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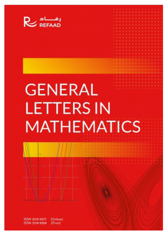

Right Central CNZ Property Skewed by Ring Endomorphisms

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Abstract

The concept of the reversible ring property concerning nilpotent elements was introduced by A.M. Abdul-Jabbar and C. A. Ahmed, who introduced the concept of commutativity of nilpotent elements at zero, termed as a CNZ ring, as an extension of reversible rings. In this paper, we extend the CNZ property through the influence of a central ring endomorphism α, introducing a new type of ring called a right α-skew central CNZ ring. This concept not only expands upon CNZ rings but also serves as a generalization of right α-skew central reversible rings. We explore various properties of these rings and delve into extensions of right α-skew central CNZ rings, along with examining several established results, which emerge as corollaries of our findings.

Keywords: central CNZ ring, α-skew CNZ ring,, matrix ring, polynomial ring, Dorroh extension.
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1. Introduction

In this document, we consider rings to be associative and equipped with identity elements unless specified otherwise. When referring to a ring R, $N(R)$ denote the set of all nilpotent elements in R, and $Z(R)$ represents the center of R, defined as $Z(R) = \{a \in R \mid ax = xa \text{ for all } x \in R\}$.

We denote the polynomial ring by $R[x]$, the power series ring by $R[[x]]$, the Laurent polynomial ring by $R[x, x^{-1}]$ and the Laurent power series ring over R by $R[[x, x^{-1}]]$. We use $\mathbb{Z}$ to denote the ring of integers and $\mathbb{Z}_n$ to denote the ring of integers modulo n. $M_n(R)$ represents the set of all n by n (where $n \geqslant 2$) matrix rings over R. while $U_n(R)$ stands for the upper triangular matrix rings over R and $D_n(R)$ is the ring of all upper triangular matrices whose diagonal elements are identical.

A ring is classified as *reduced* if it contains no nonzero nilpotent elements. The concept of reduced rings has been extended by N. Agayev [15] to central reduced rings. If every elements of R is central, then the ring R is called *central reduced* ring.

An *endomorphism* α is a ring homomorphism from the ring R into itself. As stated by Krempa [12], if whenever $a\alpha(a) = 0$ (for all $a \in R$) implies $a = 0$, then an endomorphism α of a ring R is said to be rigid. Furthermore, a ring R is referred to be α-rigid [6] if a rigid endomorphism α of R exists. It's worth noting

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that any rigid endomorphism of a ring is inherently a monomorphism, and according to [6, Proposition 5] α-rigid rings are reduced rings.

As stated in [9], a ring R is considered *α-compatible* if, for every pair of elements a and b in R the condition $ab = 0$ holds if and only if $a\alpha(b) = 0$. when R is an α-compatible, it implies that the endomorphism α is evidently a monomorphism. Note that R is α-compatible if and only if the left version holds (meaning, for each a and b in R, $ab = 0$ if and only if $\alpha(a)b = 0$).

Following H. Kose and A. HarmancA [10] A ring R is classified as central CNZ if, for any a, b in $N(R)$, the condition $ab = 0$ implies that ba is central in R.

In [2] an endomorphism α of a ring R is named right skew CNZ (or left skew CNZ) if for any a, b in $N(R)$ the condition $ab = 0$ implies $b\alpha(a) = 0$ (or $\alpha(b)a = 0$, respectively). If there is a right (or left) skew CNZ endomorphism α of a ring R, then ring R is said to be right α-skew CNZ (or left α-skew CNZ). Furthermore, R is known as α-skew CNZ if it satisfies both left and right α-skew CNZ conditions.

Due to B.Arnab and SH. Ch. Uday[5] An endomorphism α of a ring R is denoted as right skew central reversible (or left skew central reversible) if, for any a, b in $N(R)$ the condition $ab = 0$ implies to $b\alpha(a) \in Z(R)$ (or $\alpha(b)a \in Z(R)$, respectively). A ring R earns the label of being right (or left) α-skew central reversible if there exists a right (or left) skew central reversible endomorphism α of R. Furthermore, R is termed *α-skew central reversible* if it satisfies both left and right α-skew central reversible conditions.

In this paper, we expand the notion of the central CNZ ring property to include skewed CNZ ring properties via ring endomorphisms. We introduce the concept of a *right α-skew central CNZ*, which serves as both a generalization of right α-skew CNZ ring and an extension of central CNZ rings. We then delve into the structure and properties of right α-skew CNZ ring rings. Additionally, we demonstrate how several known results can be derived as corollaries of our findings.

Throughout this work, the notation α is used to denote a nonzero endomorphism of a given ring, unless stated otherwise.

2. Basic Characterization of Right α-Skew Central CNZ

In this section, our focus is on exploring the fundamental structure of right α-skew central CNZ rings, while also investigating various associated ring properties. We commence by introducing the following definition:

Definition 2.1. An endomorphism α applied to a ring R is defined as a *right skew central CNZ* if for any a, b in $N(R)$ with $ab = 0$, $b\alpha(a)$ is belong to $Z(R)$. Furthermore, the ring R is said to be a *right α-skew central CNZ* If there exists an endomorphism α of R such that α is a right skew central CNZ.

A ring R is central CNZ if R is right I_R-skew central CNZ, here I_R represents the identity endomorphism of the ring R. Additionally, every subring S fulfilling $\alpha(S) \subseteq S$ of a right α-skew central CNZ ring, also qualifies as a right α-skew central CNZ ring. We use this fact without reference in the procedure.

Theorem 2.2. *For a ring R with a compatible endomorphism α, the following statements are equivalent.*

(1) *We say that R is an α-skew central CNZ ring, whenever $N(R)^2 = 0$.*

(2) *Let R be a commutative ring. Then R is right α-skew central CNZ if and only if $ab = 0$ for $a, b \in N(R)$ implies $aR\alpha^n(b) \in Z(R)$ and $\alpha^n(a)Rb \in Z(R)$ for any non negative integer n if and only if R is left α-skew central CNZ.*

Proof. (1) Let $a, b \in N(R)$ and such that $a.b = 0$, we have $\alpha(b) \in \alpha(N(R))$ and so $\alpha(b) \in N(R)$ since $\alpha(N(R) \subseteq N(R)$. That is why $\alpha(b)a = 0$ since $N(R).N(R) = N(R)^2 = 0$. so $\alpha(b)a \in Z(R)$ Therefore R is an α-skew central CNZ ring.

