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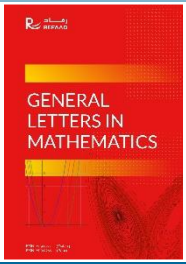

Dividing a Farm: A Simple Application of Game Theory in Geometry

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Abstract

Two farmers decide to divide a triangle farm between them. The problem is modeled as a simple application of game theory in geometry. The game is defined and is solved, theoretically, in independent case. Simulation results are proposed using the copula function, in the dependent cases. Finally, concluding remarks are proposed.

Keywords: Copula; Game theory; Geometry; Triangle
***MSC 2020**: 91A43*

1 Introduction

Game theory is the set of mathematical tools for modeling the conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions and those decisions influence on other's welfare. Some questions about elements of a game are:
(i) What are the player actions, payoffs, set of information and strategies?
(ii) What are the possible outputs of the game and how long does a game last?

Game theory is useful tool for professionals in the fields of operations research, economics, finance, regulation, military, insurance, retail marketing, politics, conflict analysis, and energy, to name a few, see [6].

It studies what happens when self-interested agents interact. Self-interest agents have their own descriptions of which states of the world they like, and which can be good for other players? As stated, it has applications in economy, negotiations, trade, resource management ecology, physics and also geometry.

Graphical models and geometrical interpretations play important role in mathematics, economy, and finance and risk management. In geometry, the preferences and utility functions of players are represented by a geometrical graphical model and game theory and concepts such as Nash equilibrium solutions are shown under this framework. Indeed, it is usual method to propose economical and financial models such as binomial trees by graphical methods and then use then define the game components on that graph. The same approach is done in Markov decision process, see [4].
Here, a simple application of game theory in geometry is discussed.

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1.1 Problem definition

Consider an equilateral triangle farm ABC^{Δ} with the size of side of l. Two farmer 1,2 decide to divide this farm between themselves. They select an arbitrary point D in the triangle and decide about three parts ADB^{Δ}, ADC^{Δ} and BDC^{Δ} as follows: Farmer 1 gets the part ADB^{Δ} and $\alpha \in (0,1)$ percent of BDC^{Δ} and the second farmer takes the second part ADC^{Δ} and $\beta = 1 - \alpha \in (0,1)$ percent of remaining part of BDC^{Δ}. This is a type of strategic game between two farmers. The following figure visualizes the above discussion:

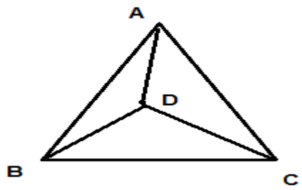

Figure 1: Pattern of dividing the farm

The point D (in the ABC^{Δ}) is arbitrary point and two farmers select it as follows: The first farmer draw the parallel line d_1 across the segment line $\overline{AB}$ and the second farmer (considering this restriction that the point D should be lie in the triangle ABC^{Δ}) draw the parallel line d_2 across the segment line $\overline{AC}$. The cross point of two lines d_1 and d_2 specifies the point D.

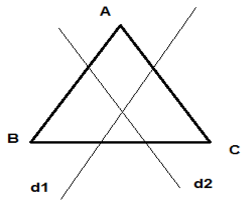

Figure 2: Pattern of selection of point D

Here, first, we prove that the sum of heights plotted from D to each side of triangle ABC^{Δ} is constant and is equal to the height of triangle ABC^{Δ}. To this end, notice that the areas of triangles ADB^{Δ}, ADC^{Δ} and BDC^{Δ} are:

$$\overline{DH}_1(l/2), \overline{DH}_2(l/2), \text{ and } \overline{DH}_3(l/2),$$

respectively. The sum of these areas is:

$$(\overline{DH}_1 + \overline{DH}_2 + \overline{DH}_3)(l/2)$$

which is equal to the area of triangle ABC^{Δ} which is $\overline{AH}(l/2)$. Hence,

$$\overline{DH}_1 + \overline{DH}_2 + \overline{DH}_3 = \overline{AH}.$$

Also,

$$\overline{AH} = \sin\left(\frac{\pi}{3}\right) l = \frac{\sqrt{3}}{2} l.$$

Let $x = \frac{\overline{DH}_1}{l}\frac{2}{\sqrt{3}}$, $y = \frac{\overline{DH}_2}{l}\frac{2}{\sqrt{3}}$, and $z = \frac{\overline{DH}_3}{l}\frac{2}{\sqrt{3}}$. Hence, $x + y + z = 1$ and $x, y, z \in (0,1)$.

