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Global Properties of the Solution of the Einstein-Maxwell-Boltzmann-Scalar Field System with Pseudo-Tensor of Pressure on a Bianchi Type I Space-Time

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Abstract. In this paper, we study the asymptotic behaviour, the geodesic completeness and the energy condition for the coupled Einstein-Maxwell-Boltzmann-Massive scalar field system with pseudo-tensor of pressure in the sources which rules the dynamincs of kind of charged pure matter in the presence of a massive scalar field which is a continuity of the work we have done. We then prove that, at late times, the Hubble variable of the space time goes to strictly positive limit.

Keywords: *pseudo-tensor; cosmological constant; Einstein system; Maxwell system; global existence; regular; asymptotic behaviour; geodesic completeness; positivity condition.*

1 INTRODUCTION

Global dynamics of various kinds of particles remain and active research area in the relativistic kinetic theory. After establishing global existence of the solution for the Einstein equation coupled to various field equations, one of the main problems is the properties of these solutions. This has been done in [11] by E. Takou in the case of Einstein-Boltzmann with Robertson-Walker space-time background; in [12] by H. Lee who studied asymptotic behaviour of the Einstein-Vlasov system with positive cosmological constant; in [4] by N. Noutchegueme and A. Nangue who studied the case of coupled Einstein-Maxwell-Massive scalar field with cosmological constant, but they did not consider the Boltzmann equation which introduce in the sources term of the Einstein equation the tensor $\widehat{T}^1_{\alpha\beta} = \int_{\mathbb{R}^3} \frac{f(t,\overline{p})\,\widehat{p}_\alpha\,\widehat{p}_\beta|\widehat{g}|^{\frac{1}{2}}}{p^o} d\bar{p}$; A. D. Rendall also studied in [13] the global properties of locally spatially homogenous cosmological models with matter.

In this paper, we study the asymptotics behaviour of the collisional evolution of a kind of charged massive particles with the same rest mass, under the influence of their commun gravitational field, represented by the metric tensor which is subjet to the Einstein equation, their self created electromagnetic forces subjet to the Maxwell equation. We also take into account the massive scalar field, which is the mathematical tool to explain the phenomenon of gravitation waves which propagate in the space at the speed of the light, just as the electromagnetic waves (even in the presence of material bodies). For more details on this question, see [1] and [4]. We endelly take into account the pseudo-tensor of pressure due to A. Lichnerowicz. The phenomenon is governed by the coupled Einstein-Maxwell-Boltzmann-Massive scalar field. The Boltzmann equation generalizes the Vlasov equation, in the sense that it takes into account the interaction between the particles. The interactions are defined by a non-linear operator Q called the

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"collision operator". In the binary and elastic scheme, due to Lichnerowicz and Chernikov [2] we adopt, at a given position, only two particles collide in an instanta neous shock, without destroying each other. The collision only affects their momenta, which are not longer the same, before and after the shock, only their sum being preserved. We consider the Einstein equation with a cosmological constant Λ, which is a classical mathematical tool to explain the accelerated phenomenon of the expansion of the Universe, as revealed by today's cosmological observations. For more details on the question, see [3] .

The purpose of this paper is to study the asymptotic behaviour which reveals an exponential growth of the gravitational potentials and we show that, at late times the Hubble variable of space-time goes to strictly positive limit, both results confirmining the accelerated expansion of the Universe, represented here by Bianchi space-time type 1. We also establih the geodesic completeness and we were able to prove that the considered model satisfies the weak and dominant energy conditions. We proved that if $\Lambda \geq \frac{9}{2}(H(0))^2$, where $H = -\frac{1}{3}g^{ij}k_{ij}$ is the Hubble variable, then the strong energy condition is also satisfied.

The paper is organized as follows:

• In section 2, we introduce the coupled system and all the transformations leading to the equivalent first order differential system on which we make our investigation.

• In section 3, we state the preliminary results and the global existence theorem proved in [**9**].

• In section 4, we study the asymptotic behaviour of the global solution at late times.

• In section 5, we study the geodesic completeness.

• In section 6, we study the energy conditions.

2 THE COUPLED SYSTEM

In this section, we recall the tools used in [**9**] and we state the coupled system as follows:

• Unless otherwise specifed, Greek indices $\alpha, \beta, \gamma, ...$, range from 0 to 3 and Latin indices $i, j, k, ...$, from 1 to 3. We adopt the Einstein summation convention

$$a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha.$$

• we consider the time-oriented space-time $(M,\widehat{g})$, where M is a four-dimensional manifold and $\widehat{g}$ the metric tensor of Lorentzian signature $(-,+,+,+)$. We consider the collisional evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type I space-time with locally rotationnal symmetric (here after L.R.S), the metric then takes the form

$$\widehat{g} = -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2). \tag{1}$$

Where the metric potentials $a > 0, b > 0$ are two continuously differentiable unknown functions of time t alone and subjet to the Einstein equations.

• The Einstein-Maxwell-Boltzmann-Massive scalar field with cosmological constant and pseudo-tensor of pressure can be written as follows:

$$\widehat{R}_{\alpha\beta} - \frac{1}{2}\widehat{R}\widehat{g}_{\alpha\beta} + \wedge\widehat{g}_{\alpha\beta} = 8\pi\left(\widehat{T}^1_{\alpha\beta} + \widehat{\tau}_{\alpha\beta} + \widehat{T}^2_{\alpha\beta} + \widehat{H}_{\alpha\beta}\right) \tag{2}$$

$$\widehat{\nabla}_\alpha \widehat{F}^{\alpha\beta} = 4\pi\widehat{J}^\beta \tag{3}$$

$$\widehat{\nabla}_\alpha \widehat{F}_{\beta\gamma} + \widehat{\nabla}_\beta \widehat{F}_{\gamma\alpha} + \widehat{\nabla}_\gamma \widehat{F}_{\alpha\beta} = 0 \tag{4}$$

$$L_X f = Q(f, f)\ . \tag{5}$$

Where (2) is the Einstein system for the unknown metric tensor $\widehat{g} = (\widehat{g}_{\alpha\beta})$; $\widehat{R}_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor, $\widehat{R} = \widehat{g}^{\alpha\beta}\widehat{R}_{\alpha\beta}$ is the scalar curvature; $\left(\widehat{T}^1_{\alpha\beta}\right)$ is the energy-momentum tensor associated to f, $(\widehat{\tau}_{\alpha\beta})$ is the Maxwell tensor associated to the electromagnetic field $\widehat{F}$; $\left(\widehat{T}^2_{\alpha\beta}\right)$ is the tensor associated to a massive scalar field ϕ which is an unknown function of the time t; (3) and (4) are the two sets of Maxwell equations written

in covariant form, for the electomagnetic field $\widehat{F} = (\widehat{F}^{0i}, \widehat{F}_{ij})$ which is the unknown. $\widehat{F}$ is a closed antisymmetric 2-form depending only on the time t, $\widehat{F}^{0i}$ and $\widehat{F}_{ij}$ are respectively its electric and magnetic parts.