(2) Let R be a commutative ring. Assume that R is right α-skew central CNZ and $ab = 0$ for $a, b \in N(R)$. It is Sufficient to demonstrate that $aR\alpha(b) \in Z(R)$ and $\alpha(a)Rb \in Z(R)$. Since R is commutative

and right α-skew central CNZ, $ab = 0$ implies both $b\alpha(a)$ and $\alpha(b)a$ are belong to $Z(R)$ by (2), and so $\alpha(a)b \in Z(R)$ and $a\alpha(b) \in Z(R)$.

□

Proposition 2.3. *Suppose thst R is an α-compatible ring. Then the following statements are equivalent::*

(1) *R is central CNZ;*

(2) *R is right α-skew central CNZ;*

Proof. Let R be an α-compatible ring. (1) $\Rightarrow$ (2): Suppose that R is central CNZ and let $ab = 0$, for $a, b \in N(R)$. Given that R is α-compatible, when $\alpha(a)b = 0$ it follows that $b\alpha(a) \in Z(R)$. Therefore R is considered as a right α-skew central CNZ.

(2) $\Rightarrow$ (1): Is obvious.

□

Every right α-skew central reversible ring is certainly right α-skew central CNZ, but the converse is not true, as illustrated by the following example.

Example 2.4. Consider a ring $R = U_2(\mathbb{Z})$ equipped with an endomorphism α defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Here R is not right α-skew reversible by [13, Example 2.2(2)], but R is a right α-skew central CNZ ring by Theorem 2.2(1), since $N(R)^2 = 0$.

Now, let's examine the opposite condition of a right α-skew central CNZ ring R:

$$(*) \qquad a\alpha(b) = 0 \text{ for } a, b \in N(R) \text{ implies } ba \in Z(R).$$

Proposition 2.5. *For a ring R equipped with a monomorphism α, the following statement holds:*

(1) *R is satisfies the condition $(*)$ above if and only if R is right α-skew central CNZ.*

(2) *For a α-compatible ring R, R is satisfies the condition $(*)$ above if and only if R is right α-skew central CNZ.*

Proof. (1) Suppose that R satisfies the condition $(*)$. Let $ab = 0$ for $a, b \in N(R)$. Then $0 = \alpha(ab) = \alpha(a)\alpha(b)$, and so $b\alpha(a) \in Z(R)$ by assumption. Thus R is right α-skew central CNZ.

Conversely, assume that R is right α-skew central CNZ. If $a\alpha(b) = 0$ for $a, b \in N(R)$ then $\alpha(ba) = \alpha(b)\alpha(a) \in Z(R)$ since $\alpha(b) \in N(R)$ and α is homomorphism. Hence $ba \in Z(R)$, concluding that R satisfies the condition $(*)$.

(2) Is an immediate consequence of (1) because α is a monomorphism when R is α-compatible.

□

The condition that "α is a monomorphism " in proposition **??** is not redundant, as demonstrated by the following example..

Example 2.6. The ring $R = U_2(\mathbb{Z}_4)$ with the endomorphism α expressed as:

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

.

It is obvious α is not injective and that R is right α-skew central CNZ but it does not satisfy the condition $(*)$. Indeed, for

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R),$$

we have $A\alpha(B) = 0$ but

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \notin Z(R).$$

In accordance with [8], a ring R accompanied by an endomorphism α is said to be a central α-rigid ring if, for any element a in R, the condition $a\alpha(a) = 0$ implies that a belongs to the center of $R(Z(R))$.

Proposition 2.7. *In a ring R with an endomorphism α, if R is categorized as central α-rigid, then it necessarily qualifies as a right α-skew central CNZ ring.*

Proof. Assume that R is central α-rigid and let $ab = 0$ for $a, b \in N(R)$. Then

$$b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) = 0.$$

Given that R is central α-rigid, it follows that, $b\alpha(a) \in Z(R)$. Indicating that R is right α-skew central CNZ.

□

Corollary 2.8. *If the ring R is central reduced , it implies that R is also central CNZ.*

Theorem 2.9. *For a ring R with an endomorphism α with $\alpha(a) = a$, the following statements are equivalent:*
(1) R *is central α-rigid;*
(2) $U_2(R)$ *is right $\bar{\alpha}$-skew central CNZ;*

Proof. It suffices to demonstrate that (2) ⇒ (1). Let $U_2(R)$ be right $\bar{\alpha}$-skew central CNZ, Suppose that R is not central α-rigid. In that case, there exists $0 \neq a \notin Z(R)$ with $a\alpha(a) = 0 = a^2$ For

$$A = \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in N(U_2(R))$$

, we have $AB = 0$ but $B\bar{\alpha}(A) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \notin Z(U_2(R))$ implies that $U_2(R)$ is not right $\bar{\alpha}$-skew central CNZ. This leads to a contradiction, and so such a doesnt exist. Thus R is central α-rigid.

□

Corollary 2.10. *A ring R is central reduced if and only if $U_2(R)$ is central CNZ.*

The concept of right α-skew central CNZ is preserved under isomorphism by the following Proposition.

Proposition 2.11. *If S is a ring and $\sigma : R \to S$ is a ring isomorphism. Then R being a right α-skew central CNZ is equivalent to S being a right $\sigma\alpha\sigma^{-1}$-skew central CNZ.*

Proof. First we have $N(S) = \sigma(N(R))$ clearly. Then for any a, b in $N(R)$, it's equivalent to state that $a' = \sigma(a)$ and $b' = \sigma(b)$ are elements of $N(S)$. This means $ab = 0$ if and only if $a'b' = 0$. Similarly, $b\alpha(a) \in Z(R)$ if and only if $b'\sigma\alpha\sigma^{-1}(a') = \sigma(b)\sigma\alpha\sigma^{-1}(\sigma(a)) = \sigma(b\alpha(a)) \in Z(S)$. The proof is complete.

