BCK-4) \(0 * x = 0\)
(BCK-5) \(x * y = 0\) and \(y * x = 0\) imply \(x = y\), for all \(x, y, z \in X\)

We can define a binary relation \(\leq\) on \(X\) by letting \(x \leq y\) if and only if \(x * y = 0\). Then \((X, \leq)\) is a partially ordered set with least element “0” and \((X, *, 0)\) is a BCK-algebra if and only if, it satisfies the following: For all \(x, y, z \in X\)

\[
\begin{align*}
(i) \quad & (x * y) * (x * z) \leq (z * y), \\
(ii) \quad & (x * (x * y)) \leq y, \\
(iii) \quad & x \leq x, \\
(iv) \quad & 0 \leq x \\
(v) \quad & x \leq y \text{ and } y \leq x \text{ imply that } x = y,
\end{align*}
\]

In a BCK-algebra \((X, *, 0)\), we have the following properties:

\[
\begin{align*}
(P1) \quad & x * 0 = x, \\
(P2) \quad & x * y \leq x, \\
(P3) \quad & (x * y) * z = (x * z) * y, \\
(P4) \quad & (x * z) * (y * z) \leq x * y, \\
(P5) \quad & x * (x * y) = x * y, \\
(P6) \quad & x \leq y \Rightarrow x * z \leq y * z \text{ and } z * y \leq z * x,
\end{align*}
\]

Throughout this paper \(X\) will always mean a BCK-algebra unless otherwise specified.

A non-empty sub-set \(I\) of \(X\) is said to be sub-algebra of \(X\) if for \(x, y \in I \Rightarrow x * y \in I\)

A non-empty subset \(I\) of \(X\) is called an ideal of \(X\) if \((I_1)\) \(0 \in I\) \((I_2)\) \(x * y \text{ and } y \in I \Rightarrow x \in I\) for every \(x, y \in X\), is said to be an n-fold positive implicative ideal of \(X\) if \((I_1)\) and \((I_2)\) there exists a fixed \(n \in \mathbb{N}\) \(X^x, y, \in I \Rightarrow x * z^n \in I\) for every \(x, y, z \in X\), is said to be an n-fold BCK-ideal of \(X\) if \((I_1)\) and \((I_2)\) there exists a fixed \(n \in \mathbb{N}\) \(X^x, y, \in I \Rightarrow x * y^n \in I\) for every \(x, y, z \in X\). For any elements \(x\) and \(y\) of \(X\), \(x * y^n\) denotes \((........(x * y) * y) * .......\) * \(y\) in which ‘\(y\)’ occurs \(n\)-times.

An interval-valued intuitionistic fuzzy set (i-v IFS, shortly) “\(\tilde{A}\)” over \(X\) is an object having the form \(\tilde{A} = \{(x, \tilde{\mu}_A, \tilde{\lambda}_A): x \in X\}\), where \(\tilde{\mu}_A(x): X \rightarrow D[0, 1]\) and \(\tilde{\lambda}_A(x): X \rightarrow D[0, 1]\), the intervals \(\tilde{\mu}_A(x)\) and \(\tilde{\lambda}_A(x)\) denotes the intervals of the degree of membership and the degree of the non-membership of the element \(x\) to the set \(\tilde{A}\), where \(\mu_A(x) = [\mu_A^-, \mu_A^+]\) and \(\lambda_A(x) = [\lambda_A^-, \lambda_A^+]\) for all \(x \in X\) with the condition \([0, 0] \leq [\tilde{\mu}_A(x) + \tilde{\lambda}_A(x)] \leq [1, 1]\) for all \(x \in X\)

. For the sake of simplicity, we use the symbol \(\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)\).

Definition 2.2. An i-v IFS \(\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)\) in \(X\) is an interval-valued intuitionistic fuzzy ideal of \(X\), if it satisfies

(i-v IF1) \(\tilde{\mu}_A(0) \geq \mu_{\tilde{\mu}_A}(x)\) and \(\tilde{\lambda}_A(0) \leq \lambda_{\tilde{\lambda}_A}(x)\)

(i-v IF2) \(\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\}\)

(i-v IF3) \(\tilde{\lambda}_A(x) \leq \min\{\tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y)\}\) for all \(x, y \in X\).