$\widehat{T}^1_{\alpha\beta}$, $\widehat{\tau}_{\alpha\beta}$ and $\widehat{T}^2_{\alpha\beta}$ are defined by:

$$\widehat{T}^1_{\alpha\beta} = \int_{\mathbb{R}^3} \frac{f(t,\overline{p})\,\widehat{p}_\alpha\,\widehat{p}_\beta\,|\widehat{g}|^{\frac{1}{2}}}{\widehat{p}^o} d\overline{p}, \tag{6}$$

$$\widehat{\tau}_{\alpha\beta} = -\frac{1}{4}\widehat{g}_{\alpha\beta}\widehat{F}^{\lambda\mu}\widehat{F}_{\lambda\mu} + \widehat{F}_{\alpha\lambda}\widehat{F}^{\lambda}_{\beta} \tag{6'}$$

$$\widehat{T}^2_{\alpha\beta} = \widehat{\nabla}_\alpha\phi\widehat{\nabla}_\beta\phi - \frac{1}{2}\widehat{g}_{\alpha\beta}\left[\widehat{\nabla}^\lambda\phi\widehat{\nabla}_\lambda\phi + m_0^2\phi^2\right] . \tag{7}$$

Where in (6),(6'), (7) as in (3) and (4), f is the distribution function which measures the probability of the presence of particles in the plasma, $\mathrm{m}_0 > 0$ is a given constant called the mass of a unknown scalar field ϕ. Notice that $\frac{1}{2}m_0^2\phi^2$ represents the potential associated to the scalar field ϕ and $\widehat{\nabla}$ stands for the covariant derivative. In (2), $\widehat{H}_{\alpha\beta}$ is the pseudo-tensor of pressure, due to A.Lichnerowicz ([**2**]) and is defined by:

$$\widehat{H}_{\alpha\beta} = -\widehat{\theta}_{\alpha\beta}, \text{ where } \widehat{\nabla}_\alpha\widehat{\theta}^{\alpha\beta} = -u^\beta, \quad \widehat{g}^{ij}\widehat{\theta}_{ij} = 0 \tag{8}$$

with, $u = (u^\beta) = (1,0,0,0)$ a unit future pointing time-like vector, tangent to the axis at any point. (5) is the Boltzmann equation, where L_X is the Lie derivative of f with respect to the vectors field $X(\widehat{F}) = \left(\widehat{p}^\alpha, \widehat{P}^\alpha(F)\right)$ and $Q(f,f)$ the collision operator (For more details on $Q(f,f)$, see ([**9**]).

The massive particles have a rest mass $m > 0$, normalized to the unity. We denote by $T(\mathbb{R}^4)$ the tangent bundle of $\mathbb{R}^4$ with coordinates $(x^\alpha, \widehat{p}^\alpha)$, where $\widehat{p} = (\widehat{p}^\beta) = (\widehat{p}^0, \widehat{\overline{p}}\ )$ stands for the momentum of each particle and $\widehat{\overline{p}}\ = (\widehat{p}^i)$, $i = 1,2,3$. In fact, the charged particles move on the mass hyperboloïd $P(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $P(\mathbb{R}^4):\ \widehat{g}_{\alpha\beta}\widehat{p}^\alpha\widehat{p}^\beta = -1$ or equivalently, using expression (1) of $\widehat{g}$.

$$\widehat{p}^0 = \sqrt{1 + a^2(p^1)^2 + b^2((p^2) + (p^3)^2)}. \tag{9}$$

The charged particles also create a current $\widehat{J} = (\widehat{J}^\beta)$ called the Maxwell current that we take in form

$$\widehat{J}^\beta = \int \frac{1}{\widehat{p}^0}\widehat{p}^\beta f\ ab^2 d\overline{p} - eu^\beta. \tag{10}$$

Where $e = e(t) \geq 0$ denotes the charge density of particles.

As was established in [**9**], The conservation laws:

$$\widehat{\nabla}_\alpha(\widehat{T}^{1,\alpha\beta} + \widehat{\tau}^{\alpha\beta} + \widehat{T}^{2,\alpha\beta} + \widehat{H}^{\alpha\beta}) = 0, \tag{11}$$

leads to the equation for the scalar field:

$$\dot{\phi}\left(\ddot{\phi} + 3H\dot{\phi} + m_0^2\phi\right) + 1 = 0, \tag{12}$$

where

$$H = -\frac{g^{ij}k_{ij}}{3}, \text{with } \ k_{ij} = -\frac{1}{2}\partial_t g_{ij}. \tag{13}$$

H is called the Hubble variable.

To study this non-linear second order equation in ϕ, we set

$$\psi = \frac{1}{2}\left(\dot{\phi}\right)^2, \tag{13'}$$

we choose to look for a non-decreasing and non constant scalar field ϕ, which means $\dot{\phi} > 0$;

$$\dot{\phi} = \sqrt{2\psi} \tag{14}$$

• Next, we set

$$\left\{\begin{array}{c} \widehat{F}^{0i}=E^i,\ \widehat{F}_{ij}=F_{ij},\ \widehat{T}^2_{00}=T^2_{00},\ \widehat{T}^2_{0i}=T^2_{0i}\ ,\ \widehat{T}^2_{ij}=T^2_{ij},\ \widehat{H}_{00}=H_{00},\ \widehat{H}_{0i}=H_{0i}, \\ \widehat{H}_{ij}=H_{ij},\ \ \widehat{T}^1_{00}=T^1_{00},\ \widehat{T}^1_{0i}=T^1_{0i}\ ,\ \widehat{T}^1_{ij}=T^1_{ij},\ \widehat{\tau}_{00}=\tau_{00},\ \widehat{\tau}_{0i}=\tau_{0i},\ \widehat{\tau}_{ij}=\tau_{ij} \\ \widehat{p}^0=p^0,\ \widehat{p}^i=p^i \end{array}\right.$$

As specified in [**9**] the components $\widehat{T}^1_{\alpha\beta}$, $\widehat{\tau}_{\alpha\beta}$, $\widehat{T}^2_{\alpha\beta}$, and $\widehat{H}_{\alpha\beta}$ defined by (6), (6'), (7) and (8) can clearly be written as follow:

$$T^1_{00} \ = \ \int_{\mathbb{R}^3} f(t,\overline{p})p^0\,|g|^{\frac{1}{2}}\,d\overline{p};\ T^1_{11}=\int_{\mathbb{R}^3}\frac{f(t,\overline{p})(g_{11})^2(p^1)^2\,|g|^{\frac{1}{2}}}{p^0}d\overline{p}$$

$$T^2_{00}=\frac{1}{2}(\dot{\phi})^2+\frac{1}{2}m_0^2\phi^2;\ \ T^2_{0i}=0;\ T^2_{ij}=\frac{1}{2}g_{ij}(2\psi-m_0^2\phi^2);\ \ i,j=1,2,3 \tag{16}$$

$$H_{00}=-\frac{C}{ab^2}+\int_{t_0}^{t_0+T}a(t)b(t)dt,\ C\in\mathbb{R}_-;\ \ H_{0i}=0;\ \ H_{ij}=0;\ \ i,j=1,2,3. \tag{17}$$