□

Theorem 2.12. *Let α_γ be an endomorphism of a ring R_γ for every γ belonging to the set Γ. then the following assertions are equivalent:*

(i) *R_γ is a right α_γ-skew central CNZ ring for every γ in the set Γ.*

(ii) *The direct product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-skew central CNZ for the endomorphism $\bar{\alpha} : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma\in\Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma\in\Gamma}$.*

(iii) *The direct sum $\bigoplus_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-skew central CNZ for the endomorphism $\bar{\alpha} : \bigoplus_{\gamma\in\Gamma} R_\gamma \to \bigoplus_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma\in\Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma\in\Gamma}$.*

Proof. It suffices to show that (i)⇒(ii) because the class of α-skew central CNZ rings is closed under subrings. Note that $N(\prod_{\gamma\in\Gamma} R_\gamma) \subseteq \prod_{\gamma\in\Gamma} N(R_\gamma)$ and $\alpha_\gamma(R_\gamma) \subseteq R_\gamma$ for each $\gamma \in \Gamma$. Suppose that R_γ is right α-skew central CNZ for each $\gamma \in \Gamma$ and let $AB = 0$ where $A = (a_\gamma)_{\gamma\in\Gamma}, B = (b_\gamma)_{\gamma\in\Gamma} \in N(\prod_{\gamma\in\Gamma} R_\gamma)$. Then $a_\gamma b_\gamma = 0$ for each $\gamma \in \Gamma$ and $b_\gamma \alpha(a_\gamma) \in Z(R_\gamma)$ for each $\gamma \in \Gamma$ by hypothesis, since $a_\gamma, b_\gamma \in N(R_\gamma)$ for each $\gamma \in \Gamma$. This implies $B\bar{\alpha}(A) \in Z(\prod_{\gamma\in\Gamma} R_\gamma)$, indicating that the direct product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-skew central CNZ.

(ii)⇒(iii) and (iii)⇒(i) since the class of right α-skew central CNZ rings is closed under subrings

□

Proposition 2.13. *If R is a ring equipped with an endomorphism α. and there exists a central idempotent e in R such that $\alpha(e) = e$ and $\alpha(1-e) = 1-e$, then both eR and $(1-e)R$ are right α-skew central CNZ if and only if R itself satisfies the property of being right α-skew central CNZ.*

Proof. This arises from the observation that R is isomorphic to the direct sum of eR and $(1-e)R$ denoted $eR \oplus (1-e)R$, Additionally, since the property of being right α-skew central CNZ is preserved under subring S where $\alpha(S) \subseteq S$, it follows that $\alpha(eR) \subseteq eR, \alpha((1-e)R) \subseteq (1-e)R$.

□

In a ring R equipped with an endomorphism α and considering an ideal I of R, if I is an α-ideal (meaning, $\alpha(I) \subseteq I$), then the map $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a) + I$ for all a in R is an endomorphism of the quotient ring R/I.

Proposition 2.14. *If R is a ring equipped with an endomorphism α and I represents an α-ideal of R. If R/I satisfies the condition of being right $\bar{\alpha}$-skew central CNZ and I is central α-rigid as a ring without an identity element, then R is a right α-skew central CNZ ring.*

Proof. Suppose that R/I is a right $\bar{\alpha}$-skew central CNZ and I is central α-rigid as a ring without identity. Let $ab = 0$ for $a, b \in N(R)$. Then $(a+I)(b+I) = I$ and $a+I, b+I \in N(R/I)$. Since R/I is right $\bar{\alpha}$-skew central CNZ, $b\alpha(a) \in Z(I)$. Hence $b\alpha(a)\alpha(b\alpha(a)) = b\alpha(ab)\alpha^2(a) = 0$ and so since I is an central α-rigid ring. Thus R is right α-skew central CNZ. □

3. Matrix Rings of Right α-Skew Central CNZ Rings

For a ring R with an endomorphism α and $n \geqslant 2$, the corresponding $(a_{ij}) \to (\alpha(a_{ij}))$ induces an endomorphism of $M_n(R)$, $U_n(R)$ and $D_n(R)$, respectively. We denote them by $\bar{\alpha}$.

Lemma 3.1. *Let R be a ring, then we have:*

$$Z(D_n(R)) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & \cdots & a_{1n} \\ 0 & a & 0 & 0 & \cdots & 0 \\ 0 & 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{1n} \in Z(R) \right\}.$$

Proof. By [[5], proposition 1.7] □

Note that both $U_2(R)$ and $D_2(R)$ over a reduced ring R with any endomorphism α are right α-skew central CNZ by Theorem 2.2(1), since

$$N(U_2(R)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\} = N(D_2(R)).$$

This illuminates that right α-skew central CNZ rings need not be abelian.

The subsequent example illustrates the existence of a reduced ring A and its endomorphism α where $Mat_2(A)$ is not right $\bar{\alpha}$-skew central CNZ.

Example 3.2. Consider the direct sum $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then A is a commutative reduced ring. Consider the endomorphism $\alpha : A \to A$ which is defined as $\alpha((a, b)) = (b, a)$. For

$$x = \begin{pmatrix} (0,0) & (1,0) \\ (0,1) & (0,0) \end{pmatrix} = y \in N(Mat_2(A)),$$

we have $xy = 0$ but

$$y\bar{\alpha}(x) = \begin{pmatrix} (0,0) & (1,0) \\ (0,1) & (0,0) \end{pmatrix} \begin{pmatrix} \alpha((0,0)) & \alpha((1,0)) \\ \alpha((0,1)) & \alpha((0,0)) \end{pmatrix} = \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (0,1) \end{pmatrix} \notin Z(\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

For $\begin{pmatrix} (1,1) & (1,0) \\ (0,1) & (1,1) \end{pmatrix} \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we get

$$\begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (0,1) \end{pmatrix} \begin{pmatrix} (1,1) & (1,0) \\ (0,1) & (1,1) \end{pmatrix} \neq \begin{pmatrix} (1,1) & (1,0) \\ (0,1) & (1,1) \end{pmatrix} \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (0,1) \end{pmatrix}.$$

Thus $Mat_2(A)$ is not right $\bar{\alpha}$-skew central CNZ.