Theorem 2.3. An intuitionistic fuzzy sub-algebra \(A = (\mu_A, \lambda_A)\) is an intuitionistic fuzzy ideal of \(X\) if and only if for \(x, y, z \in X, x * y \leq z \Rightarrow \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}\) and \(\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z)\}\).

Theorem 2.4. Let \(\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)\) be an interval valued intuitionistic fuzzy ideal of \(X\).

If \(x \leq y\) in \(X\), then \(\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y), \tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)\), that is, \(\tilde{\mu}_A\) is order-reversing and \(\tilde{\lambda}_A\) is order-preserving.
3 Interval-valued intuitionistic fuzzy n-fold BCK-ideals of BCK-algebras:

Definition 3.1. An i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in $X$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of $X$, if it satisfies

(i-vBCKI^n 1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x), \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ and there exists a fixed $n \in N$ such that

(i-vBCKI^n 2) $\tilde{\mu}_A(x \ast y^n) \geq \min\{\tilde{\mu}_A((x \ast y^{n+1}) \ast z), \tilde{\mu}_A(z)\}$

(i-vBCKI^n 3) $\tilde{\lambda}_A(x \ast y^n) \leq \max\{\tilde{\lambda}_A((x \ast y^{n+1}) \ast z), \tilde{\lambda}_A(z)\}$ for all $x, y, z \in X$.

Theorem 3.2
Every interval-valued intuitionistic fuzzy n-fold BCK-ideal of $X$ is an interval-valued intuitionistic fuzzy deal of $X$.

Proof:
Put $y = 0$ in (i-vBCKI^n 2) and (i-vBCKI^n 3) we get the proof of the result.

The following example shows that the converse of theorem 3.2 may not be true.

Example 3.3
Let $X = N \cup \{0\}$, where $N$ is the set of natural numbers, in which the operation $\ast$ is defined by $x \ast y = \max\{0, x - y\}$ for all $x, y \in X$. Then $X$ is a BCK-algebra.

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in $X$ given by

$$\tilde{\mu}_A(0) = [0.7, 0.8] > [0.3, 0.4] = \tilde{\mu}_A(x)$$

and $\tilde{\lambda}_A(0) = [0.1, 0.2] < [0.3, 0.4] = \tilde{\lambda}_A(x)$ for all $x \neq 0 \in X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy ideal of $X$ but it is not an interval-valued intuitionistic fuzzy 2-fold BCK-ideal of $X$, because $\tilde{\mu}_A(5 \ast 2) = \tilde{\mu}_A(1) = [0.3, 0.4] < [0.7, 0.8] = \tilde{\mu}_A(0) = \min\{\tilde{\mu}_A((5 \ast 2^3) \ast 0), \tilde{\mu}_A(0)\}$ and $\tilde{\lambda}_A(5 \ast 2) = \tilde{\lambda}_A(1) = [0.4, 0.45] > [0.1, 0.2] = \tilde{\lambda}_A(0) = \max\{\tilde{\lambda}_A((5 \ast 2^3) \ast 0), \tilde{\lambda}_A(0)\}$.

We give a condition for an interval-valued intuitionistic fuzzy ideal to be an intuitionistic fuzzy n-fold BCK-ideal

Proposition 3.4
Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of $X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of $X$ if and only if it satisfies the following inequalities

$$\tilde{\mu}_A(x \ast y^n) \geq \tilde{\mu}_A(x \ast y^{n+1})$$

and

$$\tilde{\lambda}_A(x \ast y^n) \leq \tilde{\lambda}_A(x \ast y^{n+1})$$

for all $x, y \in X$.

