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A series of new formulas to approximate the Sine and Cosine functions

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Abstract

In our analysis, we approximate the sine and cosine trigonometric functions within the interval $\left[0, \frac{\pi}{2}\right]$. This examination yields two distinct formulas for approximating sine and cosine. Initially, we endeavor to derive the formula that involves a square root, and then we develop an alternative formula that does not require any use of a square root. However, even after establishing the square root-free procedure, we seek to enhance its accuracy and subsequently derive yet another formula for more precise approximations of trigonometric functions within the interval $\left[0, \frac{\pi}{2}\right]$. Thus, our analysis presents two primary procedures. One is utilizing square roots, and the other abstaining from them. Our focus extends to ensuring the accuracy of these trigonometric functions specifically within the interval $\left[0, \frac{\pi}{2}\right]$. This precision assessment is visually represented through graphs, illustrating the disparity between the values generated by the formulated functions and the exact values of these functions. Consequently, these graphs serve as indicators of the error within the interval $\left[0, \frac{\pi}{2}\right]$. Finally, we conduct a comparative analysis between our approximation and the 7th-century Indian mathematician Bhaskara I's approximation formula.

Keywords: Sine, Cosine, Bhaskara Is formula.
2020 MSC: MSC 00A05, MSC 00A22, MSC 00A99.

1. Introduction

Trigonometry is a branch of mathematics that works on the relationship between the side lengths and angles of triangles. Trigonometry is highly beneficial across diverse fields, including surveying and navigation. Six fundamental functions characterize this mathematical discipline, each serving as essential tools in applications ranging from practical measurements to complex calculations. These functions mainly relate the angle of a right-angled triangle to ratios of two side lengths. The most widely used of these functions are sine, cosine, and tangent. The cosecant, secant, and cotangent are their reciprocals, respectively. This article mainly explains sine and cosine functions. The sine function means the ratio of the length of the opposite side to that of the hypotenuse in a right-angled triangle. The cosine function expresses the ratio of the adjacent leg to the hypotenuse angle, while the tangent function signifies the ratio of sine to cosine. Because the hypotenuse in a right-angled triangle is consistently weightier than

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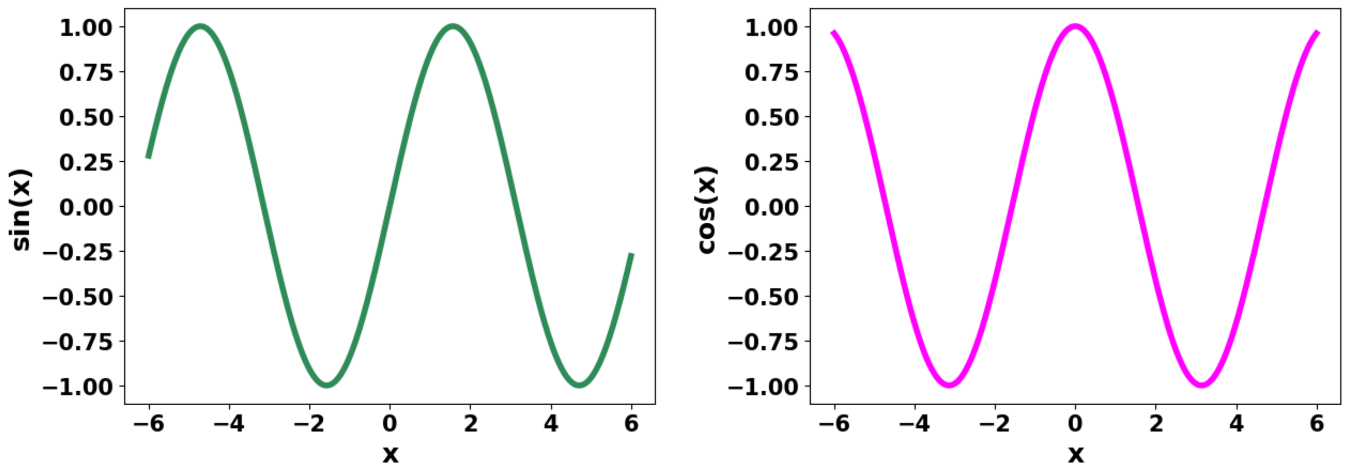
the other sides, both sine and cosine yield real numbers falling between 0 and 1. However, the tangent function can produce positive or negative real numbers because of its unique properties. Sine, cosine and tan are periodic functions, and the period of sine and cosine functions is 2π. Whether the value would be positive or negative depends on the quadrant. If we can calculate the values in one quadrant successfully, then we can quickly get the values in any quadrant. Thus, this paper shows a different approach to approximate sine and cosine on the interval $\left[0, \frac{\pi}{2}\right]$. Across the historical progression of trigonometry, numerous methods have emerged to calculate these functions more precisely. In computers, the Taylor series is used to calculate sine [1].



(a) Graphical presentation of $\sin x$. (b) Graphical presentation of $\cos x$.

Figure 1: Graphical representation of $\sin x$ and $\cos x$.

2. Derivation and Accuracy

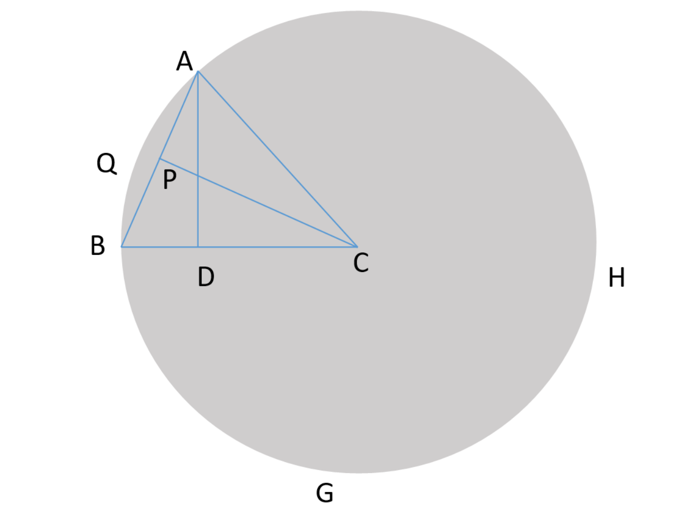


Figure 2: Geometrical structure and isosceles triangle.