Note that $M_n(R)$, $D_n(R)$ and $U_n(R)$ for $n \geqslant 3$ are not right $\bar{\alpha}$-skew central CNZ, Consider any ring R with an endomorphism α, where $\alpha(1) \neq 0$ (for instance, when α is a monomorphism), as illustrated in the following example.

Example 3.3. Consider a ring R with an endomorphism α where $\alpha(1) \neq 0$. For the ring $D_3(R)$, consider $e_{12}, e_{23} \in N(D_3((R)))$ where e_{ij} represents a matrix unit with a 1 at the $(i, j) - th$ entry and zeros elsewhere. Then $e_{23}e_{12} = 0$ but

$$e_{12}\bar{\alpha}(e_{23}) = \begin{pmatrix} 0 & 0 & \alpha(1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin Z(D_3(R)).$$

For $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in Z(D_3(R))$ we have,

$$\begin{pmatrix} 0 & 0 & \alpha(1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \alpha(1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

showing that $D_3(R)$ is not right $\bar{\alpha}$-skew central CNZ.

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is defined as the ring $T(R, M) = R \oplus M$, equipped with standard addition and multiplication defined as follows:

$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{array}{cc} r & m \\ 0 & r \end{array}\right)$, where $r \in R$ and $m \in M$ and using the standard matrix operations. Note that $T(R, R) = D_2(R)$ and for an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R, has an endomorphism $\bar{\alpha} : T(R, R) \to T(R, R)$ defined as

$$\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix},$$

is an endomorphism of $T(R, R)$. Considering that $T(R, 0)$ is isomorphic to R, we can recognize the restriction of $\bar{\alpha}$ by $T(R, 0)$ to α.

One might inquire whether the trivial extension over right α-skew central CNZ ring are right $\bar{\alpha}$-skew central CNZ ring. However the answer is negative as illustrated by the following example.

Example 3.4. Let $R = U_2(Z)$ be a ring over a reduced ring Z, with α as an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Then R is right α-skew central CNZ ring by theorem 2.2(1). But $T(R, R)$ is not right $\bar{\alpha}$-skew central CNZ ring For:

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(D_2(R)) = N(T(R, R))$$

with $A^3 = 0$ and $B^2 = 0$, we have $AB = 0$ but

$$B\bar{\alpha}(A) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \notin Z(T(R, R)).$$

Hence, $T(R, R)$ does not exhibit the property of being right $\bar{\alpha}$-skew central CNZ, indicating that even if R is right α-skew central CNZ with the endomorphism α, $T(R, R)$ might not share this property.

Proposition 3.5. *Let R be a rigid with an endomorphism α. Then R is right α-skew central CNZ if and only if the trivial extension $T(R, R)$ is right $\bar{\alpha}$-skew central CNZ.*

Proof. It is sufficient to show that the trivial extension $T(R, R)$ of a reduced and R is right α-skew central CNZ R is right $\bar{\alpha}$-skew central CNZ. Assume that $AB = 0$ for

$$A = \left(\begin{array}{cc} a & b \\ 0 & a \end{array}\right), B = \left(\begin{array}{cc} c & d \\ 0 & c \end{array}\right) \in N(T(R, R)).$$

Then, $ac = 0$ and $ad + bc = 0$. Since R is right α-skew central CNZ , we get $c\alpha(a) \in Z(R)$ and $0 = c(ad) + b(c)) = cbc$, and so $c^2b = 0$. Since R is α-rigid ring, $cb = 0$ and $b\alpha(c) \in Z(R)$, and $ad = 0$ implies $d\alpha(a) \in Z(R)$. Consequently, we have $B\bar{\alpha}(A) \in Z(T(R, R))$ and therefore $T(R, R)$ is right $\bar{\alpha}$-skew central CNZ.

□

Corollary 3.6. *If R is a reduced ring with an endomorphism α. Then $T(R,R)$ is central CNZ ring.*

For a ring R and $n \geqslant 2$, let

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \cdots, a_n \in R \right\}.$$

The center of $V_n(R)$ is defined by

$$Z(V_n(R)) = \{(a_i \in V_n(R) : a_i \in Z(R)\}.$$

Note that: $V_n(R) \cong \frac{R[x]}{x^n R[x]}$

Theorem 3.7. *Let R be reduced ring together with an endomorphism α and let n be a positive integer, then R is right α-skew central CNZ with $\alpha(1) = 1$ if and only if $\frac{R[x]}{\langle x^n \rangle}$ is right $\bar{\alpha}$-skew central CNZ ring, where $\langle x^n \rangle$ is the ideal generated by x^n.*

Proof. Based on the closure property of the class of right α-skew central CNZ rings under subring S with $\alpha(S) \subseteq S$, it follows that R is right α-skew central CNZ . Conversely, let R be reduced and $n \geqslant 2$.. Then $V_n(R)$ is right $\bar{\alpha}$-skew central CNZ by [5, Theorem 2.17]. □

The following corollary is a direct consequence of Theorem 3.7.

Corollary 3.8. *If we consider a reduced ring R with an endomorphism α. Then R is right α-skew central CNZ. if and only if $V_n(R)$ is a right $\bar{\alpha}$-skew central CNZ ring. for any $n \geqslant 2$.*

Proof. Note that $V_n(R) \cong R[x]/\langle x^n \rangle$ by [4].

□

Corollary 3.9. *If R is a reduced ring. Then $V_n(R) \cong R[x]/\langle x^n \rangle$ is a central CNZ ring for any positive integer n.*

For a ring R and for a central unit $r \in R$, define

$$R_r = \left\{ \begin{pmatrix} a & b \\ 0 & a + rb \end{pmatrix} \mid a, b \in R \right\}.$$

Consider R_r as a subring of $M_2(R)$. When α is an endomorphism of R with $\alpha(r) = r$, the restriction of $\bar{\alpha}$ to R_r gives rise to an endomorphism of R_r.