Proof:
Put $z = 0$ in (i-v BCKI^n 2) and (i-v BCKI^n 3), we get

$$\tilde{\mu}_A(x \ast y^n) \geq \min\{\tilde{\mu}_A((x \ast y^{n+1}) \ast 0), \tilde{\mu}_A(0)\} = \min\{\tilde{\mu}_A(x \ast y^{n+1}), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(x \ast y^{n+1})$$

and

$$\tilde{\lambda}_A(x \ast y^n) \leq \max\{\tilde{\lambda}_A((x \ast y^{n+1}) \ast 0), \tilde{\lambda}_A(0)\} = \max\{\tilde{\lambda}_A(x \ast y^{n+1}), \tilde{\lambda}_A(0)\} = \tilde{\lambda}_A(x \ast y^{n+1})$$

Therefore, $\tilde{\mu}_A(x \ast y^n) \geq \tilde{\mu}_A(x \ast y^{n+1})$ and $\tilde{\lambda}_A(x \ast y^n) \leq \tilde{\lambda}_A(x \ast y^{n+1})$ for all $x, y \in X$.

Conversely, (i-vIF2) and (i-vIF3) we get

$$\tilde{\mu}_A(x \ast y^n) \geq \tilde{\mu}_A(x \ast y^{n+1}) \geq \min\{\tilde{\mu}_A((x \ast y^{n+1}) \ast z), \tilde{\mu}_A(z)\}$$

and
\[ \tilde{\lambda}_A(x \ast y^n) \leq \tilde{\lambda}_A(x \ast y^{n+1}) \leq \max\{\tilde{\lambda}_A((x \ast y^{n+1}) \ast z), \tilde{\lambda}_A(z)\} \text{ for all } x, y, z \in X. \]

Thus \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of \( X \).

**Corollary 3.5**

Every interval-valued intuitionistic fuzzy n-fold BCK-ideal \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) of \( X \) satisfies the inequalities \( \tilde{\mu}_A(x \ast y^n) \geq \tilde{\mu}_A(x \ast y^{n+k}) \) and
\[ \tilde{\lambda}_A(x \ast y^n) \leq \tilde{\lambda}_A(x \ast y^{n+k}) \text{ for all } x, y \in X \text{ and } k \in \mathbb{N}. \]

**Proof:**

Using the proposition 3.4, the proof is straightforward by Induction.

**Theorem 3.6.**

An interval-valued intuitionistic fuzzy set \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) in \( X \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of \( X \) if and only if the non-empty upper \( \tilde{s} \)-level cut \( U(\tilde{\mu}_A; \tilde{s}) \) and the non-empty lower \( \tilde{t} \)-level cut \( L(\tilde{\lambda}_A; \tilde{t}) \) are n-fold BCK-ideals of \( X \) for any \( \tilde{s}, \tilde{t} \in \mathbb{D}[0,1] \).

**Proof:** The proof is straightforward.

### 4 Interval-valued intuitionistic fuzzy n-fold positive implicative ideals of BCK-algebras:

**Definition 4.1**

An i-v IFS \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) in \( X \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal (i-vIFPI\(_n\)-ideal) of \( X \) if it satisfies

(i-vIFPI\(_n\)1) \( \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x), \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x) \) and there exists a fixed \( n \in \mathbb{N} \) such that

(i-vIFPI\(_n\)2) \( \tilde{\mu}_A(x \ast z^n) \geq \min\{\tilde{\mu}_A((x \ast y) \ast z^n), \tilde{\mu}_A(y \ast z^n)\} \)

(i-vIFPI\(_n\)3) \( \tilde{\lambda}_A(x \ast z^n) \leq \max\{\tilde{\lambda}_A((x \ast y) \ast z^n), \tilde{\lambda}_A(y \ast z^n)\} \text{ for all } x, y, z \in X. \)

**Example 4.2**

Let \( X = \{0,1,2\} \) be a BCK-algebra with the following Cayley table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Define an i-v IFS \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) in \( X \) by

\( \tilde{\mu}_A(0) = [0.75, 0.8], \tilde{\mu}_A(1) = [0.6, 0.7], \tilde{\mu}_A(2) = [0.2, 0.3] \) and

\( \tilde{\lambda}_A(0) = [0.1, 0.15], \tilde{\lambda}_A(1) = [0.2, 0.25], \tilde{\lambda}_A(2) = [0.4, 0.45] \)

Then \( \tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A) \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of \( X \) for all \( n \in \mathbb{N} \).
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Theorem 4.3
Every interval-valued intuitionistic fuzzy n-fold positive implicative ideal X must be interval-valued intuitionistic fuzzy ideal of X.