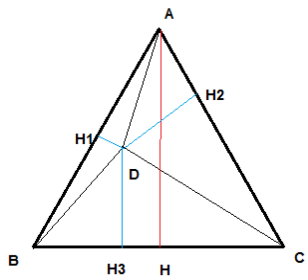

Figure 3: Relations between heights

1.2 Utility functions

In economics and game theory literatures, agent's preferences are captured by a utility function $u(.)$ which assigns a single number to express desirability of a state for agent. A rational agent should choose the action which maximizes the agent's utility. This principle formalizes the general notion that agent should "do the right things". The axioms of utility theory are order-ability, transitivity, continuity, substitutability, and decomposability; see [5] for more details. When there are at least two agents which make decisions about the same problem, the game theory is born. Here, the utility functions of both players are formalized, as follows.

One can notice that the share of first farm is:

$$\left(\frac{l}{2}\right)(\overline{DH}_1+\alpha\overline{DH}_2)=\left(\frac{l}{2}\right)\left(\frac{l\sqrt{3}}{2}\right)(x+\alpha z)=$$
$$=\frac{\sqrt{3}}{4}l^2\big(x+\alpha(1-x-y)\big)=\frac{\sqrt{3}}{4}l^2(\alpha+(1-\alpha)x-\alpha y).$$

Similarly, the share of the second farmer is:

$$\frac{\sqrt{3}}{4}l^2(\beta+(1-\beta)y-\beta x).$$

Thus, the utility functions of two farmers are:

$$\begin{cases} u_1(x,y)=\alpha+(1-\alpha)x-\alpha y \\ u_2(x,y)=\beta+(1-\beta)y-\beta x, \end{cases}$$

where $0<x<1$, $\beta=1-\alpha$ and $\alpha,\beta\in(0,1)$. Since,

$$0<z=1-x-y,$$

therefore $0<y<1-x$.

Hereafter, it is interested to find the variation area (feasible set) of x,y. To this end, consider the following figure.

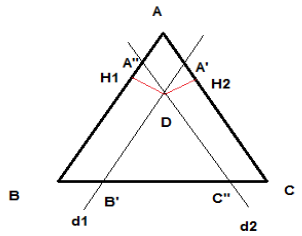

Figure 4: Feasible set of x,y

In the triangle $DA"H_1{}^{\Delta}$, we have

$$\frac{\overline{DH_1}}{\overline{DA"}}=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

(since d_2 and $\overline{AC}$ are parallel). Hence,

$$\overline{DA"}=\frac{2}{\sqrt{3}}\overline{DH}_1=\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2}lx=lx.$$

Also, $\overline{A'A}=\overline{DA"}=lx$. Thus,

$$\overline{A'C}=\overline{AC}-\overline{A'A}=l-xl=(1-x)l.$$

According to the Thales theorem in the triangle ABC^{Δ}, we have

$$\frac{\overline{AH}}{\overline{DH_2}}=\frac{\overline{AC}}{\overline{AC}-\overline{A"C"}}.$$

Remark 1. Here a simple description is proposed about the Thales theorem. The intercept theorem (known as the Thales theorem) is important theorem in elementary geometry which is about the ratios of various lines segments that are created if two intersecting lines are intercepted by a pair of parallel lines, see Bailey *et al*. (2007). In the following figure it is stated as

$$\frac{\overline{SA}}{\overline{SC}}=\frac{\overline{SB}}{\overline{SD}}=\frac{\overline{AB}}{\overline{CD}}$$