Here, ABC is an isosceles triangle; AQBGH is a circle, and AC and BC are its radii. Let AC=BC=a, and AD is perpendicular to BC which means AD is the height of this triangle. Let $< ACB = x$ radian, the chord AB=k, the arc AQB=s, and AD=h. So, the length of the arc AQB= ax [2]. Now, s= ax. If we apply the cosine law [3],

$$AB = \sqrt{AC^2 + BC^2 - 2(AC)(BC)\cos x} = \sqrt{a^2 + a^2 - 2(a)(a)\cos x} = a\sqrt{2(1-\cos x)}.$$

Let $\frac{s}{k} = f \Rightarrow k = \frac{s}{f} \Rightarrow k = \frac{ax}{f}$. Here, PC is perpendicular to AB, and as it is an isosceles, AP=BP [4], and applying the Pythagorean theorem [5], $BP = \sqrt{BC^2 - CP^2} = \sqrt{a^2 - CP^2}$. Now,

$$AB = AP + BP \Rightarrow AB = BP + BP \Rightarrow AB = 2BP \Rightarrow 2BP = AB$$

$$\Rightarrow 2\sqrt{(a^2 - CP^2)} = AB \Rightarrow 4(a^2 - CP^2) = AB^2 \Rightarrow 4a^2 - 4CP^2 = k^2 \Rightarrow CP = \frac{\sqrt{4a^2 - k^2}}{2}.$$

Area of the isosceles $ABC = \frac{1}{2}AB.CP = \frac{1}{2}k\frac{\sqrt{4a^2-k^2}}{2} = \frac{k\sqrt{4a^2-k^2}}{4} = \frac{ax\sqrt{4a^2 - \frac{a^2x^2}{f^2}}}{4f}$ [as $k = \frac{ax}{f}$].

Also, the area of isosceles ABC is $\frac{BC.AD}{2}$ or $\frac{ah}{2}$ [6].

So, $\frac{ax\sqrt{4a^2 - \frac{a^2x^2}{f^2}}}{4f} = \frac{ah}{2} \Rightarrow \frac{x\sqrt{4a^2 - \frac{a^2x^2}{f^2}}}{4f} = \frac{h}{2} \Rightarrow \frac{x\sqrt{4a^2f^2 - a^2x^2}}{4f^2} = \frac{h}{2}$

$\Rightarrow \frac{ax\sqrt{4f^2-x^2}}{4f^2} = \frac{h}{2} \Rightarrow \frac{2x\sqrt{4f^2-x^2}}{4f^2} = \frac{h}{a} \Rightarrow \frac{x\sqrt{4f^2-x^2}}{2f^2} = \sin x.$ [according to Figure 2, $\frac{h}{a} = \frac{AD}{AC} = \sin x$]

Thus, $\sin x = \frac{x\sqrt{4f^2-x^2}}{2f^2}$.

Now,

$$f = \frac{s}{k} \Rightarrow f = \frac{ax}{a\sqrt{2(1-\cos x)}} \Rightarrow f = \frac{x}{\sqrt{2(1-\cos x)}}. \tag{2.1}$$

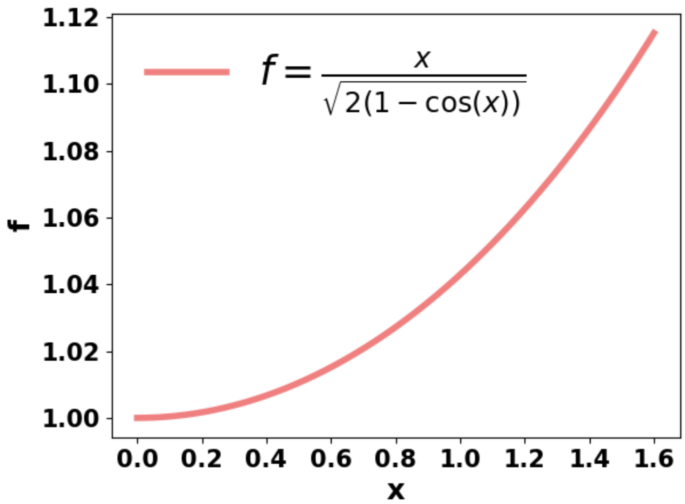


Figure 3: Graphical presentation of $f = \frac{x}{\sqrt{2(1-\cos x)}}$.

In Figure 3, the part of the graph where x is between 0 and $\frac{\pi}{2}$ radian can be considered as a part of a parabola [7, 10, 11]. The equation of that parabola would be like:

$$(x-0)^2 = m(y-1) \Rightarrow x^2 = my - m \Rightarrow y = \frac{x^2+m}{m}. \tag{2.2}$$

Where m is a constant. Now, it can be assumed that when x is between 0 and $\frac{\pi}{2}$ radian, $f \approx \frac{x^2+m}{m}$ because this parabola approximates the function of f on the interval $\left[0, \frac{\pi}{2}\right]$. Hence,

$$\sin x \approx \frac{x\sqrt{4\left(\frac{x^2+m}{m}\right)^2 - x^2}}{2\left(\frac{x^2+m}{m}\right)^2}. \tag{2.3}$$