Proof:
Put \( z = 0 \) in (i-vIFPI \(^2\)) and (i-vIFPI \(^3\)), we get the proof of the result.

The following example shows that the converse of the Theorem 4.3 not true in general.

Example 4.4
Let \( A = (\hat{\mu}_A, \hat{\lambda}_A) \) be an i-v IFS in X given by
\[
\mu_A(13*5^2) = \mu_A(3) = [0.3,0.4] < [0.7,0.8] = \mu_A(0)
\]

Then \( A \) is an interval-valued intuitionistic fuzzy ideal of X but \( A \) is not an interval-valued intuitionistic fuzzy 2-fold Positive implicative ideal of X, because
\[
\mu_A((x*z) (y*z)) \geq \mu_A((x*y) z)\]

\[
\lambda_A((x*z) (y*z)) \leq \lambda_A((x*y) z)
\]

Theorem 4.5
Let \( \hat{A} = (\hat{\mu}_A, \hat{\lambda}_A) \) be an interval-valued intuitionistic fuzzy ideal of X. Then \( \hat{A} \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of X if and only if it satisfies the inequalities
\[
\mu_A((x z^n) (y z^n)) \geq \mu_A((x y) z^n)\]

\[
\lambda_A((x z^n) (y z^n)) \leq \lambda_A((x y) z^n)
\]

for all \( x, y, z \in X \).
and so $\tilde{\lambda}_A((a*b)*z^n) = \tilde{\lambda}_A(0)$.

Using (P3), (i-vIFPI²) and (i-vIFPI³) we obtain

\[
\tilde{\mu}_A((x*z^n)*(y*z^n)) = \tilde{\mu}_A((x*(y*z^n))*z^n) = \tilde{\mu}_A(a*z^n) \\
\geq \min\{\tilde{\mu}_A((a*b)*z^n),\tilde{\mu}_A(b*z^n)\} \\
= \min\{\tilde{\mu}_A(0),\tilde{\mu}_A(b*z^n)\} \\
= \tilde{\mu}_A(b*z^n) = \tilde{\mu}_A((x*y)*z^n) \text{and}
\]

\[
\tilde{\lambda}_A((x*z^n)*(y*z^n)) = \tilde{\lambda}_A((x*(y*z^n))*z^n) = \tilde{\lambda}_A(a*z^n) \\
\leq \max\{\tilde{\lambda}_A((a*b)*z^n),\tilde{\lambda}_A(b*z^n)\} \\
= \max\{\tilde{\lambda}_A(0),\tilde{\lambda}_A(b*z^n)\} \\
= \tilde{\lambda}_A(b*z^n) = \tilde{\lambda}_A((x*y)*z^n)
\]

Thus $\tilde{\mu}_A((x*z^n)*(y*z^n)) \geq \tilde{\mu}_A((x*y)*z^n)$ and

\[
\tilde{\lambda}_A((x*z^n)*(y*z^n)) \leq \tilde{\lambda}_A((x*y)*z^n) \text{ for all } x,y \in X.
\]

Covertly, For any $x,y,z \in X$. Using (i-vIF-2) and (i-vIF-3), we obtain

\[
\tilde{\mu}_A(x*z^n) \geq \min\{\tilde{\mu}_A((x*z^n)*(y*z^n)),\tilde{\mu}_A(y*z^n)\} \geq \min\{\tilde{\mu}_A((x*y)*z^n),\tilde{\mu}_A(y*z^n)\} \text{ and}
\]

\[
\tilde{\lambda}_A(x*z^n) \leq \max\{\tilde{\lambda}_A((x*z^n)*(y*z^n)),\tilde{\lambda}_A(y*z^n)\} \leq \max\{\tilde{\lambda}_A((x*y)*z^n),\tilde{\lambda}_A(y*z^n)\} \text{ for all } x,y,z \in X.
\]

Thus $\tilde{A} = (\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionist fuzzy n-fold positive implicative ideal of $X$.