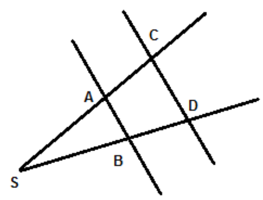

Figure 5: Thales theorem

Hence, $\frac{(\frac{\sqrt{3}}{2})l}{(\frac{\sqrt{3}}{2})ly}=\frac{l}{l-\overline{A"C"}}$. Therefore,

$$\overline{A"C"}=(1-y)l.$$

Since $\overline{DA"}=lx$, then

$$\overline{DC}"=(1-x-y)l=zl.$$

Notice that $\overline{DC}"$ moves from zero (at the point B') to $\overline{AC'}$. Indeed,

$$0<zl<(1-x)l,$$

at which $z<1-x$ is clear and $z>0$ implies that

$$x+y<1.$$

Thus, $0<x<1-y$.

It is natural to assume that $\overline{DC}"$ is far from zero. Thus, assume that its minimum value is γl, where γ is a small positive number. Thus, $zl > \gamma l$ and then

$$z > \gamma.$$

Therefore,

$$x + y < 1 - \gamma$$
$$0 < y < 1 - \gamma - x.$$

An important question is how the portion coefficient is chosen? The next section answers this question.

2 Choosing α.

In this section, methods of choosing the portion of α are discussed. For example, it is natural to assume that the share of farmer 1 of part BDC^{Δ} (and consequently, farmer 2) is the function of actions of both farmers, *i.e.*, $\alpha = \alpha(x, y)$ (the share of farmer 1) is a function of x where

$$\alpha: (0,1) \times (0,1) \to (0,1)$$

has the first derivative over (0,1). For example, two farmers can decide that if one of them selects bigger height, then he choose smaller part of BDC^{Δ}. In this case, then

$$\alpha = \alpha(x, y) = \theta 1(x < y) + (1 - \theta) 1(x > y),$$

where $\theta > 0.5$. Notation $1(x < y)$ is one if $x < y$ and zero otherwise.

2.1 Exact distribution

In this sub-section, the portion α is considered as a random variable and its distribution is derived. Here, six steps are done as follows:

<u>a) α is functions of $u_i, i = 1,2$</u>. Denote $u_i(x, y) = u_i, i = 1,2$. Notice that

$$u_1 = \alpha + (1 - \alpha)x - \alpha y =$$
$$x - \alpha(x + y - 1) =$$
$$= x + \alpha z.$$

Thus, $\alpha = \frac{u_1 - x}{z}$. Notice again that

$$u_2 = y - \beta(x + y - 1) = y + (1 - \alpha)z.$$

One can see that $u_1 + u_2 = 1$. Thus,

$$z = 1 - x - y = (u_1 - x) + (u_2 - y).$$

In this way,

$$\alpha = \frac{u_1 - x}{z} = \frac{u_1 - x}{(u_1 - x) + (u_2 - y)}.$$

To make sure that $\alpha \in (0,1)$, it is enough to assume that $0 \le A \le u_i, A = x, y$.

<u>b) Transformation</u>. Define,

$$v_i = \frac{u_i - A_i}{u_i} \in (0,1), i = 1,2, A_1 = x, A_2 = y.$$

Then,

$$\alpha = \frac{u_1 v_1}{u_1 v_1 + u_2 v_2}.$$

Suppose that v_1 and v_2 are independent and identically distributed uniformly on (0,1). It is easy to see that

$$P(\alpha \le \omega) = P\left(\frac{v_1}{v_2} \le \frac{\omega}{1-\omega} \times \frac{u_2}{1-u_2}\right).$$