Proposition 3.10. *If a ring R, with an endomorphism α where $\alpha(r) = r$ for a central unit r is right α-skew central CNZ, then R_r is also right $\bar{\alpha}$-skew central CNZ, and vice versa.*

Proof. We first establish the following Claim.

Claim1. $Z(R_r) = \left\{ \begin{pmatrix} a & b \\ 0 & a + rb \end{pmatrix} \mid a, b \in R \right\}$.

proof Claim 1. If a and b be elements of the center of R $(Z(R))$, then for any x and y in R, the following holds: Let $a, b \in Z(R)$. Then for any $x, y \in R$, we have

$$\begin{pmatrix} a & b \\ 0 & a + rb \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x + ry \end{pmatrix}$$

$=$

$$= \begin{pmatrix} ax & ay+b(x+ry) \\ 0 & (a+rb)(x+ry) \end{pmatrix}$$
$$= \begin{pmatrix} xa & xb+y(a+rb) \\ 0 & (x+ry)(a+rb) \end{pmatrix}$$
$$= \begin{pmatrix} x & y \\ 0 & x+ry \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix}.$$

as $r \in Z(R)$ and $a, b \in N(R)$. This gives:

$$\begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix} \in Z(R_r).$$

Then for any $m \in R$, we have:

$$\begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix}.$$

This gives $am = ma$ and $bm = mb$, leading to $a, b \in Z(R)$. Hence
$Z(R_r) = \left\{\begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix} \mid a, b \in R\right\}$.

Claim2. If for $a, b \in N(R)$, $a, a+rb \in Z(R)$, then $b \in Z(R)$.

Proof of Claim 2. Since $Z(R)$ is a subring of R, take any $a, b \in N(R)$, such that a and $a+rb$ belong to $Z(R)$, we can conclude that $rb = (a+rb) - a \in Z(R)$.

$$rb = (rb)m = m(rb) = r(mb),$$

as $r \in Z(R)$. Furthermore, as r is unit, it follows that $br = rb$, and this implying that b belongs to $Z(R)$. Suppose R is a right α-skew central CNZ and let $A, B \in R_r$ such that $AB = 0$, we have

$$A = \begin{pmatrix} a & b \\ 0 & a+rb \end{pmatrix}$$

and

$$B = \begin{pmatrix} x & y \\ 0 & x+ry \end{pmatrix},$$

for $a, b, x, y \in N(R)$.

Therefore, $(a+rb)(x+ry) = 0$ and $ax = 0$. Since R is right α-skew central CNZ, thus $(x+ry)(\alpha(a) + r\alpha(b)) = (x+ry)\alpha(a+rb) \in Z(R)$. and $x\alpha(a) \in Z(R)$.

$$B\bar{\alpha}(A) = \begin{pmatrix} x & y \\ 0 & x+ry \end{pmatrix}\begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a)+r\alpha(b) \end{pmatrix}$$
$$= \begin{pmatrix} x\alpha(a) & (x+ry)\alpha(b)+y\alpha(a) \\ 0 & (x+ry)(\alpha(a)+r\alpha(b)) \end{pmatrix} \in Z(R_r).$$

So, R_r is also right $\bar{\alpha}$-skew central CNZ. Now, consider $a, b \in N(R)$ such that $ab = 0$. Then

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in N(R_r),$$

such that $AB = 0$. under the assumption that, $B\alpha(A) \in Z(R_r)$. Hence R_r is right α-skew central CNZ. □

Corollary 3.11. *A ring R is considered to be central CNZ if and only if, for any central unit r in R, the ring R_r retains the property of being central CNZ.*

4. Extensions of Right α-Skew Central CNZ Rings

Let R be a ring and S be a subring of R . We define $T[R,S]$ as the set $\{(r_1, r_2, \ldots, r_n, s, s, \ldots)\}$ where each r_i belongs to R, s belongs to S, n is greater than or equal to 1, and i ranges from 1 to n. $T[R,S]$ is endowed with ring structure through component-wise addition and multiplication. Note that, the center of $T[R,S]$ denoted by $Z(T[R,S])$ is defined as the set $\{(r_1, r_2, \ldots, r_n, s, s, \ldots)\}$ where each r_i belongs to $Z(R)$, s belongs to $Z(S)$, n is greater than or equal to 1, and i ranges from 1 to n. For an endomorphism α where $\alpha(S)$ is contained within S, the mapping $\bar{\alpha} : T[R,S] \to T[R,S]$ is defined as follow:

$$\bar{\alpha}((r_1, r_2, \ldots, r_n, s, s, \ldots)) = (\alpha(r_1), \alpha(r_2), \ldots, \alpha(r_n), \alpha(s), \alpha(s), \ldots.)$$

Induces an endomorphism of $T[R,S]$.

In the upcoming proposition, we outline the conditions that are necessary and sufficient for $T[R,S]$ to be classified as a right α-skew central CNZ.

Proposition 4.1. *Consider a ring R and a subring S of R, sharing the same identity as R, along with an endomorphism α of R such that $\alpha(S)$ is a subset of S. Then, the following statements are equivalent:*
(1) $T[R,S]$ *is a right $\bar{\alpha}$-skew central CNZ,*
(2) *Both R and S are right α-skew central CNZ.*

Proof. (1) $\Rightarrow$ (2) Suppose a and b are nilpotent elements in a ring R, with the property $ab = 0$. Let $A = (a, 0, 0, 0, \ldots)$, $B = (b, 0, 0, 0, \ldots)$. It follows that both A and B are nilpotent elements in $T[R,S]$. Consequently, $b\alpha(a)$ is central in R, rendering R a right α-skew central CNZ. Consider s and t as nilpotent elements in S such that $st = 0$. Let $X = (0, s, s, s, \ldots)$, $Y = (0, t, t, t, \ldots) \in T[R,S]$. Consequently, X and Y belong to the set of nilpotent elements $N(T[R,S])$, and their product $XY = 0$. Additionally, as $T[R,S]$ is a right $\bar{\alpha}$-skew central CNZ, $\bar{\alpha}(Y)X$ is central. This implies that $t\alpha(s)$ is central in S, leading to the conclusion that S is a right α-skew central CNZ.

(2) $\Rightarrow$ (1) Consider $A = (a_1, a_2, \ldots, a_n, b, b, b, \ldots)$ and $B = (b_1, b_2, \ldots, b_m, d, d, \ldots)$ in $T[R,S]$ both being nilpotent with $AB = 0$. As a consequence, all components of A and B are nilpotent. Assuming $n \leqslant m$, we observe $a_ib_i = 0$ for $1 \leqslant i \leqslant n$, implying $b_i\alpha(a_i) \in Z(R)$ due to R being a right α-skew central CNZ. Furthermore, $\alpha(S) \subseteq S$ signifies that S is also a right α-skew central CNZ. consequently, $r\alpha(s) \in Z(S)$ which implies $B\bar{\alpha}(A) \in Z(T[R,S])$.

□

Corollary 4.2. *If S is a subring of a ring R, both R and S are central CNZ if and only if $T[R,S]$ is central CNZ for $n \geqslant 1$.*

Now, we show that for right α-skew central CNZ ring R , polynomial rings (power series rings) need not be right $\bar{\alpha}$-skew central CNZ, as Illustrated by the subsequent example.

Example 4.3. We modify the ring discussed in [16, Example 2.8], derived from [18, Example 2.1]. Retaining the same A, we introduce an automorphism δ of A defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively.

Consider the set C, containing all polynomials in A with zero constant terms. We consider the ideal I of A generated by

$$
\begin{aligned}
&a_0b_0, a_0b_1+a_1b_0, a_0b_2+a_1b_1+a_2b_0, a_1b_2+a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\
&b_0a_0, b_0a_1+b_1a_0, b_0a_2+b_1a_1+b_2a_0, b_1a_2+b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\
&(a_0+a_1+a_2)r(b_0+b_1+b_2), (b_0+b_1+b_2)r(a_0+a_1+a_2),\\
&a_0a_0, a_2a_2, a_0ra_0, a_2ra_2, b_0b_0, b_2b_2, b_0rb_0, b_2rb_2, r_1r_2r_3r_4,\\
&a_0a_1+a_1a_0, a_0a_2+a_1a_1+a_2a_0, a_1a_2+a_2a_1, b_0b_1+b_1b_0, b_0b_2+b_1b_1+b_2b_0, b_1b_2+b_2b_1,\\
&(a_0+a_1+a_2)r(a_0+a_1+a_2), (b_0+b_1+b_2)r(b_0+b_1+b_2),
\end{aligned}
$$

where $r, r_1, r_2, r_3, r_4 \in C$. Certainly, C^4 is contained within the ideal I ($C^4 \subseteq I$). Let R be the quotient ring A/I. As $\delta(I) \subseteq I$, we can define an automorphism α of R by mapping $\alpha(s+I) = \delta(s)+I$ for $s \in A$. For simplicity, we represent each element of A by its corresponding image in R.

For $p(x) = a_0 + a_1x^2 + a_2x^4, q(x) = b_0c + b_1cx^2 + b_2cx^4 \in N(R[x;\alpha])$ since $C^4 \subseteq I$, we have $p(x)q(x) \in I[x]$, however $q(x)\bar{\alpha}(p(x)) = (b_0c + b_1cx^2 + b_2cx^4)(b_0 + b_1x^2 + b_2x^4) \nsubseteq Z(R[x])$ since $b_0cb_1 + b_1cb_0$is not central. Thus R[x] is not right $\bar{\alpha}$-skew central CNZ.

Now, we demonstrate that R is a right α-skew central CNZ ring. A monomial typically denotes a product involving the indeterminates $a_0, a_1, a_2, b_0, b_1, b_2$ and c. and a monomial of degree n denotes a product involving precisely n indeterminates. Let H_n represent the set of all linear combinations of such monomials of degree n over $\mathbb{Z}_2$.Note that for any n the set H_n is finite. Moreover, the ideal I of R is homogeneous, signifying that if $\sum_{i=1}^{s} r_i \in I$ with each r_i belonging to H_i then every r_i is also a member of I.

We apply the argument in [18, Example 2.1] in the following

`Claim 1.` If $f_1g_1 \in I$ for $f_1, g_1 \in H_1$ then $g_1\alpha(f_1) \in I$.

Proof. Suppose that $f_1g_1 \in I$ for $f_1, g_1 \in H_1$. According to the definition of I, we are limited to the following scenarios:

$$
\begin{gathered}
(f_1 = a_0, g_1 = b_0), (f_1 = a_2, g_1 = b_2)\\
(f_1 = a_0, g_1 = a_0), (f_1 = a_2, g_1 = a_2)\\
(f_1 = b_0, g_1 = a_0), (f_1 = b_2, g_1 = a_2),\\
(f_1 = b_0, g_1 = b_0), (f_1 = b_2, g_1 = b_2),\\
(f_1 = a_0+a_1+a_2, g_1 = b_0+b_1+b_2), (f_1 = b_0+b_1+b_2, g_1 = a_0+a_1+a_2),\\
(f_1 = a_0+a_1+a_2, g_1 = a_0+a_1+a_2), (f_1 = b_0+b_1+b_2, g_1 = b_0+b_1+b_2).
\end{gathered}
$$

So we get `Claim 1`, using the definition of I again.

`Claim 2.` If $fg \in I$ for each f and g in A, then $g\alpha(f) \in Z(R)$.

Proof. Let $f = f_1 + f_2 + f_3 + f_4$ and $g = g_1 + g_2 + g_3 + g_4$, where $f_1, g_1 \in H_1$, $f_2, g_2 \in H_2$, $f_3, g_3 \in H_3$, and $f_4, g_4 \in I$. Note that $H_i \subseteq I$ for $i \geqslant 4$ and $fg \in I$ implies $f_1g_1 \in I$ and $f_1g_2 + f_2g_1 \in I$ since I is homogeneous. By Claim1, $g_1\alpha(f_1) \in I$. We show that $g_1\alpha(f_2) + g_2\alpha(f_1) \in I$, we have the following cases:

$$
\begin{gathered}
(f_1 = a_0, g_1 = b_0), (f_1 = a_2, g_1 = b_2)\\
(f_1 = a_0, g_1 = a_0), (f_1 = a_2, g_1 = a_2)\\
(f_1 = b_0, g_1 = a_0), (f_1 = b_2, g_1 = a_2),\\
(f_1 = b_0, g_1 = b_0), (f_1 = b_2, g_1 = b_2),\\
(f_1 = a_0+a_1+a_2, g_1 = b_0+b_1+b_2), (f_1 = b_0+b_1+b_2, g_1 = a_0+a_1+a_2),\\
(f_1 = a_0+a_1+a_2, g_1 = a_0+a_1+a_2), (f_1 = b_0+b_1+b_2, g_1 = b_0+b_1+b_2).
\end{gathered}
$$

If $f_2, g_2 \in I$ then we get the result, Hence, we need to examine other situations for f f_2 and g_2. When $f_1 = a_0, g_1 = b_0$, we get the following cases:

$$(f_2 = sh, g_2 = kt), (f_2 = hs, g_2 = tk)$$
$$(f_2 = hs, g_2 = kt), (f_2 = sh, g_2 = tk)$$
$$(f_2 = sh, g_2 \in I), (f_2 = hs, g_2 \in I)$$
$$(f_2 \in I, g_2 = kt), (f_2 \in I, g_2 = tk).$$

where $h, k \in \{a_0, b_0\}$ and each of s, t is a sum of monomials of degree 1. Then $g_1\alpha(f_1), g_2\alpha(f_1) + g_1\alpha(f_2) \in I$.

The computations of other cases are almost same. Thus $g\alpha(f) \in I$.

Now let $yz \in I$ for $y, z \in A$. Write $y = u + y', z = v + z'$ for some $u, v \in \mathbb{Z}_2$ and some $y', z' \in C$. So $uv + uz' + y'v + y'z' = yz \in I$; hence $u = 0$ or $v = 0$.

Assume $u = 0$. Then $y'v + y'z' \in I$ and $y'z' \in I$ as I is homogenous and $v = 1$ then $y' \in Z_2$, so by claim 2 $z\alpha(y) = z\alpha(y') \in Z(R)$ and consequently, $z\alpha(y) \in Z(R)$. If $v = 0$, then $y'z' \in Z(R)$ and so $z\alpha(y) = z'\alpha(y') \in Z(A)$ by `Claim 2`.

The proof for the case when $v = 0$ follows a similar reasoning. Thus, R is confirmed to be a right α-skew central CNZ ring

A ring R is termed *power-serieswise Armendariz* as defined by [17], if for every pair of power series $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0} b_j x^j$ in $R[[x]]$, their product $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. It's important to note that while power-serieswise Armendariz rings are indeed Armendariz rings by definition, the converse is not necessarily true by [17, Example2.4].

Lemma 4.4. *Let* R *be a power-serieswise-Armendariz ring and* α *an endomorphism of* R.

(1) R *is central CNZ if and only if* R *is right* α*-skew central CNZ.*

(2) *When* S *represents any of the following:* $[x], [x, x^{-1}], [[x]]$ *or* $[[x, x^{-1}]]$, *the set of nilpotent elements in* RS *equals the product of the nilpotent elements in* R *and* S $(N(RS) = N(R)S)$, *in case* α *is an automorphism.*

Proof. (1) In a central CNZ ring R, if the product of two nilpotent elements a and b is zero, then under an endomorphism α, the product $\alpha(a)b = 0$ is also zero by [14, Theorem 3.3(3)] and so $b\alpha(a) \in Z(R)$ since R is central CNZ. Thus R is right α-skew central CNZ. Conversely, suppose that R is right α-skew central CNZ and let $ab = 0$ for $a, b \in N(R)$. Then $b\alpha(a) \in Z(R)$ and so $ba \in Z(R)$ by [14, Theorem 3.3(3)]. Hence R is central CNZ.

(2) It directly follows from [3, Theorem 2.13].

□

Theorem 4.5. *Suppose* R *is a power-serieswise Armendariz ring. The following statements are equivalent*

(1) *The ring* R *is right* α*-skew central CNZ.*

(2) *The polynomial ring* $R[x]$ *is right* $\bar{\alpha}$*-skew central CNZ.*

(3) *The Laurent polynomial ring* $R[x, x^{-1}]$ *is a right* $\bar{\alpha}$*-skew central CNZ.*

(4) *The power series ring* $R[[x]]$ *is right* $\bar{\alpha}$*-skew central CNZ.*

(5) *The Laurent power series ring* $R[[x, x^{-1}]]$ *is right* $\bar{\alpha}$*-skew central CNZ.*

Proof. It is enough to show that (1)⇒(5): Assume that (5) holds. Consider $p(x)q(x) = 0$ where $p(x) = \sum_{i=0}^{\infty} a_i x^i$ and $q(x) = \sum_{j=0}^{\infty} b_j x^j \in N(R[[x, x^{-1}]])$. Using Lemma 4.4(2), we find that $a_i, b_j \in N(R)$ and thus $a_i b_j = 0$ for all i and j. Consequently, because R is α-compatible, we have $b_j a_i \in Z(R)$. This leads to $q(x)\bar{\alpha}(p(x)) \in Z(R[[x, x^{-1}]])$, demonstrating that $R[[x, x^{-1}]]$ is right $\bar{\alpha}$-skew central CNZ.

□

cinsider a R with a monomorphism α . We now explore the Jordan's construction, introduced [7], which involves creating an over-ring of R using α. Denote this over-ring as $A(R,\alpha)$, defined as the set of elements of the form $\{x^{-i}rx^{i} \mid r \in R \text{ and } i \geqslant 0\}$ within the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Note that for $j \geqslant 0$, $x^j r = \alpha^j(r)x^j$ implies $rx^{-j} = x^{-j}\alpha^j(r)$ for $r \in R$. This yields that for each $j \geqslant 0$ we have $x^{-i}rx^{i} = x^{-(i+j)}\alpha^j(r)x^{i+j}$. Hence, $A(R,\alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following operations:

(1) addition: $x^{-i}rx^{i} + x^{-j}sx^{j} = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$.

(2) Multiplication: $(x^{-i}rx^{i})(x^{-j}sx^{j}) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$.

$r, s \in R$ and $i, j \geqslant 0$. Note that $A(R,\alpha)$ acts as an over-ring of R. The mapping $\bar{\alpha} : A(R,\alpha) \to A(R,\alpha)$ defined as $\bar{\alpha}(x^{-i}rx^{i}) = x^{-i}\alpha(r)x^{i}$ constitutes an automorphism of $A(R,\alpha)$. Jordan's work in 1982 [7] illustrated that, utilizing left localization of the skew polynomial $R[x;\alpha]$ concerning the set of powers of x, that for any given pair (R,α), The extension $A(R,\alpha)$ is always guaranteed to exist . This ring, denoted as $A(R,\alpha)$, is commonly known as the Jordan extension of R by α.

Proposition 4.6. *For a ring R equipped with a monomorphism α, R satisfies the property of being right α-skew central CNZ if and only if it's Jordan extension $A = A(R,\alpha)$ under α also adheres to the condition of being right $\bar{\alpha}$-skew central CNZ.*

Proof. The prove is started with two claims.

Claim 1. If $r \in Z(R)$, then $\alpha^r \in Z(R)$, for all $r \geqslant 1$.
proving that $\alpha(r) \in Z(R)$ is sufficient.

$$\alpha(r)x = \alpha(rx_j) = \alpha(x_j r) = \alpha(x_j)\alpha(r) = x\alpha(r).$$

Leads to $\alpha(r) \in Z(R)$.

Claim 2. Let $r \in Z(R)$ and $x^{-j}sx^{j} \in N(A)$ where s is in $N(R)$. According to Claim 1, we get:

$$(x^{-i}rx^{i})(x^{-j}sx^{j}) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} = x^{-(i+j)}(\alpha^i(s)\alpha^j(r))x^{(i+j)} = (x^{-j}sx^{j})(x^{-i}rx^{i}),$$

implies to $x^{-i}rx^{i} \in Z(A(R,\alpha))$.

If R adheres to being right α-skew central CNZ and $AB = 0$ for $A = x^{-i}rx^{i}$ and $B = x^{-j}sx^{j}$ in $N(A)$ with $i, j \geqslant 0$ and $r, s \in N(R)$. From the condition $AB = 0$, we deduce $\alpha^j(r)\alpha^i(s) = 0$ and thus $0 = \alpha^i(s)\alpha(\alpha^j(r)) = \alpha^i(s)\alpha^{j+1}(r)$ by assumption. Hence

$$\begin{aligned} d\bar{\alpha}(c) &= (x^{-j}sx^{j})\bar{\alpha}(x^{-i}rx^{i}) = (x^{-j}sx^{j})(x^{-i}\alpha(r)x^{i}) \\ &= x^{-(j+i)}\alpha^i(s)\alpha^j(\alpha(r))x^{i+j} = x^{-(j+i)}\alpha^i(s)\alpha^{j+1}(r))x^{i+j} \in Z(A(R,\alpha)). \end{aligned}$$

Therefore the Jordan extension $A = A(R,\alpha)$ is right $\bar{\alpha}$-skew central CNZ. The conserve is obvious since right α-skew central CNZ rings are closed under subrings.

□

In the context where R acts as an algebra over a commutative ring S, due to [11], the *Dorroh extension* of R by S can be described as the Abelian group $R \times S$. Here, the multiplication operation is defined as $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where r_i belongs to R and s_i belongs to S. We denote the Dorroh extension of R by S as D. In this framework, considering an S-endomorphism α of R, the mapping $\bar{\alpha} : D \to D$ is defined as $\bar{\alpha}(r, s) = (\alpha(r), s)$, is an S-algebra homomorphism.

Theorem 4.7. *Given that R is an algebra over a commutative reduced ring Z with a Z-endomorphism α, R being a right α-skew central CNZ ring is equivalent to the right Dorroh extension D of R by Z being a right $\bar{\alpha}$-skew central CNZ.*

Proof. It's straightforward to verify that $N(D) = (N(R), 0)$ as Z is a commutative reduced ring. This implies that every nilpotent element in D can be represented as $(r, 0)$ for some nilpotent element r of R.Therefore:

$$(r_1, 0)(r_2, 0) = (0, 0) \text{ if and only if } r_1 r_2 = 0,$$

since R is right α-skew central CNZ, $r_2\alpha(r_1) \in Z(R)$. For any $(r, s) \in D(R, Z)$ we have:

$$\begin{aligned}(r_2, 0)(\bar{\alpha}((r_1, 0))(r, s) = (r_2, 0)(\alpha(r_1), 0)(r, s) &= (r_2\alpha(r_1)r + sr_2\alpha(r_1), 0)\\ &= (r(r_2\alpha(r_1)) + s(r_2\alpha(r_1), 0)\\ &= (r, s)(r_2, 0)(\alpha(r_1), 0)\\ &= (r, s)(r_2, 0)(\bar{\alpha}(r_1, 0)).\end{aligned}$$

So D is right $\bar{\alpha}$-skew central CNZ. For sufficiency , let $r_1, r_2 \in N(R)$ with $r_1 r_2 = 0$,. This implies $(r_1, 0)(r_2, 0) = (0, 0)$ where $(r_1, 0), (r_2, 0)$ are nilpotents of D, by hypothesis $(r_2, 0), (\alpha(r_1), 0) \in Z(D)$, and so $r_2, \alpha(r_1) \in Z(R)$. Therefore R is right α-skew central CNZ.

□

Conclusion

In this paper, the notion of right α-skew central CNZ is introduced. Right α-skew central Property is generalized the notion of right α-skew central reversible in [5] and extended the notion of right α-skew CNZ published in [2]. Additionally, This work specified several characteristics and extensions of right α-skew central CNZ. Besides the relation between right α-skew central CNZ and other rings was also examined. Ultimately, it demonstrated that for right α-skew central CNZ ring R , polynomial rings (power series rings) need not be right $\bar{\alpha}$-skew central CNZ, as Illustrated by the subsequent in Example 4.3.

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