**Proposition 4.6**

Let $\tilde{A} = (\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of $X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$ then it satisfies the inequalities

\[
\tilde{\mu}_A(x*y^n) \geq \tilde{\mu}_A((x*y)*y^n) \text{ and } \tilde{\lambda}_A(x*y^n) \leq \tilde{\lambda}_A((x*y)*y^n) \text{ for all } x,y \in X.
\]

**Proof:**

Put $z = y$ in (i-vIFPI²) and (i-vIFPI³), we get the proof of the result.

**Proposition 4.7**

Let $\tilde{A} = (\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionist fuzzy set of $X$. Then $\tilde{A}$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$ if and only if it is an intuitionistic fuzzy n-fold BCK-ideal of $X$.

**Proof:**

Putting $z = y$ in (i-vIFPI²) and (i-vIFPI³), we get

\[
\tilde{\mu}_A(x*y^n) \geq \min\{\tilde{\mu}_A((x*y)*y^n),\tilde{\mu}_A(y*y^n)\} = \min\{\tilde{\mu}_A(x*y^{n+1}),\tilde{\mu}_A(0)\}
\]

\[
= \tilde{\mu}_A(x*y^{n+1}) \text{ and}
\]

\]
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\[ \lambda_A(x \ast y^n) \leq \max \{\lambda_A((x \ast y) \ast y^n)\}, \lambda_A(y \ast y^n) = \max \{\mu_A(x \ast y^{n+1}), \lambda_A(0)\} \]

Therefore, \( \mu_A(x \ast y^n) \geq \mu_A(x \ast y^{n+1}) \) and \( \lambda_A(x \ast y^n) \leq \lambda_A(x \ast y^{n+1}) \) for all \( x, y \in X \).

By proposition 3.4, \( \tilde{A} \) is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of \( X \).

Conversely, it follows from (P3) and (P4) that

\[ \mu_A((x \ast z^n \ast (y \ast z^n)) = \lambda_A(((x \ast z^n) \ast (y \ast z^n))) \leq \lambda_A(((x \ast z^n) \ast (y \ast z^n))) \]

Using Corollary 3.5, (i-vIF2) and (i-vIF3) we get

\[ \mu_A((x \ast z^n) \geq \mu_A(x \ast z^{2n}) \geq \min \{\mu_A((x \ast z^n) \ast (y \ast z^n)), \mu_A(y \ast z^n)\} \]

\[ \lambda_A(x \ast z^n) \leq \lambda_A(x \ast z^{2n}) \leq \max \{\lambda_A((x \ast z^n) \ast (y \ast z^n)), \lambda_A(y \ast z^n)\} \]

Thus \( \tilde{A} \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of \( X \).

**Theorem 4.9**

Let \( \tilde{A} = (X, \mu_A, \lambda_A) \) be an i-v IF set of \( X \), then the following conditions are equivalent.

(i) \( \tilde{A} \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of \( X \).

(ii) The non-empty sets \( \bigcup (\mu_A; [s_1, s_2]) \) and \( L(\lambda_A; [t_1, t_2]) \) are interval valued n-fold positive implicative ideals of \( X \), for all \( [s_1, s_2], [t_1, t_2] \in D[0,1] \)

**Proof:**

The proof is straight forward.

**Theorem 4.10**

Let \( \tilde{A} = (\mu_A, \lambda_A) \) be an interval-valued intuitionistic fuzzy ideal of \( X \), then the following conditions are equivalent:

(i) \( \tilde{A} = (\mu_A, \lambda_A) \) is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal.

(ii) \( \tilde{A} = (\mu_A, \lambda_A) \) is an interval-valued intuitionistic fuzzy n-fold BCK-ideal of \( X \).

(iii) \( \mu_A(x \ast y^n) \geq \mu_A(x \ast y^{n+1}) \) and \( \lambda_A(x \ast y^n) \leq \lambda_A(x \ast y^{n+1}) \) for all \( x, y \in X \).