<u>c) Another transformation</u>. Define

$$L = -\log(v_2) - (-\log(v_1)).$$

Then,

$$P(\alpha \le \omega) = P\left(\log\left(\frac{v_1}{v_2}\right) \le \log\left(\frac{\omega}{1-\omega}\right) + \log\left(\frac{u_2}{1-u_2}\right)\right) =$$
$$= P(L \le G),$$

where $G = \text{logit}(\omega) + \text{logit}(u_2)$ and

$$\text{logit}(A) = \log\left(\frac{A}{1-A}\right), A = \omega, u_2.$$

<u>d) Distribution of L</u>. One can see that $-\log(v_2)$ and $-\log(v_1)$ have common distribution exponential with unit rate and they are statistically independent. Then, L has the Laplace distribution with unit rate, see Johnson and Kotz (2018). That is,

$$P(L \le G) = \begin{cases} 0.5e^{G} & G \le 0, \\ 1 - 0.5e^{-G} & G \ge 0, \end{cases}$$

Where:

$$e^{G} = \frac{\omega}{1-\omega} \times \frac{u_2}{1-u_2}.$$

e) Distribution of α. Notice that $P(\alpha \leq \omega) = P(L \leq G)$. Notice that $G \leq 0$ is equivalent to the

$$\omega \leq 1 - u_2.$$

Therefore, given u_2, it is seen that:

$$P(\alpha \leq \omega) = \begin{cases} 0.5\frac{\omega}{1-\omega} \times \frac{u_2}{1-u_2} & \omega \leq 1 - u_2 \\ 1 - 0.5\frac{1-\omega}{\omega} \times \frac{1-u_2}{u_2} & \omega \geq 1 - u_2 \end{cases}$$

When $u_1 = u_2 = 0.5$, then no player want to change his position to the better one, then, in this case,

$$P(\alpha \leq \omega) = \begin{cases} 0.5\frac{\omega}{1-\omega} & \omega \leq 0.5 \\ 1 - 0.5\frac{1-\omega}{\omega} & \omega \geq 0.5 \end{cases}$$

f) Density and mean of α. Considering $P(\alpha \leq \omega)$ as the distribution function of α computed at ω, it is seen that the density function is given by:

$$f_{\alpha|u_2}(x) = \begin{cases} 0.5\frac{u_2}{1-u_2} \times \frac{1}{(1-x)^2} & x \leq 1 - u_2 \\ 0.5\frac{1-u_2}{u_2} \times \frac{1}{x^2} & x \geq 1 - u_2 \end{cases}$$

The mean of α, as a function of u_2, is given by:

$$E(\alpha|u_2) = 0.5\frac{u_2}{1-u_2}\int_0^{1-u_2} \frac{xdx}{(1-x)^2} + 0.5\frac{1-u_2}{u_2}\int_{1-u_2}^{1} \frac{x}{x^2}dx =$$
$$0.5\frac{u_2}{1-u_2}\log(u_2) - 0.5\frac{1-u_2}{u_2}\log(1-u_2) + 0.5$$

By replacing $u_1 = 1 - u_2$, in the above formula, it is seen that:

$$E(\alpha|u_1) = 0.5\frac{1-u_1}{u_1}\log(1-u_1) - 0.5\frac{u_1}{1-u_1}\log(u_1) + 0.5$$

The conditional expected utility of u_2 given u_1 is

$$E(y|u_1) + E((1-\alpha)|u_1)(1 - E(x|u_1) - E(y|u_1)) =$$
$$0.5u_2 + E((1-\alpha)|u_1)(1 - 0.5u_1 - 0.5u_2) =$$
$$0.5(1 - u_1 + 1 - E(\alpha|u_1)) =$$
$$0.5\left(2 - u_1 + \frac{u_1}{1-u_1}\log(u_1) - \frac{1-u_1}{u_1}\log(1-u_1) - 1\right).$$

The following figure shows that behavior of mean function

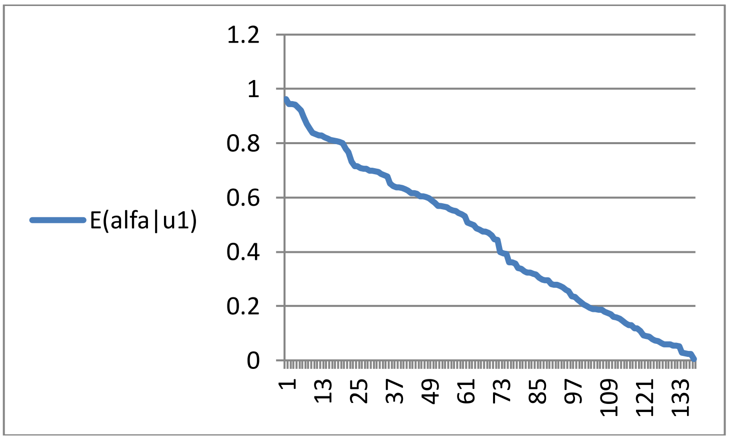

Figure 6: The mean function

Here, as soon as the second player decides about the lower bound of his/her expected utility, then, the value of u_1 is chosen and then, $u_2 = 1 - u_1$ is derived. As soon as, u_1 and u_2 is derived, then, distribution of x, y and α is fully determined and this problem is a completely simulation-based problem.

2.2 Simulations

In the case of correlated v_1 and v_2, use the copula function.

Remark 2. Copula is a powerful way to model the dependencies in a random vector. One key insight is due to the famous Sklar theorem. It implies that the distribution of any continuous random vector can e expressed by copula and marginal distributions. It is easy to estimate the marginal of random variables and most of time exist in practice, so all we need is to estimate the copula function and this would lead to joint distribution of random variables exist in random vector, see Czado (2010). Mathematically, function $c: [0,1]^d \to [0,1]$ is a copula function if

$$c(u_1, \ldots, u_d) = P\left(X_1 \leq F_1^{-1}(u_1), \ldots, X_1 \leq F_d^{-1}(u_d)\right),$$

where random vector $(X_1, \dots, X_d)$ have marginal distributions $(F_1, \dots, F_d)$, respectively. In this way, the joint distribution of $(X_1, \dots, X_d)$ is captured.
There are many types of copula functions like normal, t, Archimedean copulas, however, here, normal type of copula function is surveyed and its results are presented, as follows.
To simulate $\frac{v_1}{v_2}$, let

$$n_i = \Phi^{-1}(v_i), i = 1,2,$$

where Φ is the distribution function of standard normal distribution. Let

$$\rho = \text{corr}(n_1, n_2),$$

be the correlation of n_1, n_2. Then, n_1, n_2 are distributed normal jointly with zero means and correlation ρ. Thus,

$$V = \frac{v_1}{v_2} = \frac{\Phi(n_1)}{\Phi(n_2)}.$$

To simulate n_1, n_2, it is enough to let

$$n_1 = z_1 \text{ and } n_2 = \rho z_1 + \sqrt{1-\rho^2} z_2,$$

where z_1, z_2 are independent and identically distributed with common standard normal distributions. Notice that

$$E(\alpha|u_1) = \int_0^1 P(\alpha > \omega) d\omega = 1 - \int_0^1 P(\alpha \le \omega) d\omega =$$
$$1 - \int_0^1 P\left(V \le \frac{\omega}{1-\omega} \times \frac{1-u_1}{u_1}\right) d\omega =$$
$$= 1 - \frac{u_1}{1-u_1} \int_0^\infty \frac{P(V \le v)}{(1+\frac{u_1}{1-u_1}v)^2} dv = 1 - \frac{u_1}{1-u_1} \int_0^\infty \frac{1-P(V>v)}{(1+\frac{u_1}{1-u_1}v)^2} dv =$$
$$1 - \frac{u_1}{1-u_1} \int_0^\infty \frac{dv}{(1+\frac{u_1}{1-u_1}v)^2} + \frac{u_1}{1-u_1} \int_0^\infty \frac{P(V>v)}{(1+\frac{u_1}{1-u_1}v)^2} dv =$$
$$1 - \frac{u_1}{1-u_1} A + \frac{u_1}{1-u_1} B.$$

Here, two components A and B are given by

$$A = \int_0^\infty \frac{dv}{(1+\frac{u_1}{1-u_1}v)^2}$$
$$B = \int_0^\infty \frac{P(V>v)}{(1+\frac{u_1}{1-u_1}v)^2} dv.$$

It is easy to see that

$$A = \frac{1-u_1}{u_1}.$$

Thus, $\frac{u_1}{1-u_1} A = 1$ and hence,

$$E(\alpha|u_1) = \frac{u_1}{1-u_1} B.$$

To compute B, apply he integration by part, it is seen that

$$B = \frac{1-u_1}{u_1}\left(1 - E\left(\frac{1}{1+\frac{u_1}{1-u_1}V}\right)\right).$$

Thus, $\frac{u_1}{1-u_1} B = 1 - E\left(\frac{1}{1+\frac{u_1}{1-u_1}V}\right)$. Therefore, $E(\alpha|u_1) = 1 - E\left(\frac{1}{1+\frac{u_1}{1-u_1}V}\right)$. To reduce, the complexity of computation, using the second order Taylor expansion, it is seen that

$$E(\alpha|u_1) \approx \frac{u_1}{1-u_1} E(V) + \left(\frac{u_1}{1-u_1}\right)^2 E(V^2).$$

As follows, using the response surface methodology technique, we derive functional forms of $E(V)$ and $E(V^2)$, based on ρ. To this end, first, using the Monte Carlo simulation with 1000 repetitions, V is simulated for various values of ρ, then its non-central moments are computed and then by plotting $E(V)$ and $E(V^2)$, based on ρ, their functional forms are derived.
The following plots give the visualizations of these functions

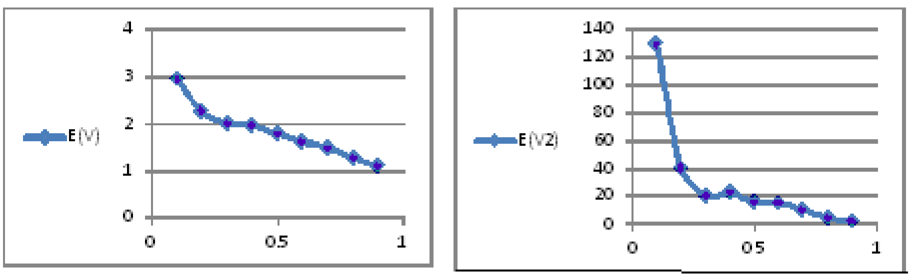

Figure 6: $E(V)$ and $E(V^2)$

It is seen that:

$$\begin{cases} E(V) = 2.8 - 1.96\rho \\ E(V^2) = \frac{0.9}{\rho^2} \end{cases}$$

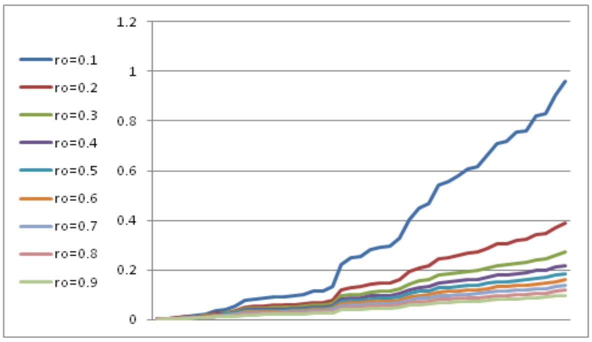

Figure 7: The simulated $E(\alpha|u_1)$

3 Concluding Remarks

The current paper studies the farm dividing problem between two farmers from a game theory point of view. Two independent and dependent selections of players are considered and the game is solved.

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