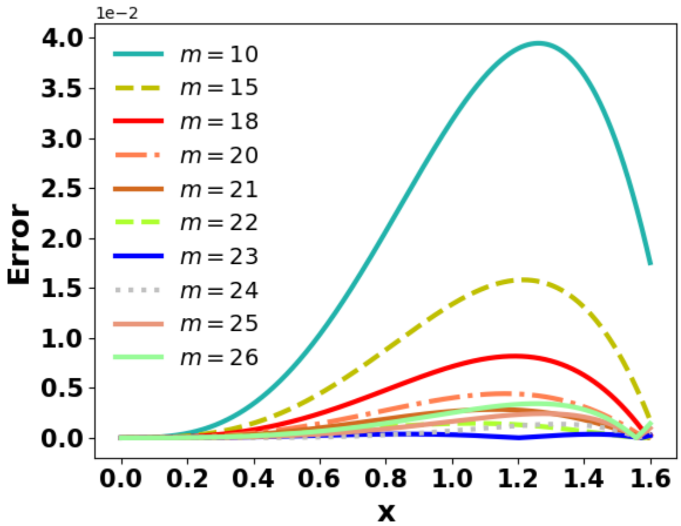


Figure 4: Absolute difference between exact $\sin x$ and Equation (2.3) for different values of m.

For m=23, it can be noted from Figure 4 that Equation (2.3) provides a relatively good estimate when the angle is between 0 and $\left[0, \frac{\pi}{2}\right]$ radian.

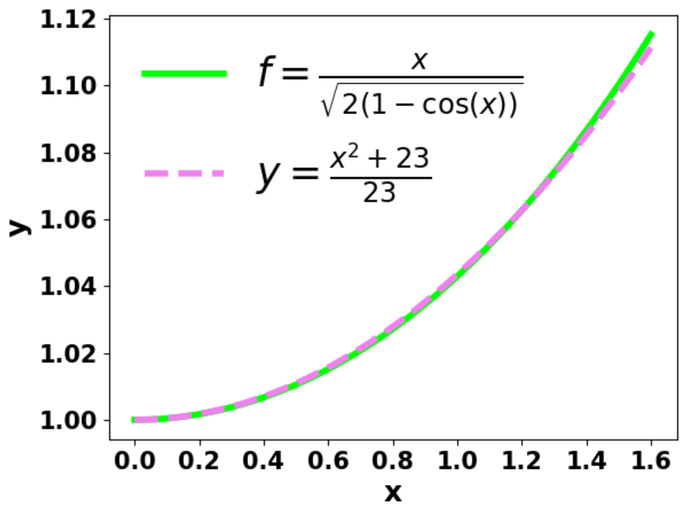
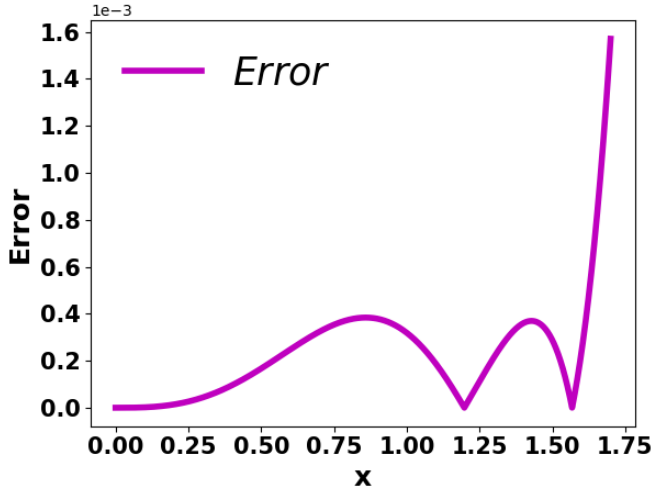


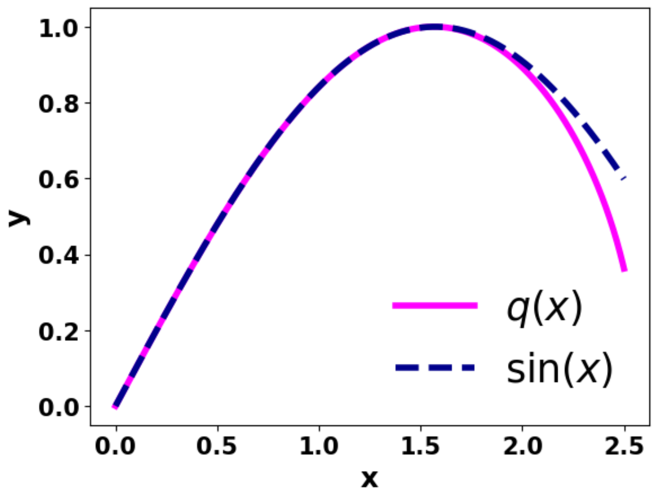
Figure 5: Comparison of $f = \frac{x}{\sqrt{2(1-\cos x)}}$ and $y = \frac{x^2+23}{23}$ on the interval $\left[0, \frac{\pi}{2}\right]$.

From Figure 5, $f \approx \frac{x^2+23}{23}$. Let $q(x) = \frac{x\sqrt{4(\frac{x^2+23}{23})^2 - x^2}}{2(\frac{x^2+23}{23})^2}$. Replacing x with $(\frac{\pi}{2} - x)$,

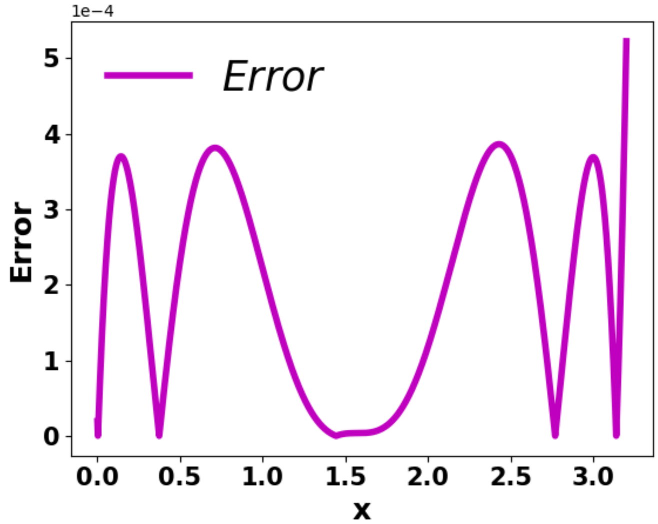
$$q(\frac{\pi}{2} - x) = \frac{23(\frac{\pi}{2} - x)\sqrt{4\big((\frac{\pi}{2} - x)^2 + 23\big)^2 - 529(\frac{\pi}{2} - x)^2}}{2\big((\frac{\pi}{2} - x)^2 + 23\big)^2}.$$



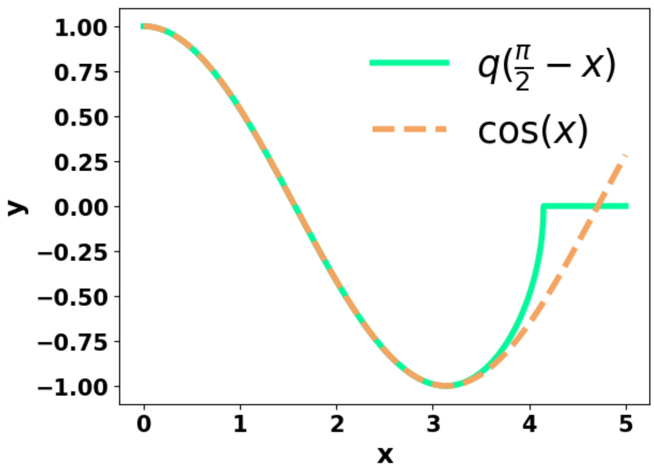
(a) The absolute difference between $q(x)$ and $\sin x$.



(b) An evaluation of $q(x)$ and $\sin x$.



(c) The absolute difference between $q(\frac{\pi}{2} - x)$ and $\cos x$.



(d) An evaluation of $q(\frac{\pi}{2} - x)$ and $\cos x$.

Figure 6: Absolute difference and an evaluation of $\sin x$ and $\cos x$.

From Figures 6a, 6b, 6c, and 6d, it is examined that the error is very negligible on the interval $[0, \frac{\pi}{2}]$.

So, $q(x) \approx \sin x$, and $q(\frac{\pi}{2} - x) \approx \cos x$. Hence,

$$\sin x \approx \frac{23x\sqrt{4(x^2+23)^2 - 529x^2}}{2(x^2+23)^2}. \tag{2.4}$$

$$\cos x \approx \frac{23(\frac{\pi}{2}-x)\sqrt{4\big((\frac{\pi}{2}-x)^2+23\big)^2-529(\frac{\pi}{2}-x)^2}}{2\big((\frac{\pi}{2}-x)^2+23\big)^2}. \tag{2.5}$$

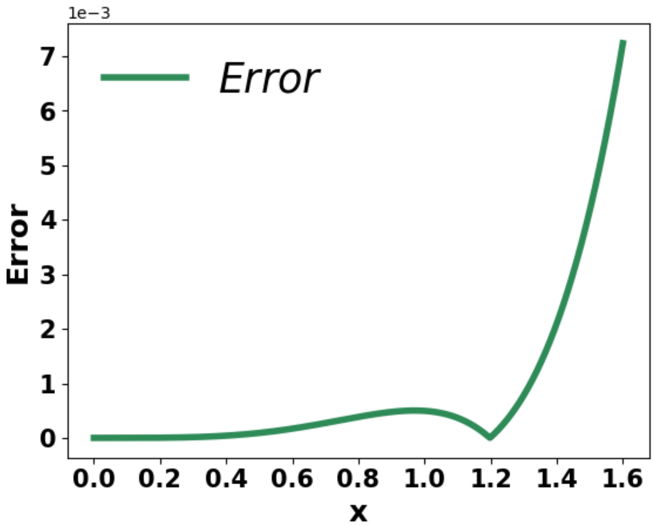
Now, $f \approx \frac{x^2+23}{23} \approx \frac{x}{\sqrt{2(1-\cos x)}}$. $\left[\text{Since } f = \frac{x}{\sqrt{2(1-\cos x)}}\right]$
$\Rightarrow \frac{x^2+23}{23} \approx \frac{x}{\sqrt{2(1-\cos x)}} \Rightarrow \frac{(x^2+23)^2}{529} \approx \frac{x^2}{2(1-\cos x)} \approx 1-\cos x \approx \frac{529x^2}{2(x^2+23)^2} \Rightarrow \cos x \approx 1-\frac{529x^2}{2(x^2+23)^2}$.

Thus,

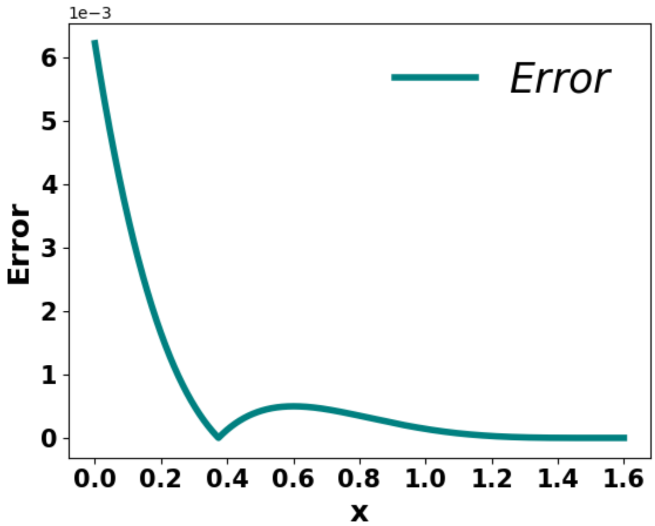
$$\cos x \approx 1-\frac{529x^2}{2(x^2+23)^2}. \tag{2.6}$$

Replacing x with $(\frac{\pi}{2}-x)$ gives,

$$\sin x \approx 1-\frac{529(\frac{\pi}{2}-x)^2}{2\big((\frac{\pi}{2}-x)^2+23\big)^2}. \tag{2.7}$$



(a) The absolute difference between $\cos x$ and Equation (2.6).

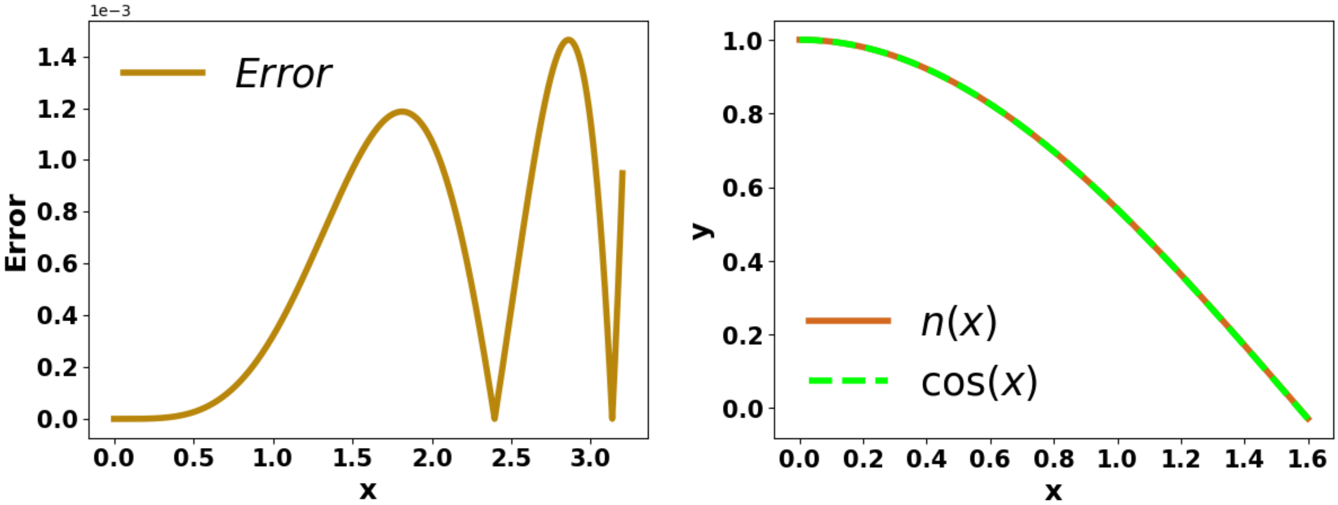


(b) The absolute difference between $\sin x$ and Equation (2.7).

Figure 7: Absolute difference of $\cos x$ and $\sin x$ with respect to Equations (2.6) and (2.7).

In Figure 7a, when x is between 0 and $\frac{\pi}{4}$ radian, Equation (2.6) provides a better approximation value. So, in order to approximate sine and cosine more accurately, we can use this formula: $\cos x = 2(\cos\frac{x}{2})^2-1$ [8, 13]. This formula is helpful because $\frac{\pi}{4} = \frac{1}{2}.\frac{\pi}{2}$. So, replacing x with $\frac{x}{2}$, $\cos\frac{x}{2} \approx 1-\frac{529(\frac{x}{2})^2}{2\big((\frac{x}{2})^2+23\big)^2}$. Since, $\cos x = 2(\cos\frac{x}{2})^2-1$, we get the following expression,

$$\cos x \approx 2\Big(1-\frac{1058x^2}{(x^2+92)^2}\Big)^2-1 = n(x). \tag{2.8}$$



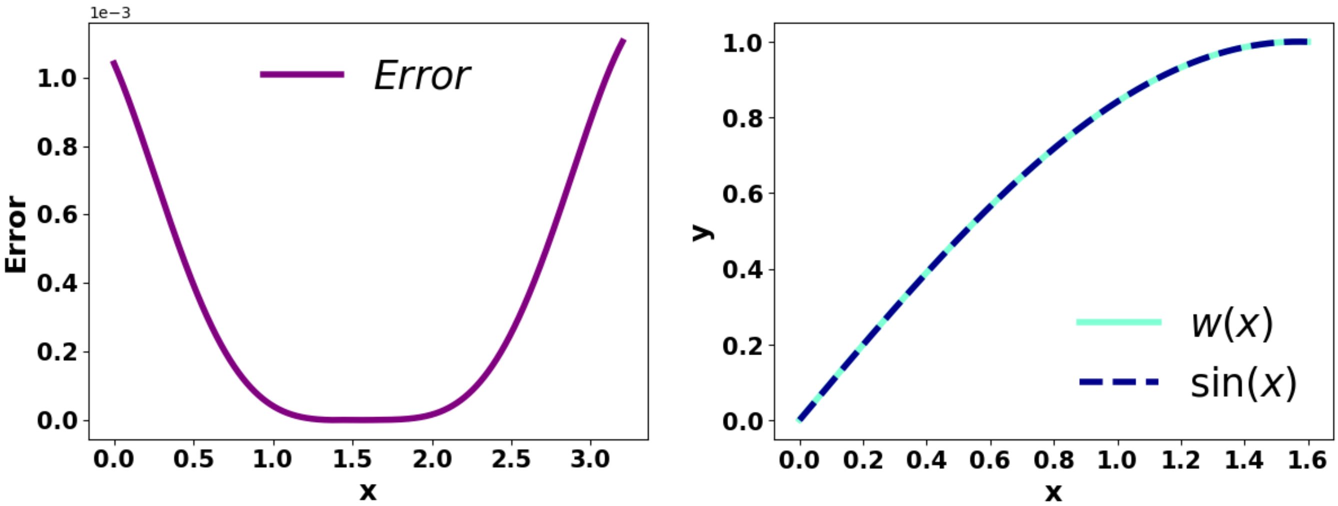
(a) The absolute difference between $\cos x$ and $n(x)$.

(b) An evaluation of $\cos x$ and and $n(x)$.

Figure 8: The absolute difference and an evaluation of $\cos x$ and Equation (2.8).

Replacing x with $(\frac{\pi}{2}-x)$,

$$\sin x \approx 2\Big(1-\frac{1058(\frac{\pi}{2}-x)^2}{((\frac{\pi}{2}-x)^2+92)^2}\Big)^2-1 = w(x). \tag{2.9}$$



(a) The absolute difference between $\sin x$ and $w(x)$.

(b) An evaluation of $\sin x$ and and $w(x)$.

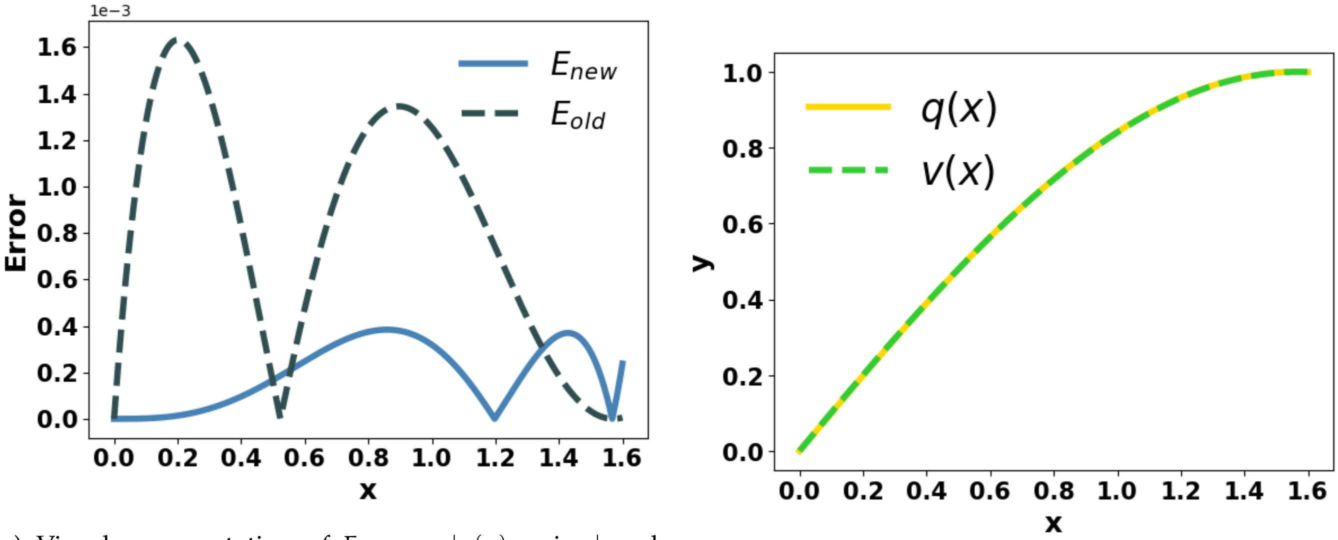
Figure 9: The absolute difference and an evaluation of $\sin x$ and Equation (2.9).

All these formulas approximate well when x is between 0 and $\frac{\pi}{2}$ radian.

3. Comparing with Bhaskara Is formula to approximate Sine

There is a formula of 7th century Indian Mathematician Bhaskara (c.600 - c.680) to approximate sine [9]. That formula is $\sin x \approx \frac{16x(\pi-x)}{5\pi^2-4x(\pi-x)}$, and Equation (2.4) is $\sin x \approx \frac{23x\sqrt{4(x^2+23)^2-529x^2}}{2(x^2+23)^2}$. Let

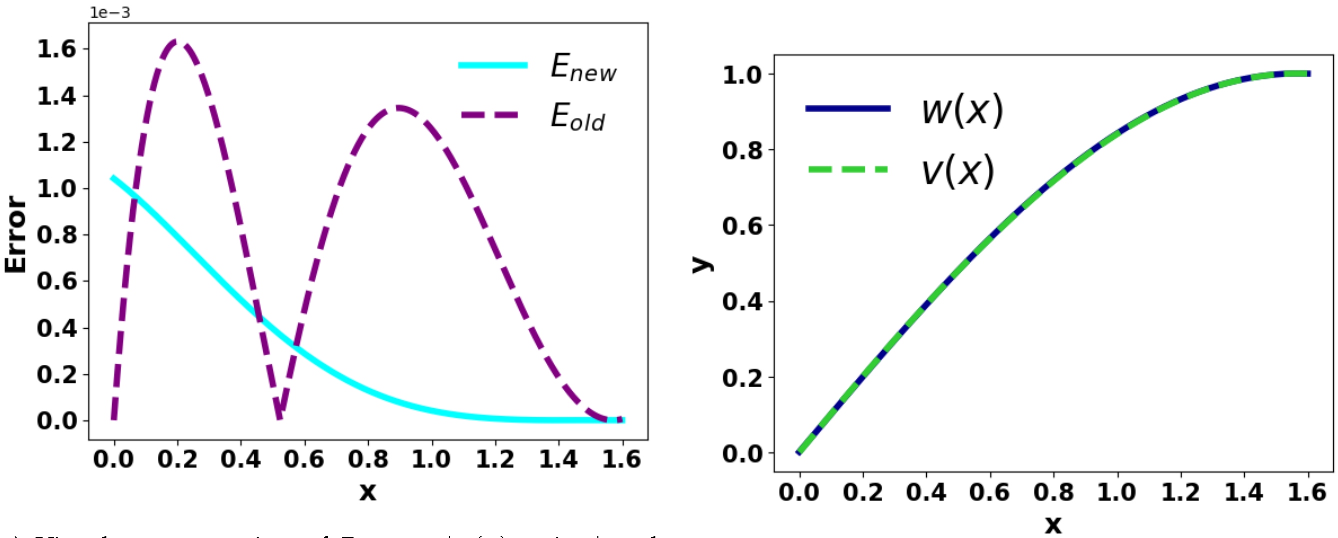
$$v(x) = \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}.$$



(a) Visual representation of $E_{\text{new}} = |q(x) - \sin x|$ and $E_{\text{old}} = |v(x) - \sin x|$.