(iv) \( \mu_A((x \ast z^n) \ast (y \ast z^n)) \geq \mu_A((x \ast y) \ast z^n) \) and \( \lambda_A((x \ast z^n) \ast (y \ast z^n)) \leq \lambda_A((x \ast y) \ast z^n) \) for all \( x, y, z \in X \).

(v) \( U(\mu_A; [s_1, s_2]) \) and \( L(\lambda_A; [t_1, t_2]) \) are n-fold positive implicative ideals of \( X \) for all \( [s_1, s_2], [t_1, t_2] \in D[0,1] \).
Proof:
The proof is follows from the Proposition 4.7, Proposition 3.4, Theorem 4.5 and Theorem 4.9

**Theorem 4.11**

If $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy $n$-fold positive implicative ideal of $X$ then

(i) for $x,y,a,b \in X$, $((x*y)*y^n) \leq a \implies \mu_A (x*y)^n \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\}$ and
\[ \tilde{\lambda}_A (x*y^n) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\} \]

(ii) for $x,y,a,b \in X$, $((x*y*z^n) \leq a \implies \mu_A ((x*z^n)*(y*z^n)) \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\}$ and
\[ \tilde{\lambda}_A ((x*z^n)*(y*z^n)) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\} \]

**Proof:**

(i) Let $x,y,z \in X$ be such that $((x*y)*y^n) \leq a \implies b$. Using Theorem 2.3, We have $\tilde{\mu}_A ((x*y)*y^n) \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\}$ and $\tilde{\lambda}_A ((x*y)^n) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\}$.

Put $z = y$ in (i-vIFPI 2) and (i-vIFPI 3) we get
\[ \tilde{\mu}_A ((x*y)^n) \geq \min \{\tilde{\mu}_A (x*y)^n, \tilde{\mu}_A (y*y^n)\} = \min \{\tilde{\mu}_A ((x*y)*y^n), \tilde{\mu}_A (0)\} \]
\[ = \tilde{\mu}_A ((x*y)*y^n) \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\} \]
and
\[ \tilde{\lambda}_A ((x*y)^n) \leq \max \{\tilde{\lambda}_A (x*y)^n, \tilde{\lambda}_A (y*y^n)\} = \max \{\tilde{\lambda}_A ((x*y)*y^n), \tilde{\lambda}_A (0)\} \]
\[ = \tilde{\lambda}_A ((x*y)*y^n) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\} \]

Therefore, $\tilde{\mu}_A ((x*y)^n) \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\}$ and $\tilde{\lambda}_A ((x*y)^n) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\}$

(ii) Let $x,y,z \in X$ be such that $((x*y)*z^n) \leq a \implies b$. It follows from Theorems 4.5, Theorems 4.6 and By (i).

We obtain $\tilde{\mu}_A ((x*z^n)*(y*z^n)) \geq \tilde{\mu}_A ((x*y)*z^n) \geq \min \{\tilde{\mu}_A (a), \tilde{\mu}_A (b)\}$ and
\[ \tilde{\lambda}_A ((x*z^n)*(y*z^n)) \leq \tilde{\lambda}_A ((x*y)*z^n) \leq \max \{\tilde{\lambda}_A (a), \tilde{\lambda}_A (b)\} \]

This completes the proof.

**Theorem 4.16.**

(Extension property for an interval-valued intuitionistic fuzzy $n$-fold positive implicative ideal). Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ and $\tilde{B} = (\tilde{\mu}_B, \tilde{\lambda}_B)$ be an interval-valued intuitionistic fuzzy ideals of $X$ such that $\tilde{\lambda}(0) = \tilde{\lambda}(0)$ and $\tilde{\lambda}(0) \leq \tilde{\lambda}(0)$, that is, $\mu_A (0) = \mu_B (0)$, $\lambda_A (0) = \lambda_B (0)$ and $\mu_A (x) \leq \mu_B (x)$, $\lambda_A (x) \geq \lambda_B (x)$ for all $x \in X$. If $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy $n$-fold positive implicative ideal of $X$ then so is $\tilde{B}$.