$$\left\{\begin{array}{c} \tau_{00}=\frac{1}{2}\left[(a_0E^1)^2(\frac{b_0}{b})^4+(b_0E^2)^2(\frac{b_0^2}{ab})^2+(a_0E^3)^2(\frac{b_0^2}{ab})^2\right] \\ +\left[(\frac{\varphi_{12}}{ab})^2+(\frac{\varphi_{13}}{ab})^2+(\frac{\varphi_{23}}{b^2})^2\right] \\ \tau_{11}=\frac{1}{2}\left[-(aa_0E^1)^2(\frac{b_0}{b})^4+(a_0^2b_0^2E^2)^2(\frac{b_0}{b})^2+(b_0^2E^3)^2(\frac{b_0}{b})^2\right] \\ +\frac{1}{2}\left[(\frac{\varphi_{12}}{b})^2+(\frac{\varphi_{13}}{b})^2-(\frac{a\ \varphi_{23}}{b^2})^2\right] \\ \tau_{22}=\frac{1}{2}\left[(a_0b_0E^1)^2(\frac{b_0}{b})^2-(b_0^2E^2)^2(\frac{a_0}{a})^2+(b_0^2E^3)^2(\frac{a_0}{a})^2\right] \\ +\frac{1}{2}\left[(\frac{\varphi_{12}}{a})^2-(\frac{\varphi_{13}}{a})^2+(\frac{a\ \varphi_{23}}{b})^2\right] \\ \tau_{33}=\frac{1}{2}\left[(a_0b_0E^1)^2(\frac{b_0}{b})^2+(b_0^2E^2)^2(\frac{a_0}{a})^2-(b_0^2E^3)^2(\frac{a_0}{a})^2\right] \\ +\frac{1}{2}\left[-(\frac{\varphi_{12}}{a})^2+(\frac{\varphi_{13}}{a})^2+(\frac{\varphi_{23}}{b})^2\right]. \\ \tau_{0i}=-(\frac{a_0}{a})(\frac{b_0}{b})^2E^j\varphi_{ij},\qquad \tau_{12}=\frac{1}{b^2}(-a_0^2b_0^2E^1E^2+\varphi_{13}\varphi_{23}) \\ \tau_{13}=\frac{-1}{b^2}(a_0^2b_0^4E^1E^3+\varphi_{12}\varphi_{23}),\ \tau_{23}=\frac{1}{a^2}(-a_0^2b_0^4E^2E^3+\varphi_{12}\varphi_{13}) \end{array}\right. \tag{18}$$

and we have the inequalities,

$$0\leq\frac{1}{2}g_{ij}E^iE^j\leq\tau_{00};\qquad 0\leq\frac{1}{4}F^{ij}F_{ij}\leq\tau_{00} \tag{19}$$

Because from (6') and (18) we have, $\tau_{00}=\frac{1}{4}F^{ij}F_{ij}+\frac{1}{2}g_{ij}E^iE^j\geq 0$.

• Using the change of variables as in [**9**], The Einstein-Maxwell-Boltzmann-Scalar field system is equivalent to the following system (20)-(28), which is a first order differential system:

$$\frac{dH}{dt}=-\frac{3}{2}(1+\Sigma_+^2)H^2-\frac{P_1+2P_2}{6}\ +\frac{\Lambda}{2} \tag{20}$$

$$\frac{ds}{dt}=6s(1-s)\Sigma_+H \tag{21}$$

$$\frac{dz}{dt}=2z(1-z)(1+\Sigma_+-3s\Sigma_+)H \tag{22}$$

$$\frac{d\Sigma_+}{dt}=-\frac{3}{2}(1-\Sigma_+^2)H\Sigma_++\frac{P_1}{6H}(\Sigma_+-2)+\frac{P_2}{3H}(\Sigma_++1)\ -\frac{\Lambda\Sigma_+}{2H}. \tag{23}$$

$$\frac{d\phi}{dt}=\sqrt{2\psi} \tag{24}$$

$$\frac{d\psi}{dt} = -6H\psi - m_0^2\phi\sqrt{2\psi} \ - 1 \tag{25}$$

$$\frac{df}{dt} = \frac{1}{p^o}\, Q(f, f, \overline{p}) \tag{26}$$

$$\frac{dp^i}{dt} = -2\ \Gamma^i_{0j}p^j + (-a_0 b_0^2 E^i + \frac{ab^2 g^{ij}p^k\varphi_{ki}}{p^0})\int_{\mathbb{R}^3} f(t, \overline{p}\ )d\overline{p} \tag{27}$$

$$F^{0i} = \frac{a_0 b_0^2 E^i}{ab^2}, \qquad F_{ij} = \varphi_{ij}\ , \quad i, j = 1, 2, 3 \tag{28}$$

Where:

$$H = -\frac{trk}{3}, \quad z = \frac{a^2b^2}{2a^2 + b^2 + a^2b^2}\ , \ s = \frac{b^2}{a^2 + 2b^2}, \ \ \Sigma_+ = -\frac{3}{tr\,k}\frac{\dot{b}}{b} - 1, \ trk = -\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)$$

and

$$H_1^0 < H \leq H_0; \ 0 < s < 1; \quad 0 < z < 1; \quad -1 < \Sigma_+ < 1.$$

The equation (25) in ψ is given by (12) using the change of variable (13') which provides in the same time equation (24) in ϕ.

- Next, we also have the following set of four constraints equations from [**9**]:

$$R + (tr\,k)^2 = k_{ij}k^{ij} + 2\Lambda + 16\pi(T_{00}^1 + T_{00}^2 + H_{00} + \tau_{00}) \tag{29}$$

$$S_i^o + \Lambda g_i^0 = 8\pi(T_i^{1,0} + \tau_i^0 + T_i^{2,0} + H_i^0), \ S_{ij} + \Lambda g_{ij} = 8\pi\,(T_{ij}^1 + \tau_{ij} + T_{ij}^2 + H_{ij}). \tag{30}$$

$$E^i\varphi_{ij} = 0 \tag{31}$$

$$\sum_{k=1}^{3}\varphi_{ik}\varphi_{jk} - a_0^2 b_0^4 E^i E^j = 0, \ i \neq j \tag{32}$$

$$\varphi_{12}^2 - \varphi_{13}^2 - a_0^2 b_0^4\left[(E^2)^2 - (E^3)^2\right] = 0 \tag{33}$$

Where (31), (32) and (33) are given by the Maxwell's equations (3) and (4); whereas (29) called the Hamiltonian constraint, and (30) are given by the Einstein's equation (2). Recall that, from [5], we have

$$R = g^{ij}R_{ij} < 0. \tag{33'}$$

3 THE PRELIMINARY RESULTS

We recall the useful notion of relative norm staded in [**10**]. Define the norm of the $n \times n$ matrix A by

$$\|A\| = \sup\left\{\frac{\|Ax\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}}, \ x \neq 0, \ x \in \mathbb{R}^n\right\}.$$

and If A_1 and A_2 are two $n \times n$ symmetric matrices with A_1 positive definite, the norm of A_2 with respect to A_1 is:

$$\|A_2\|_{A_1} = \sup\left\{\frac{\|A_2x\|_{\mathbb{R}^n}}{\|A_1x\|_{\mathbb{R}^n}}, \ x \neq 0, \ x \in \mathbb{R}^n\right\}.$$

then we have :
$\|A_2\| \leq \|A_2\|_{A_1} \|A_1\|$ and $\|A_2\|_{A_1} \leq \left[tr(A_1^{-1}A_2A_1^{-1}A_2\right]^{\frac{1}{2}}$.
Now taking $A_1=(g_{ij})_{1\leq i,j\leq n}$, and $A_2=(a_{ij})$, we obtain:

$$\|A_2\|_{A_1} \leq (a_{ij}a^{ij})^{\frac{1}{2}} \tag{34}$$

So that we have

$$\|A_2\| \leq \|A_1\| (a_{ij}a^{ij})^{\frac{1}{2}} \tag{35}$$

Also recall the following result of [**10**]

$$\left|\frac{p^i}{p^0}\right| \leq C|g|^{\frac{3}{2}}, \quad \left|F^{0i}\right| \leq (g_{rs}F^{0r}F^{0s})^{\frac{1}{2}}|g|^{\frac{3}{2}}. \tag{36}$$

where $C>0$ is a constant.

In the following, we will denote by L_{ij} the traceless tensor associated to k_{ij} and defined by

$$L^{ij} = k_{ij} + g_{ij}H \tag{36'}$$

which gives the relation

$$k_{ij}k^{ij} = L_{ij}L^{ij} + 3H^2 \tag{36''}$$

Proposition 3.1
(a) The Hubble variable H satisfies the equation:

$$\frac{dH}{dt} = -\frac{1}{3}\left(R + (tr\,k)^2 + 4\pi g^{ij}(T^1_{ij}+T^2_{ij}) - 12\pi(T^1_{00}+T^2_{00}+H_{00}) - 8\pi\tau_{00} - 3\Lambda\right) \tag{37}$$

b) Denote by L_{ij} the traceless tensor assocated to k_{ij} , then

$$\frac{dH}{dt} = \frac{1}{3}\left(-3H^2 + \Lambda + 4\pi m_0^2\phi^2 - L_{ij}L^{ij} - 4\pi g^{ij}T^1_{ij} - 4\pi(T^1_{00}+\tau_{00}+T^2_{00}+H_{00}+2\tau_{00}) - 12\pi\psi\right)$$

(c) We then obtain:
i) The equality

$$-6H^2 + 2\Lambda + 8\pi m_0^2\phi^2 = -L_{ij}L^{ij} - 16\pi(T^1_{00}+\tau_{00}+H_{00}) + R - 16\pi\psi \tag{38}$$

ii) and the inequalities

$$\frac{dH}{dt} \leq \frac{1}{3}\left(-3H^2 + \Lambda + 4\pi m_0^2\phi^2\right) \tag{39}$$

$$-3H^2 + \Lambda + 4\pi m_0^2\phi^2 \leq 0 \tag{40}$$

$$\frac{dH}{dt} \leq 0 \tag{41}$$

Proof see [9].

Theorem 1 Let $\phi_0,\ \psi_0 \in \mathbb{R},\ \Lambda \geq 0,\ r>0,\ d>\frac{5}{2},\ a_0, b_0, \dot{a}_0$ *and* $\dot{b}_0 \in \mathbb{R},\ \overline{p}_0 \in \mathbb{R}^3,\ f_0 \in H^3_{d,r}(\mathbb{R}^3),\ F^{0i}(0) = E^i \in \mathbb{R},\ F_{ij}(0) = \varphi_{ij} \in \mathbb{R}$ *such that* $a_0, b_0, \dot{a}_0, \dot{b}_0,\ \phi_0,\ \psi_0$, $f_0,\ E^i,\ \varphi_{ij}$ *verify the constraints (29), (31), (32) and (33). Then:*

1. H *is uniformly bounded and we have*

$$\sqrt{\frac{\Lambda}{3}} \leq H(t) \leq H(0). \tag{42}$$

2- Differential system (20)$-(21)-(22)-(23)-(24)-(25)-(26)-(27)$ *has a unique global solution* $(H,s,z,\Sigma_+,\phi,\psi,f,\overline{p})$ *defined on* $[0,+\infty[$ *and verifying* $(H,s,z,\Sigma_+,\phi,\psi,f,\overline{p})(0) = (H_0,s_0,z_0,\Sigma_{+0},\phi_0,\psi_0,f_0,\overline{p}_0)$.

3- The coupled Einstein-Maxwell-Boltzmann-Massive scalar field system, in a locally rotationally symmetric Bianchi type 1 space-time, has a unique global regular solution $(a,b,F^{0i},F_{ij},f,\phi)$ *defined on* $[0,+\infty[$ *and verifying* $a(0)=a_0,\ b(0)=b_0,\ F^{0i} = \frac{a_0b_0^2}{ab^2}E^i,\ F^{0i}(0)=E^i,\ F_{ij}(0)=\varphi_{ij},\ \ f(0)=f_0$ *and* $\phi(0)=\phi_0$.

4 ASYMPTOTIC BEHAVIOUR

We consider the global solution over $[0, +\infty[$ and we study the asymptotic behaviour of the different elements at late times.

Theorem 2 At late times

$$H = \lambda + 0\left(e^{-2\lambda t}\right) \tag{43}$$

$$g_{ij} = \left(C_{ij} + 0\left(e^{-\lambda t}\right)\right) e^{2\lambda t} \tag{44}$$

$$g^{ij} = \left(C^{ij} + 0\left(e^{-\lambda t}\right)\right) e^{-2\lambda t} \tag{45}$$

$$L_{ij} L^{ij} = 0\left(e^{-2\lambda t}\right) \tag{46}$$

$$\tau_{00} = 0\left(e^{-2\lambda t}\right) \tag{47}$$

$$H_{00} = 0\left(e^{-2\lambda t}\right) \tag{48}$$

$$|R| = 0\left(e^{-2\lambda t}\right) \tag{49}$$

$$\left(\dot{\phi}\right)^2 = 0\left(e^{-2\lambda t}\right) \tag{50}$$

$$\psi = 0\left(e^{-2\lambda t}\right) \tag{51}$$

$$E^i E_i = 0\left(e^{-2\lambda t}\right) \tag{52}$$

$$F^{ij} F_{ij} = 0\left(e^{-2\lambda t}\right) \tag{53}$$

$$F^{0i} = 0\left(e^{-2\lambda t}\right) \tag{54}$$

$$L_{ij} = 0\left(e^{\lambda t}\right) \tag{55}$$

$$L^{ij} = 0\left(e^{-3\lambda t}\right) \tag{56}$$

Where C_{ij} and C^{ij} are positive definite matrices independent of t and

$$\lambda = (k_0)^{\frac{1}{2}}, \quad k_0 = \frac{4\pi m_0^2 \left(\phi(0)\right)^2 + \Lambda)}{3}. \tag{57}$$

Proof

Proof (43)

From inequality (38), we can write:

$$\frac{dH}{dt} \leq \frac{1}{3}\left(-3H^2 + \Lambda + 4\pi m_0^2 \phi^2\right) = \left(H - \sqrt{\frac{\Lambda + 4\pi m_0^2 \phi^2}{3}}\right)\left(-H - \sqrt{\frac{\Lambda + 4\pi m_0^2 \phi^2}{3}}\right) \tag{58}$$

from (40) and since ϕ is non decreasing, we have

$$-H \leq -\sqrt{\frac{\Lambda + 4\pi m_0^2 \phi^2}{3}} \leq -\sqrt{\frac{\Lambda + 4\pi m_0^2 (\phi(0))^2}{3}}$$

and (58) can be written

$$\frac{d\left(H-\sqrt{\frac{\Lambda+4\pi m_0^2(\phi(0))^2}{3}}\right)}{dt}+2\lambda\left(H-\sqrt{\frac{\Lambda+4\pi m_0^2(\phi(0))^2}{3}}\right)\leq 0. \tag{59}$$

With

$$\lambda=\frac{1}{3}\sqrt{3\Lambda+12\pi m_0^2(\phi(0))^2}.$$

Multiplying (59) by $e^{2\lambda t}$ and integrating over $[0,t]\,,\ t\geq 0$, we obtain:

$$e^{2\lambda t}\left(H-\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}}\right)\leq H(0)-\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}}.$$

From (40), we have

$$H\geq\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}},$$

which gives

$$0\leq H-\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}}\leq\left(H(0)-\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}}\right)e^{-2\lambda t}.$$

Finaly

$$H=\sqrt{\frac{\Lambda+4\pi m_0^2\phi^2(0)}{3}}+0\left(e^{-2\lambda t}\right).$$

This proves (43) with λ given by (57).

Proof of (44), (45)

The metric of De-Sitter can be written in cartesian coordinates, as

$$g_s=-dt^2+\alpha^2ch^2(\frac{t}{\alpha})\left[dx^2+dy^2+dz^2\right]. \tag{60}$$

(44) shows that the metric $\widehat{g}$ defined by (1) can be written in the form :

$$\widehat{g}=-dt^2+e^{2\lambda t}\left(C_{11}+0\left(e^{-2\lambda t}\right)\right)dx^2+e^{2\lambda t}\left(C_{22}+0\left(e^{-2\lambda t}\right)\right)\left[dy^2+dz^2\right]. \tag{61}$$

With C_{11}, C_{22} the positive constantes. But we have $ch^2(\frac{t}{\alpha})$ which gives :

$$\begin{aligned}ch^2(\frac{t}{\alpha}) &= \frac{1}{4}\left(e^{\frac{t}{\alpha}}+e^{-\frac{t}{\alpha}}\right)^2\\ &= e^{2\frac{t}{\alpha}}\left(\frac{1}{2}+\frac{e^{-2\frac{t}{\alpha}}}{2}\right)^2\\ &= e^{2\frac{t}{\alpha}}\left(\frac{1}{4}+\frac{e^{-2\frac{t}{\alpha}}}{2}+\frac{e^{-4\frac{t}{\alpha}}}{4}\right)\end{aligned}$$

which, for the large time t, gives:

$$ch^2(\frac{t}{\alpha})=e^{2\frac{t}{\alpha}}\left(\frac{1}{4}+0(e^{-\frac{t}{\alpha}})\right)$$

So, for the large time t, the metric of De-Sitter (60) takes the form:

$$g_s=-dt^2+\alpha^2e^{2\frac{t}{\alpha}}\left(\frac{1}{4}+0(e^{-\frac{t}{\alpha}})\right)\left[dx^2+dy^2+dz^2\right] \tag{62}$$

$$=-dt^2+e^{2\frac{t}{\alpha}}\left(\frac{1}{4}\alpha^2+0(e^{-\frac{t}{\alpha}})\right)dx^2+e^{2\frac{t}{\alpha}}\left(\frac{1}{4}\alpha^2+0(e^{-\frac{t}{\alpha}})\right)\left[dy^2+dz^2\right]$$

We can then conclude that our model (61) approch asymptotically the De-Sitter model if we take $C_{11} = C_{22} = \frac{1}{4}\alpha^2$ and $\lambda = \frac{1}{\alpha}$.

Finally, we have

$$g_{ij} = \left(C_{ij} + 0\left(e^{-\lambda t}\right)\right) e^{2\lambda t}$$

and (45) is a direct consequence of (44), since $g^{ij} = (g_{ij})^{-1}$.

Proof (46), (47), (48), (49), (50) and (51)

From equality (39), we have:

$$-6H^2 + 2\Lambda + 8\pi m_0^2\phi^2 = -L_{ij}L^{ij} - 16\pi(T_{00}^1 + \tau_{00} + H_{00}) + R - 16\pi\psi$$

since, from the definition (13'), (15), (16), (17) and (18), we have respectively:

$$\psi > 0;\ \ T_{00}^1 > 0;\ T_{00}^2 > 0;\ H_{00} > 0 \text{ and } \tau_{00} > 0;$$

and according to (36′) and (36"), we also have $L_{ij}L^{ij} > 0$; and finally acording to Jantzen R.T in ([5]), where we have $R \leq 0$, we obtain the following inequalities:

$$L_{ij}L^{ij} \leq 6H^2 - 2\Lambda - 8\pi m_0^2\phi_0^2,$$

$$\tau_{00} \leq 6H^2 - 2\Lambda - 8\pi m_0^2\phi_0^2,$$

$$H_{00} \leq 6H^2 - 2\Lambda - 8\pi m_0^2\phi_0^2,$$

$$16\pi\psi \leq 6H^2 - 2\Lambda - 8\pi m_0^2\phi_0^2,$$

$$-R \leq 6H^2 - 2\Lambda - 8\pi m_0^2\phi_0^2.$$

(Recall that we have used the fact that ϕ is non decreasing). Now, using the relation (43), one has:

$$L_{ij}L^{ij} = 0\left(e^{-2\lambda t}\right)$$

$$\tau_{00} = 0\left(e^{-2\lambda t}\right)$$

$$H_{00} = 0\left(e^{-2\lambda t}\right)$$

$$\psi = \frac{1}{2}\left(\dot{\phi}\right)^2 = 0\left(e^{-2\lambda t}\right)$$

$$|R| = 0\left(e^{-2\lambda t}\right).$$

and consequently the avowed relations (46), (47), (48), (49), (50) and (51).

Proof (52), (53)

From (19), we have

$$\begin{aligned} 0 &\leq \frac{1}{2}g_{ij}E^iE_j \leq \tau_{00}, \\ 0 &\leq \frac{1}{4}F^{ij}F_{ij} \leq \tau_{00}. \end{aligned}$$

and (52) and (53) are then consequences of (47).

v) Proof (54)

The second relation (36) gives, using (52) and (44),

$$\left|F^{0i}\right| \leq \left(g_{rs}F^{0r}F^{0s}\right)^{\frac{1}{2}} |g|^{\frac{3}{2}} \leq e^{-\lambda t}e^{3\lambda t}$$

which proves (54);

For the **Proof (55) and (56)** see LEE. Hayoung in [**6**]. ■

Remark 1

From relation (43), we have the fact that the Hubble variable H goes to a strictly positive limit at late time; which then shows that our Universe is in acccelerated expansion.

5 GEODESIC COMPLETENESS

In this section, it will be shown that, the global in time solution proved in theorem 1 is future complete; This means that all inextendible causal geodesics are complete in the future direction.

Theorem 3

Under conditions of theorem 1, the space-time which exists globally is future geodesically complete.

Proof

The geodesic equations for the metric (1) imply that along geodesics, the variables t, p^0, p^i satisfy a differential system which contains, between others, the equation:

$$\frac{dt}{ds} = p^0 \tag{63}$$

where s is an affine parameter.The space-time will be geodesically future complete if the affine parameter s tends to $+\infty$ as the time t tends to $+\infty$. Since $(\mathrm{p}^0)^2 = 1 + g_{ij}p^ip^j$, it will be enough if we prove that there exists a constant $K > 0$ such that:

$$\frac{ds}{dt} = \left(1 + g_{ij}p^ip^j\right)^{-\frac{1}{2}} \geq K > 0 \tag{64}$$

Since by integration (64), $s > Kt + K_0$, which shows that $s \to +\infty$ as $t \to +\infty$. Our goal is then to prove that

$$\left(1 + g_{ij}p^ip^j\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 + a^2\left(p^1\right)^2 + b^2\left(\left(p^2\right)^2 + \left(p^3\right)^2\right)}} \geq K > 0.$$

It should be enough to show that $g_{ij}p^ip^j$ is bounded. We have $p_i = g_{ij}p^j$, in order to evaluate the quantity $g^{ij}p_ip_j$, we use the usual formula:

$$\frac{d}{dt}(g^{ij}p_ip_j) = \frac{d}{dt}\left(g_{ij}p^ip^j\right) = p^ip^j\frac{d}{dt}\left(g_{ij}\right) + 2g_{ij}p^j\frac{d}{dt}p^i.$$

We can write this equation following (13) as

$$\frac{d}{dt}(g^{ij}p_ip_j) = -2k_{ij}p^ip^j + 2g_{ij}p^j\frac{d}{dt}p^i. \tag{65}$$

Now, using equation (27) in p^i, (taking account of the fact that $F_{kj}p^kp^j = 0$), we obtain:

$$\frac{d}{dt}(g^{ij}p_ip_j) = 2k_{ij}p^ip^j - 2a_0b_0^2E^ip_i. \tag{66}$$

Then, using the traceless tensor $L_{ij} = k_{ij} + Hg_{ij}$, which implies that

$k_{ij} = L_{ij} - Hg_{ij}$,(66) gives:

$$\frac{d}{dt}(g^{ij}p_ip_j) = -2Hg^{ij}p_ip_j + 2L^{ij}p_ip_j - 2a_0b_0^2E^ip_i. \tag{67}$$

Since by (52), $L^{ij} = 0(e^{-3\lambda t})$, then there exists $\mathrm{C}_1 \geq 0$, such that $e^{3\lambda t}L^{ij} \leq C_1$. The matrix C^{ij} of (44) being constant and positive definite, we have

$$e^{3\lambda t}L^{ij}p_ip_j \leq C_1C^{ij}p_ip_j \leq Ce^{2\lambda t}g^{ij}p_ip_j. \tag{68}$$

$$L^{ij}p_ip_j \leq Ce^{-\lambda t}g^{ij}p_ip_j. \tag{69}$$

Since g is a scalar product, we have

$$-2a_0b_0^2E^ip_i \leq C'\left(E^iE_i\right)^{\frac{1}{2}}\left(g^{ij}p_ip_j\right)^{\frac{1}{2}}$$

by (49), we can write

$$-2a_0b_0^2E^ip_i \leq Ce^{-\lambda t}\left(g^{ij}p_ip_j\right)^{\frac{1}{2}} \tag{70}$$

Using (43), we obtain,

$$-2Hg^{ij}p_ip_j = (-2\lambda + 0(e^{-\lambda t}))g^{ij}p_ip_j$$

which can be written in the form:

$$-2Hg^{ij}p_ip_j \leq (-2\lambda + C(e^{-\lambda t}))g^{ij}p_ip_j \tag{71}$$

So, by (68), (69), (70) and (71), we can write

$$\frac{d}{dt}(g^{ij}p_ip_j) \leq (-2\lambda + 0(e^{-\lambda t}))g^{ij}p_ip_j + Ce^{-\lambda t}g^{ij}p_ip_j + Ce^{-\lambda t}\left(g^{ij}p_ip_j\right)^{\frac{1}{2}}.$$

Which yields to

$$\frac{d}{dt}(g^{ij}p_ip_j) \leq (-2\lambda + C(e^{-\lambda t}))g^{ij}p_ip_j + Ce^{-\lambda t}\left(g^{ij}p_ip_j\right)^{\frac{1}{2}} \tag{72}$$

Let us set:

$$Y = e^{\lambda t}g^{ij}p_ip_j$$

We then have

$$\frac{dY}{dt} = \lambda Y + e^{\lambda t}\frac{d}{dt}(g^{ij}p_ip_j)$$

hence, from (72), we have:

$$\frac{dY}{dt} \leq \lambda Y + (-2\lambda + C(e^{-\lambda t}))Y + Ce^{-\frac{\lambda}{2}t}Y^{\frac{1}{2}} = -\lambda Y + C(e^{-\lambda t})Y + Ce^{-\frac{\lambda}{2}t}Y^{\frac{1}{2}}.$$

But $\lambda \geq 0$, from where we have:

$$\frac{dY}{dt} \leq C(e^{-\lambda t})Y + Ce^{-\frac{\lambda}{2}t}Y^{\frac{1}{2}}.$$

Which gives:

$$\frac{dY}{dt} \leq C(e^{-\frac{\lambda}{2}t})Y + Ce^{-\frac{\lambda}{2}t}Y^{\frac{1}{2}} \tag{73}$$

But

$$e^{-\frac{\lambda}{2}t}Y^{\frac{1}{2}} = e^{-\frac{\lambda}{4}t}\left(e^{-\frac{\lambda}{2}t}Y\right)^{\frac{1}{2}} \leq \frac{1}{2}\left(e^{-\frac{\lambda}{2}t} + e^{-\frac{\lambda}{2}t}Y\right).$$

We then deduced, from (73) that:

$$\frac{d\,(Y+1)}{dt} \leq e^{-\frac{\lambda}{2}t}\,(Y+1)\,C \tag{74}$$

(74) proves that , $Y = e^{\lambda t}g^{ij}p_ip_j$ is bounded, and then, $g^{ij}p_ip_j = g_{ij}p^ip^j$ is bounded. This ends the proof of theorem 3. ■

6 ENERGY CONDITIONS

In this part we prove that the global solution satisfies the weak energy condition, the dominant energy condition and under conditions some hypothesis, the strong energy condition. Recall that a viable physical theory is supposed to fulfill at least one of the energy conditions (Hawking). In fact notice that considering the stress-energy-matter tensor of the Einstein equation (2), and keeping the hat to avoid any confusion, the quantity $\left(\widehat{T}^1_{\alpha\beta} + \widehat{T}^2_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)\widehat{X}^\alpha\widehat{X}^\beta$, represents physically the energy density of the charged particles, measured by an observer whose velocity is $\widehat{X}^\alpha$ and so must be non negative, $\widehat{X}^\alpha$ being a futur pointing time-like vector, see [**7**].

We recall below the three types of energy conditions due to Hawking in [8]. Let $(\widehat{X}^{\alpha})$, $(\widehat{Y}^{\alpha})$ be any two future pointing time-like vectors, in our case .

1) The weak energy condition is fulfilled if

$$\left(\widehat{T}^{1}_{\alpha\beta}+\widehat{T}^{2}_{\alpha\beta}+\widehat{H}_{\alpha\beta}+\widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{X}^{\beta}\geq 0.$$

2) The dominante energy condition is fulfilled if

$$\left(\widehat{T}^{1}_{\alpha\beta}+\widehat{T}^{2}_{\alpha\beta}+\widehat{H}_{\alpha\beta}+\widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{Y}^{\beta}\geq 0.$$

3) The strong positivity condition is satisfied if

$$\widehat{R}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{X}^{\beta}\geq 0,$$

where $\widehat{R}_{\alpha\beta}$ is the Ricci tensor..

Lemma 1

Let $(\widehat{X}^{\alpha})$ and $(\widehat{Y}^{\alpha})$ be two future pointing time-like or null vectors. Then

$$\widehat{X}^{\alpha}\widehat{Y}_{\alpha}\leq 0. \tag{75}$$

Proof

Since $\widehat{X}^{\alpha}\widehat{Y}_{\alpha}=\widehat{g}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{Y}^{\alpha}$, and given the definition (1) of the metric $\widehat{g}$, (75) is equivalent to:

$$\widehat{X}^{\alpha}\widehat{Y}_{\alpha}=-\widehat{X}^{0}\widehat{Y}^{0}+g_{ij}\widehat{X}^{i}\widehat{Y}^{j}\leq 0 \tag{76}$$

now $(\widehat{X}^{\alpha})$, $(\widehat{Y}^{\alpha})$ being future pointing time-like or null vectors, we have $\widehat{X}^{0}\geq 0,\ \widehat{Y}^{0}\geq 0$ and $\widehat{X}^{\alpha}\widehat{X}_{\alpha}\leq 0,\ \widehat{Y}^{\alpha}\widehat{Y}_{\alpha}\leq 0$, or equivalently:

$0\leq g_{ij}\widehat{X}^{i}\widehat{X}^{j}\leq\left(\widehat{X}^{0}\right)^{2}$ and $0\leq g_{ij}\widehat{Y}^{i}\widehat{Y}^{j}\leq\left(\widehat{Y}^{0}\right)^{2}$,

hence $0\leq\left(g_{ij}\widehat{X}^{i}\widehat{X}^{j}\right)^{\frac{1}{2}}\leq\widehat{X}^{0}$ and $0\leq\left(g_{ij}\widehat{Y}^{i}\widehat{Y}^{j}\right)^{\frac{1}{2}}\leq\widehat{Y}^{0}$. Which gives

$$0\leq\left(g_{ij}\widehat{X}^{i}\widehat{X}^{j}\right)^{\frac{1}{2}}\left(g_{ij}\widehat{Y}^{i}\widehat{Y}^{j}\right)^{\frac{1}{2}}\leq\widehat{X}^{0}\widehat{Y}^{0} \tag{77}$$

but since (g_{ij}) is a scalar product, we have

$$g_{ij}\widehat{X}^{i}\widehat{Y}^{j}\leq\left(g_{ij}\widehat{X}^{i}\widehat{X}^{j}\right)^{\frac{1}{2}}\left(g_{ij}\widehat{Y}^{i}\widehat{Y}^{j}\right)^{\frac{1}{2}}\leq\widehat{X}^{0}\widehat{Y}^{0} \tag{78}$$

(76) then follows from (77) and (78). ■

Next we prove this important result for the Maxwell tensor defined by (6').

Lemma 2

Let $(\widehat{X}^{\alpha}),(\widehat{Y}^{\alpha})$ be two future pointing time-like or null vectors. Then the Maxwell tensor $\widehat{\tau}_{\alpha\beta}$ verifies

$$\widehat{\tau}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{Y}^{\beta}\geq 0. \tag{79}$$

Proof: Let us consider the frame of the four vectors $l=(l^{\alpha}),\ n=(n^{\alpha}),\ x=(x^{\alpha}),\ y=(y^{\alpha})$, satisfying the following properties

$$l_{\alpha}l^{\alpha}=n_{\alpha}n^{\alpha}=l_{\alpha}x^{\alpha}=l_{\alpha}y^{\alpha}=n_{\alpha}y^{\alpha}=n_{\alpha}x^{\alpha}=0 \tag{80}$$

and such that $l=(l^{\alpha})$,

$n=(n^{\alpha})$ are future pointing time-like. Now inspired for instance by the case where the electromagnetic fields $\widehat{F}_{\alpha\beta}$ derives from a potential vector, the antisymmetric two-form $\widehat{F}_{\alpha\beta}$ can be written in one of the two following general forms:

$$\widehat{F}_{\alpha\beta}=\frac{S}{2}\left(l_{\alpha}n_{\beta}-l_{\beta}n_{\alpha}\right)+\frac{V}{2}\left(x_{\alpha}y_{\beta}-x_{\beta}y_{\alpha}\right) \tag{81}$$

Or

$$\widehat{F}_{\alpha\beta} = \frac{C}{2}\left(l_\alpha x_\beta - l_\beta x_\alpha\right) \tag{82}$$

Where S,V,C are constants. It is important to choose the constants S,V,C such that:

$$l_\alpha n^\alpha = -1,\ x_\alpha x^\alpha = y_\alpha y^\alpha = 1,\ x_\alpha y^\alpha = 0. \tag{83}$$

Let us consider the Maxwell tensor

$$\widehat{\tau}_{\alpha\beta} = -\frac{1}{4} g_{\alpha\beta}\widehat{F}_{\lambda\mu}\widehat{F}^{\lambda\mu} + \widehat{F}_{\alpha\lambda}\widehat{F}_\beta{}^\mu \tag{84}$$

(82) gives, using (80), : $\widehat{F}_{\lambda\mu}\widehat{F}^{\lambda\mu} = 0, \quad \widehat{F}_{\alpha\lambda}\widehat{F}_\beta{}^\mu = \frac{C^2}{4} l_\alpha l_\beta$ and (84) then gives

$$\widehat{\tau}_{\alpha\beta} = \frac{C^2}{4} l_\alpha l_\beta \tag{85}$$

But since $(\widehat{X}^\alpha), (\widehat{Y}^\beta)$,$(l^\alpha)$ are future pointing vectors, by (75) we have:
$l_\alpha X^\alpha \leq 0,\ l_\beta Y^\beta \leq 0$, then $\widehat{\tau}_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta = \frac{C^2}{4} l_\alpha l_\beta \widehat{X}^\alpha\widehat{Y}^\beta = \frac{C^2}{4}\left(l_\alpha\widehat{X}^\alpha\right)\left(l_\beta\widehat{Y}^\beta\right) \geq 0$, then $\widehat{\tau}_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta \geq 0$. ■

Theorem 4

The global solution of the Einstein-Maxwell-Boltzmann-Massive scalar field system with pseudo-tensor of pressure verifies:

1)The dominante energy condition .

2) The weak energy condition.

3) The strong positivity condition if $\Lambda \geq \frac{9}{2}\left(H(0)\right)^2$.

Proof

proof of 1°)

Let $(\widehat{X}^\alpha), (\widehat{Y}^\beta)$ be two future pointing time-like vectors, By (79) we have

$$\widehat{\tau}_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta \geq 0 \tag{86}$$

• Next, we have , since $(\widehat{p}^\alpha)$ is a time-like future pointing vector and using (75):

$$\begin{aligned}\widehat{T}^1_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta &= \int_{\mathbb{R}^3} \frac{f(t,\overline{p})}{p^0}\widehat{p}_\alpha\widehat{p}_\beta\,|g|^{\frac{3}{2}}\,\widehat{X}^\alpha\widehat{Y}^\beta d\overline{p} \\ &= \int_{\mathbb{R}^3} \frac{f(t,\overline{p})}{p^0}\,|g|^{\frac{3}{2}}\left(\widehat{p}_\alpha\widehat{X}^\alpha\right)\left(\widehat{p}_\beta\widehat{Y}^\beta\right) d\overline{p} \geq 0\end{aligned}$$

Since $\left(\widehat{p}_\alpha\widehat{X}^\alpha\right) \leq 0,\ \left(\widehat{p}_\beta\widehat{Y}^\beta\right) \leq 0$ and $f(t,\overline{p}) \geq 0$, $p^0 > 0, |g|^{\frac{3}{2}} > 0$, we obtain

$$\widehat{T}^1_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta \geq 0. \tag{87}$$

• Now the expression (7) of $\widehat{T}^2_{\alpha\beta}$ and using (16), we have

$$\begin{aligned}\widehat{T}^2_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta &= \widehat{T}^2_{00}\widehat{X}^0\widehat{Y}^0 + \widehat{T}^2_{ij}\widehat{X}^i\widehat{Y}^j \\ &= (\psi + \frac{1}{2}m_0^2\phi^2)\widehat{X}^0\widehat{Y}^0 + \frac{1}{2}g_{ij}(2\psi - m_0^2\phi^2)\widehat{X}^i\widehat{Y}^j \\ &= \psi\left(\widehat{X}^0\widehat{Y}^0 + g_{ij}\widehat{X}^i\widehat{Y}^j\right) + \frac{1}{2}m_0^2\phi^2\left(\widehat{X}^0\widehat{Y}^0 - g_{ij}\widehat{X}^i\widehat{Y}^j\right)\end{aligned}$$

From (76) and (77) we have $X^0Y^0 \geq 0,\ g_{ij}X^iY^j \geq 0$ and
$X^0Y^0 - g_{ij}X^iY^j = -X^\alpha Y_\alpha \geq 0$, we then obtain

$$\widehat{T}^2_{\alpha\beta}\widehat{X}^\alpha\widehat{Y}^\beta \geq 0. \tag{88}$$

• Since $\widehat{H}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{Y}^{\beta} = \widehat{H}_{00}\widehat{X}^{0}\widehat{Y}^{0} + \widehat{H}_{ij}\widehat{X}^{i}\widehat{Y}^{j} = \widehat{H}_{00}\widehat{X}^{0}\widehat{Y}^{0}$, because $\widehat{H}_{ij} = 0$. But from (17), one has $\widehat{H}_{00} \geq 0$, then $\widehat{H}_{00}\widehat{X}^{0}\widehat{Y}^{0} \geq 0$, we obtain

$$\widehat{H}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{Y}^{\beta} \geq 0. \tag{89}$$

From (86), (87), (88) and (89), we obtain the dominante energy condition

$$\left(\widehat{T}^{1}_{\alpha\beta} + \widehat{T}^{2}_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{Y}^{\beta} \geq 0. \tag{90}$$

.

Proof of 2°)

Setting in (90), $\widehat{X}^{\alpha} = \widehat{Y}^{\beta}$, we have $\left(\widehat{T}^{1}_{\alpha\beta} + \widehat{T}^{2}_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{X}^{\beta} \geq 0$. Then the weak energy condition is satisfied.

Proof of 3°) : Let $(\widehat{X}^{\alpha})$ be a future pointing time-like vector. We deduce from the Einstein equation (2) that $\widehat{R}_{\alpha\beta} = \left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{g}_{\alpha\beta} + 8\pi\left(\widehat{T}^{1}_{\alpha\beta} + \widehat{T}^{2}_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)$, so

$$\widehat{R}_{\alpha\beta}\widehat{X}^{\alpha}\widehat{X}^{\beta} = \left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{X}^{\alpha}\widehat{X}_{\alpha} + 8\pi\left(\widehat{T}^{1}_{\alpha\beta} + \widehat{T}^{2}_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{X}^{\beta} \tag{91}$$

By the weak energy condition, we know that

$$\left(\widehat{T}^{1}_{\alpha\beta} + \widehat{T}^{2}_{\alpha\beta} + \widehat{H}_{\alpha\beta} + \widehat{\tau}_{\alpha\beta}\right)\widehat{X}^{\alpha}\widehat{X}^{\beta} \geq 0. \tag{92}$$

Let us see at which condition we will also have

$$\left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{X}^{\alpha}\widehat{X}_{\alpha} \geq 0$$

First of all, we have

$$\widehat{R} = \widehat{g}^{\alpha\beta}\widehat{R}_{\alpha\beta} = -\widehat{R}_{00} + g^{ij}\widehat{R}_{ij} \tag{93}$$

From the classical formula linking $\widehat{R}_{ij}$ and R_{ij},

$$\widehat{R}_{ij} = R_{ij} - \partial_t k_{ij} - 3Hk_{ij} - 2k_{ij}k_{j}^{\ l} \tag{94}$$

Contracting with g^{ij}, we have:

$$g^{ij}\widehat{R}_{ij} = R - g^{ij}\partial_t k_{ij} + 9H^2 - 2k_{ij}k^{ij} \tag{95}$$

And recall that we have, using (13):

$$g^{ij}\partial_t k_{ij} = \partial_t\left(g^{ij}k_{ij}\right) - k_{ij}\partial_t g^{ij} = -3\partial_t H + 2k_{ij}k^{ij}, \tag{96}$$

since $\partial_t g^{ij} = 2k^{ij}$. Now using (96), (95) takes the form

$$g^{ij}\widehat{R}_{ij} = R + 3\partial_t H + 9H^2 \tag{97}$$

Now setting in Einstein's equation (2), $\alpha = \beta = 0$, one has:

$$\widehat{R}_{00} = -\frac{1}{2}\widehat{R} + \Lambda + 8\pi\left(\widehat{T}^{1}_{00} + \widehat{T}^{2}_{00} + \widehat{H}_{00} + \widehat{\tau}_{00}\right) \tag{98}$$

Using (97) and (98), (93) take the form,

$$\widehat{R} = \frac{1}{2}\widehat{R} - \Lambda - 8\pi\left(\widehat{T}^{1}_{00} + \widehat{T}^{2}_{00} + \widehat{H}_{00} + \widehat{\tau}_{00}\right) + R + 3\partial_t H + 9H^2$$

Using $\left(\widehat{T}^{1}_{00} + \widehat{T}^{2}_{00} + \widehat{H}_{00} + \widehat{\tau}_{00} > 0\right)$, (33′) and (41) which give: $R \leq 0$ and $\partial_t H \leq 0$, we deduce

$$\frac{1}{2}\widehat{R} \leq -\Lambda + 9H^2,$$

but H is decreasing and positive, then $H^2 \leq H^2(0)$;hence

$$\frac{1}{2}\widetilde{R} - \Lambda \leq -2\Lambda + 9\left(H(0)\right)^2.$$

So we will have, $\frac{1}{2}\widehat{R} - \Lambda \leq 0$ if $-2\Lambda + 9\left(H(0)\right)^2 \leq 0$, that gives $2\Lambda \geq 9\left(H(0)\right)^2$. Since, $(\widehat{X}^\alpha)$ a pointing time like vector , we have $\widehat{X}^\alpha \widehat{X}_\alpha \leq 0$. In conclusion if $2\Lambda \geq 9\left(H(0)\right)^2$, then we get

$$\left(\frac{1}{2}\widehat{R} - \Lambda\right)\widehat{X}^\alpha \widehat{X}_\alpha \geq 0. \tag{99}$$

We conclude that we have $\widehat{R}_{\alpha\beta}\widehat{X}^\alpha \widehat{X}^\beta \geq 0$. This completes the proof of theorem 4. ■

7 CONCLUSION

In this paper, we have determined the asymptotic behaviour (of the space-time) in the neighborhood of $+\infty$ in the case of global existence and we have proved the geodesic completeness. We have finally proved that the global solution of the Einstein-Maxwell-Boltzmann-Massive scalar field satisfied the three types of energy conditions.

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