(b) An evaluation of $q(x)$ and $v(x)$.

Figure 10: Absolute difference of $\sin x$ with respect to $q(x)$ and $v(x)$.



(a) Visual representation of $E_{\text{new}} = |w(x) - \sin x|$ and $E_{\text{old}} = |v(x) - \sin x|$.

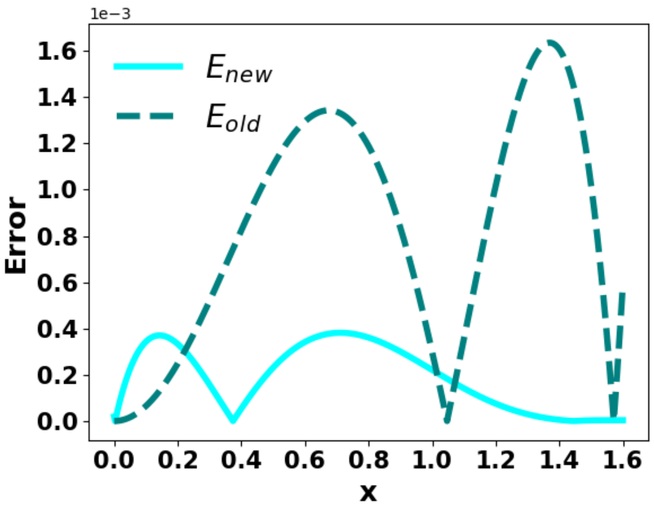
(b) An evaluation of $w(x)$ and $v(x)$.

Figure 11: Absolute difference of $\sin x$ with respect to $w(x)$ and $v(x)$.

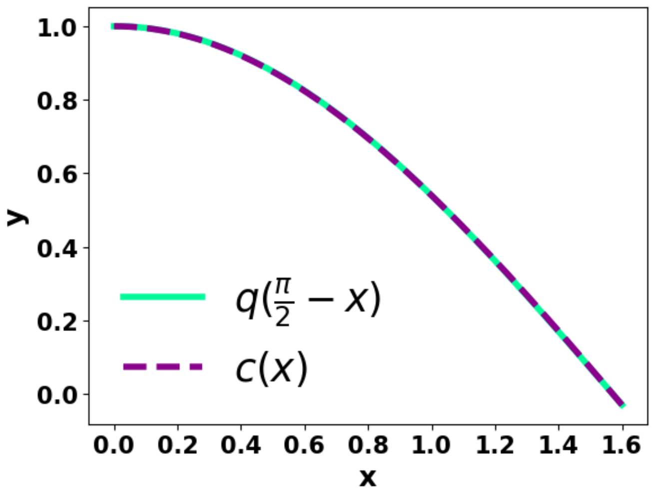
Figures 10a and 11a show that Equations (2.4) and (2.9) approximate Sine better than Bhaskara Is formula in many points, respectively.

4. Comparing with Bhaskara Is formula to approximate Cosine

The cosine approximation of Mathematician Bhaskara I [9] is $\cos x \approx \frac{\pi^2 - 4x^2}{\pi^2 + x^2}$ and Equation (2.5) is $\cos x \approx \frac{23(\frac{\pi}{2} - x)\sqrt{4\left((\frac{\pi}{2} - x)^2 + 23\right)^2 - 529(\frac{\pi}{2} - x)^2}}{2\left((\frac{\pi}{2} - x)^2 + 23\right)^2}$. Let $c(x) = \frac{\pi^2 - 4x^2}{\pi^2 + x^2}$.

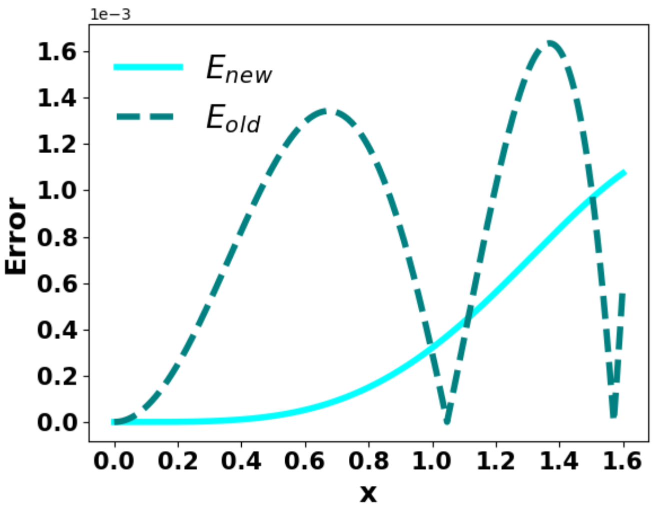


(a) Visual representation of $E_{new} = |q(\frac{\pi}{2} - x) - \cos x|$ and $E_{old} = |c(x) - \cos x|$.

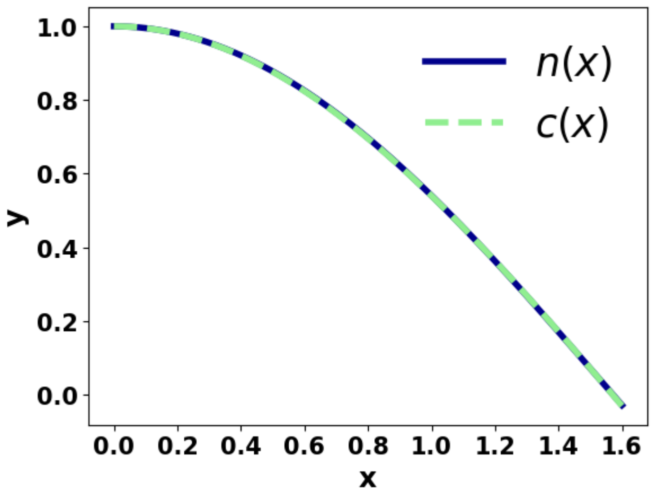


(b) An evaluation of $q(\frac{\pi}{2} - x)$ and $c(x)$.

Figure 12: Absolute difference of $\cos x$ with respect to $q(\frac{\pi}{2} - x)$ and $c(x)$.



(a) Visual representation of $E_{new} = |n(x) - \cos x|$ and $E_{old} = |c(x) - \cos x|$.



(b) An evaluation of $n(x)$ and $c(x)$.

Figure 13: Absolute difference of $\cos x$ with respect to $n(x)$ and $c(x)$.

According to Figures 12a and 13a, the cosine is more closely approximated by Equations (2.5) and (2.8) than by Bhaskara I's formula in many instances.

5. Conclusion

In this paper, we have derived two types of formulas to approximate sine and cosine. One type involves square root and another type does not require any use of square root. The formulas are

1. $\sin x \approx \dfrac{23x\sqrt{4(x^2+23)^2-529x^2}}{2(x^2+23)^2}$.
2. $\cos x \approx \dfrac{23(\frac{\pi}{2}-x)\sqrt{4\big((\frac{\pi}{2}-x)^2+23\big)^2-529(\frac{\pi}{2}-x)^2}}{2\big((\frac{\pi}{2}-x)^2+23\big)^2}$.
3. $\cos x \approx 1-\dfrac{529x^2}{2(x^2+23)^2}$.
4. $\sin x \approx 1-\dfrac{529(\frac{\pi}{2}-x)^2}{2\big((\frac{\pi}{2}-x)^2+23\big)^2}$.
5. $\cos x \approx 2\Big(1-\dfrac{1058x^2}{(x^2+92)^2}\Big)^2-1$.
6. $\sin x \approx 2\Big(1-\dfrac{1058(\frac{\pi}{2}-x)^2}{\big((\frac{\pi}{2}-x)^2+92\big)^2}\Big)^2-1$.

All the formulas approximate well when x is between 0 and $\frac{\pi}{2}$ radian. As sine and cosine are periodic functions, their value keeps getting the same after a certain angle, which is 2π radian. The value of both sine and cosine is positive when x is between 0 and $\frac{\pi}{2}$ radian, which means the first quadrant. Moreover, the value of sine and cosine is also positive when x is in the second and fourth quadrants, respectively. So, the value of sine is negative when x is in the third or fourth quadrant. Conversely, the value of cosine is negative when x is in the second or third quadrant. But as both are positive in the first quadrant, these approximations achieved in this paper can be considered as the approximation of their absolute value. So, if we use these formulas to approximate their absolute value and then consider the quadrant, we can get a proper approximation for any value of x by deciding whether it is negative or positive. Thus, we can use these formulas to approximate sine and cosine in any quadrant. Moreover, these formulas provide a more accurate value than Bhaskara Is formula in most of the cases on the interval $\big[0, \frac{\pi}{2}\big]$. Thus, these formulas can be used as a better substitution for the Mathematician Bhaskara Is formula. For this article, we have formulated a new series of expressions to approximate the trigonometric functions sine and cosine for the interval $\big[0, \frac{\pi}{2}\big]$ and conclude that those formulas provide a better approximation than the Bhaskara Is formula on that interval. In the future, such experiments can be extended to all generalized intervals and examine their behavior.

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Conflict of Interest

The authors declare no conflict of interest.

References

[1] V. Postolica, An introduction to trigonometry and its applications, Bentham Science Publishers (2018). 1

[2] Length of an arc:
https://www.google.com/url?sa=t&source=web&rct=j&url=https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-radians-2009-1.pdf&ved=2ahUKEwjP37HH1-n7AhWfVmwGHZiyAlo4KBAWegQIBBAB&usg=AOvVaw17NfJOZgHCXOgjj6rjxJHk 2

[3] J. C. Bansal, P. Bajpai, A. Rawat, and A. K. Nagar, Sine and cosine algorithm for optimization, Springer (2023). 2

[4] S. Grover, History of Development of Mathematics in India, Atma Ram & Sons, Delhi and Lucknow, 1994. 2

[5] Cosine Law:
https://www.google.com/url?sa=t&source=web&rct=j&url=https://personal.math.ubc.ca/~feldman/m100/trigId.pdf&ved=2ahUKEwjM-ouZyun7AhVMcGwGHV8WBxc4WhAWegQIBxAB&usg=AOvVaw348WZ_cZr9eKC7czlEX6zp 2

[6] E. Maor, The Pythagorean Theorem: A 4,000-Year History, Princeton University Press, (2007). 3

[7] D. Coray, C. Manoil, and J. Steinig, Notes on geometry and arithmetic, Springer (2020). 2

[8] H. S. M. Coxeter, Introduction to Geometry, 2^{nd} Edition, Wiley, (1991). 2

[9] J. R. Taylor, The calculus with analytical geometry handbook. ISBN: 978-0-930622-01-4. 2

[10] Introduction to Conics: Parabolas. https://www.cbsd.org/cms/lib/PA01916442/Centricity/Domain/2780/333202_1002_735-743.pdf 3, 4

[11] S. Gonzalez, Double-angle and half-angle identification.
https://www.alamo.edu/contentassets/35e1aad11a064ee2ae161ba2ae3b2559/analytic/math2412-double-angle-power-reducing-half-angle-identities.pdf 2

[12] K. Stroethoff, Bhaskaras approximation for the Sine, The Mathematics Enthusiast, 11(3), 2014. DOI: https://doi.org/10.54870/1551-3440.1313. 2

[13] S. L. Loney, The elements of coordinate geometry, Arihant Publications (I) Ltd, (2016). ISBN: 978-93-5176-223-2. 2