**Proof:**

It is sufficient to show that $B = (\mu_B, \lambda_B)$ satisfies the inequalities $\mu_B ((x*y)^n) \geq \mu_B ((x*y)^{n+1})$ and $\lambda_B ((x*y)^n) \leq \lambda_B ((x*y)^{n+1})$ for all $x,y \in X$. 
Characterizations of Interval-valued intuitionistic fuzzy n-fold positive implicative ideal of BCK-algebras

Let $x, y \in X$. Using (BCK-3), (P3) and Proposition 3.4, we get

\[
\tilde{\mu}_B(0) = \tilde{\mu}_A(0) = \tilde{\mu}_A((x \ast (x \cdot y^{n+1})) \ast y^n) \\
= \tilde{\mu}_A((x \cdot y^n) \ast (x \cdot y^{n+1})) \\
\leq \tilde{\mu}_B((x \cdot y^n) \ast (x \cdot y^{n+1}) \text{ and } \tilde{\lambda}_B(0) = \tilde{\lambda}_A(0) = \tilde{\lambda}_A((x \ast (x \cdot y^{n+1})) \ast y^n) \\
\geq \tilde{\lambda}_A((x \ast (x \cdot y^{n+1})) \ast y^n) \\
= \tilde{\lambda}_A((x \cdot y^n) \ast (x \cdot y^{n+1})) \\
\geq \tilde{\lambda}_B((x \cdot y^n) \ast (x \cdot y^{n+1})).
\]

Therefore, $\tilde{\mu}_B(0) \leq \tilde{\mu}_B((x \cdot y^n) \ast (x \cdot y^{n+1})$ and $\tilde{\lambda}_B(0) \geq \tilde{\lambda}_B((x \cdot y^n) \ast (x \cdot y^{n+1})$ for all $x, y \in X$. It follows from (i-vIF1), (i-vIF2) and (i-vIF3) that

\[
\tilde{\mu}_B(x \cdot y^n) \geq \min\{\tilde{\mu}_B((x \cdot y^n) \ast (x \cdot y^{n+1})), \tilde{\mu}_A(x \cdot y^{n+1})\} \\
\geq \min(\tilde{\mu}_B(0), \tilde{\mu}_A(x \cdot y^{n+1})) = \tilde{\mu}_B(x \cdot y^{n+1})
\]

And $\tilde{\lambda}_B(x \cdot y^n) \leq \max\{\tilde{\lambda}_B((x \cdot y^n) \ast (x \cdot y^{n+1})), \tilde{\lambda}_A(x \cdot y^{n+1})\}$

\[
\leq \max(\tilde{\lambda}_B(0), \tilde{\lambda}_A(x \cdot y^{n+1})) = \tilde{\lambda}_B(x \cdot y^{n+1}).
\]

Therefore, $\tilde{\mu}_B(x \cdot y^n) \geq \tilde{\mu}_B(x \cdot y^{n+1})$ and $\tilde{\lambda}_B(x \cdot y^n) \leq \tilde{\lambda}_B(x \cdot y^{n+1})$ for all $x, y \in X$.

Thus $\tilde{B} = (\tilde{\mu}_B, \tilde{\lambda}_B)$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$.

5. Conclusion:

Every interval-valued intuitionistic fuzzy n-fold BCK-ideal and interval-valued intuitionistic fuzzy n-fold positive implicative BCK-ideal of $X$ is an interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$. An interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$ satisfies the inequalities

\[
\tilde{\mu}_A((x \ast z^n) \ast (y \ast z^n)) \geq \tilde{\mu}_A((x \ast y) \ast z^n) \text{ and } \\
\tilde{\lambda}_A((x \ast z^n) \ast (y \ast z^n)) \leq \tilde{\lambda}_A((x \ast y) \ast z^n) \text{ for all } x, y, z \in X.
\]

Interval-valued intuitionistic fuzzy n-fold positive implicative ideal of $X$ if and only if it is an intuitionistic fuzzy n-fold BCK-ideal of $X$. Characterizations of these notions and extension property of interval-valued intuitionistic fuzzy n-fold positive implicative ideals are proved.
References: