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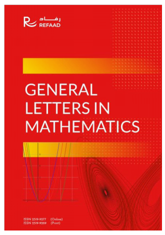

New explorations and remarkable inequalities related to *Fortune*'s conjecture and fortunate numbers

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Abstract

***Fortune*'s conjecture** (named after the social anthropologist *Reo Franklin Fortune*) is an extremely elegant mathematical conjecture that always remains an open problem in number theory. It is a conjecture about prime numbers, which leads to the so-called "fortunate numbers" (not to be confused with "lucky numbers"): *Reo F. Fortune* predicted that no fortunate number is composite. This conjecture impresses us all as mathematicians, that's why we decided that it will be the subject of this paper, which has many objectives and very interesting findings, among them:
- Highlighting numerous properties of the fortunate numbers.
- Giving a proof of ***Fortune*'s conjecture** in a particular case, by two different methods one of which is original.
- Presenting many counterexamples that reinforce the previous point, when the satisfied hypothesis in that particular case, is not met.
- Proving a new remarkable inequality that all the 3000 first (known until now) fortunate numbers perfectly fulfill.

Despite our continuous research (quite recently) on the subject of ***Fortune*'s conjecture**, we have never found a mathematical reference with a variety of ideas dealing with this conjecture, so we hope that this paper will be the first scientific work containing multiple ideas, comments, results and goals, and contributing significantly to find a definitive solution of ***Fortune*'s conjecture**, therefore this paper may advance the field..

Keywords: *Fortune*'s conjecture, fortunate numbers, prime factorials, the fundamental theorem of arithmetic, prime numbers, *Bertrand*'s postulate.
2020 MSC: 11-XX, 11-02, 11A41, 97Fxx, 11Axx.

Notations and terminology

In this paragraph, we introduce the notations required for the rest of the paper.
Let $n, r \in \mathbb{N}^*$. In all what follows, we will denote by:
a_n the n^{th} prime number, for example:
$a_1 = 2$, $a_2 = 3$, $a_3 = 5$, ...
$P_n = a_1.a_2.a_3...a_n$ the n^{th} primorial number, for example:
$P_1 = 2$, $P_2 = 6$, $P_3 = 30$, ...
And, we will put:
$I_{n,r} = \left]a_{n+r} + P_n, a_{n+r+1} + P_n\right[$.
$J_n = \left[a_{n+1} + P_n, a_{n+1}^2 + P_n\right[$.
$K_n =]1 + P_n, a_{n+1} + P_n[$.

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1. Introduction

Before getting to the subject of this article, we point out that it is impressive and surprising at the same time, that the author of this mathematical conjecture (*R. F. Fortune*: Figure 1) was a social anthropologist, psychologist, a specialist in Melanesian culture and language. However, it came in [1], that *R. F. Fortune* was for several years, fond of attending the *Cambridge* mathematics seminars. He was recognized for his study of number theory.

Figure 1: The newly married young anthropologists *R. F. Fortune* and *Margaret Mead, New Guinea*, in 1928[18]On the right**R. F. Fortune** with his wife *Margaret Mead, Sydney*[16]

It was stated in [8] that ***Fortune*'s conjecture** first appeared in the print edition of [7] in 1980 (after the death of *R. F. Fortune*), then it was discussed in more details and analysis in [9]. Furthermore, all fortunate numbers found manually or using high definition computers, are actually prime numbers, and as it was said in [8], many researchers in arithmetic acknowledged that it is unlikely that not all fortunate numbers will be prime numbers, so ***Fortune*'s conjecture** is generally thought to be true, and no counterexample has been found (to date) to disprove the rightness of this conjecture. Both amateur and professional mathematicians, researchers in arithmetic, combinatory and number theory, and computer scientists have contributed to checking or proving that all the fortunate numbers that have been found until this moment, are indeed prime numbers, consequently, these people and others, were mentioned in **"The On-line Encyclopedia of Integer Sequences"**, among them: *Joerg Arndt*, *Pierre Cami*, *Cyril Banderier*, *Chris K. Caldwell*, *Stan Wagon*, *Zhao Hui Du*, *Sean A. Irvine*, *Robert G. Wilson v.* and of course the famous researcher in combinatory and number theory *Neil James Alexander Sloane* (Figure 2) (aged of 83 years old) called **"the most influential mathematician alive"**, he was the founder of **"The On-line Encyclopedia of Integer Sequences"** (The **OEIS** for short), 59 years ago. In June 27, 2021, Russ Cox became the new president of **OEIS** Foundation, Neil James Alexander Sloane is now its Chairman (for further details, we refer the reader to [28]).

Despite all that, we note that there is a great dearth of mathematics references dealing with the subject of ***Fortune*'s conjecture**, apart from a few short papers as [6], modestly containing a few ideas that have not yet helped to find a solution to this conjecture, consequently, all mathematicians who are tempted to find a solution to ***Fortune*'s conjecture**, encounter great obstacles and difficulties, due to this paucity of mathematics references about this conjecture, so we seek, through this article, to create, from now on, a complete, basic and important reference, useful in practice for anyone who wants to learn about ***Fortune*'s conjecture** or dig deeper into this topic.

The main objectives of this paper are:

- Highlighting too many properties of the fortunate numbers.
- Giving a proof of ***Fortune*'s conjecture** in a particular case, by two different methods. The mathematical tools used in the second method, have never been used before (to our knowledge) to partially prove ***For-***

Figure 2: Neil James Alexander Sloane
[28]

***tune*'s conjecture**, so this method is **original**.
- Presenting numerous counterexamples that complement the second point (previously mentioned), when the satisfied hypothesis in that particular case, is not met.
- Proving a new remarkable inequality that all the 3000 first (known until now) fortunate numbers perfectly fulfill.
The basic design of this paper consists of 9 main parts, starting with a tiny introductory paragraph in which we adopt the mathematical symbols necessary to delve into our subject, and then a general introduction. Next, we will prove various mathematical inequalities that fortunate numbers fulfill. In a subsequent section, we will demonstrate a fundamental theorem which states that open intervals of the form $I_{n,r}$ (previously mentioned) do not contain any prime number when $a_{n+r+1} < a_{n+1}^2$, which leads us to the following result: If the n^{th} fortunate number is strictly less than a_{n+1}^2, then it is prime, therefore ***Fortune*'s conjecture** is right in this case. We come to another result in next sections, that is: If for every $n \in \mathbb{N}^*$, the interval J_n contains at least a prime number, thus ***Fortune*'s conjecture** will be proven, once and for all. Then, we prove a new remarkable inequality that all the 3000 first (known until now) fortunate numbers verify, and this is according to the results obtained by **"The On-line Encyclopedia of Integer Sequences"**. Before concluding, we give some examples related to the topic of this paper, and counterexamples showing that the open interval $I_{n,r}$ may contain prime numbers when $a_{n+r} > a_{n+1}^2$, without forgetting the appendix containing a numerical program in *Fortran* language (with some of its numerical results) which can determine the fortunate numbers and confirm that they are indeed prime. We point out that this program is simple, so it is not intended only for professionals in mathematics, but even novices in the field of computer programming or amateurs, can also benefit from it. We will end this paper with a conclusion in which we make additional comments and we summarize the various results that have been reached in this paper, as well as the prospects we hope to attain in the future in order to contribute to a solution of ***Fortune*'s conjecture**.

2. A technical definition (The fortunate number)

For a given positive integer n, the n^{th} fortunate number, denoted by F_n [31], named after the New Zealand-born *Reo Franklin Fortune* (March 27, 1903-November 25, 1979) [14, 30] is the smallest integer $m \geqslant 2$ such that: $P_n + m$ is a prime number, so:
$F_n = \min\Big\{ m \in \mathbb{N}^* : \ m \geqslant 2 \text{ and } (P_n + m) \text{ is prime} \Big\}$.

3. Approach (How to compute a fortunate number?)

To find the n^{th} fortunate number F_n:
Calculate the product of the first n prime numbers (primorial). Let this product be p, then find a prime number greater than p and return the difference between the found prime number and p, for example:
$P_1 + 3 = 5$ is a prime number, where $5 - P_1 = 3 = F_1$ is the first fortunate number.
$P_2 + 5 = 11$ is a prime number, where $11 - P_2 = 5 = F_2$ is the second fortunate number.
$P_3 + 7 = 37$ is a prime number, where $37 - P_3 = 7 = F_3$ is the third fortunate number.
$P_4 + 13 = 223$ is a prime number, where $223 - P_4 = 13 = F_4$ is the fourth fortunate number.
$P_5 + 23 = 2333$ is a prime number, where $2333 - P_5 = 23 = F_5$ is the fifth fortunate number.

3.1. *The rightness of inequalities* "$a_{n+1} \leqslant F_n < a_{n+1}^2$"$(1 \leqslant n \leqslant 200)$

In Table 1, we give the list of the 100 first fortunate numbers F_n computed by *Stan Wagon*, and in Table 2, we expose the fortunate numbers F_n for: $101 \leqslant n \leqslant 200$ computed by *Pierre Cami*. From these two tables, we remark that: $a_{n+1} \leqslant F_n < a_{n+1}^2$, $\forall n \in \mathbb{N}$: $1 \leqslant n \leqslant 200$, while Table 3 represents the list of the 500 first fortunate numbers computed by *Pierre Cami* [17].
By the way, the complete list contains 3000 fortunate numbers, and each n from this 3000 numbers verifies: $a_{n+1} \leqslant F_n < a_{n+1}^2$.

Table 1: **The rightness of inequalities "$a_{n+1} \leqslant F_n < a_{n+1}^2$" ($1 \leqslant n \leqslant 100$) [11], [17]**

n	a_{n+1}	F_n	a_{n+1}^2
1	3	3	9
2	5	5	25
3	7	7	49
4	11	13	121
5	13	23	169
6	17	17	289
7	19	19	361
8	23	23	529
9	29	37	841
10	31	61	961
11	37	67	1369
12	41	61	1681
13	43	71	1849
14	47	47	2209
15	53	107	2809
16	59	59	3481
17	61	61	3721
18	67	109	4489
19	71	89	5041
20	73	103	5329
21	79	79	6241
22	83	151	6889
23	89	197	7921
24	97	101	9409
25	101	103	10201
26	103	233	10609
27	107	223	11449
28	109	127	11881
29	113	223	12769
30	127	191	16129
31	131	163	17161
32	137	229	18769
33	139	643	19321
34	149	239	22201
35	151	157	22801
36	157	167	24649
37	163	439	26569
38	167	239	27889
39	173	199	29929
40	179	191	32041
41	181	199	32761
42	191	383	36481
43	193	233	37249
44	197	751	38809
45	199	313	39601
46	211	773	44521
47	223	607	49729
48	227	313	51529
49	229	383	52441
50	233	293	54289

n	a_{n+1}	F_n	a_{n+1}^2
51	239	443	57121
52	241	331	58081
53	251	283	63001
54	257	277	66049
55	263	271	69169
56	269	401	72361
57	271	307	73441
58	277	331	76729
59	281	379	78961
60	283	491	80089
61	293	331	85849
62	307	311	94249
63	311	397	96721
64	313	331	97969
65	317	353	100489
66	331	419	109561
67	337	421	113569
68	347	883	120409
69	349	547	121801
70	353	1381	124609
71	359	457	128881
72	367	457	134689
73	373	373	139129
74	379	421	143641
75	383	409	146689
76	389	1061	151321
77	397	523	157609
78	401	499	160801
79	409	619	167281
80	419	727	175561
81	421	457	177241
82	431	509	185761
83	433	439	187489
84	439	911	192721
85	443	461	196249
86	449	823	201601
87	457	613	208849
88	461	617	212521
89	463	1021	214369
90	467	523	218089
91	479	941	229441
92	487	653	237169
93	491	601	241081
94	499	877	249001
95	503	607	253009
96	509	631	259081
97	521	733	271441
98	523	757	273529
99	541	877	292681
100	547	641	299209

Table 2: **The rightness of inequalities "$a_{n+1} \leqslant F_n < a_{n+1}^2$" ($101 \leqslant n \leqslant 200$) [17]**

n	a_{n+1}	F_n	a_{n+1}^2
101	557	877	310249
102	563	1423	316969
103	569	929	323761
104	571	839	326041
105	577	641	332929
106	587	839	344569
107	593	971	351649
108	599	859	358801
109	601	1019	361201
110	607	643	368449
111	613	733	375769
112	617	743	380689
113	619	653	383161
114	631	1031	398161
115	641	1069	410881
116	643	983	413449
117	647	653	418609
118	653	769	426409
119	659	691	434281
120	661	1213	436921
121	673	991	452929
122	677	1091	458329
123	683	2087	466489
124	691	733	477481
125	701	1307	491401
126	709	1481	502681
127	719	883	516961
128	727	1123	528529
129	733	1523	537289
130	739	1109	546121
131	743	1171	552049
132	751	769	564001
133	757	1801	573049
134	761	1031	579121
135	769	1597	591361
136	773	829	597529
137	787	1201	619369
138	797	1453	635209
139	809	937	654481
140	811	1091	657721
141	821	1031	674041
142	823	857	677329
143	827	1187	683929
144	829	863	687241
145	839	937	703921
146	853	1163	727609
147	857	919	734449
148	859	911	737881
149	863	1187	744769
150	877	1153	769129

n	a_{n+1}	F_n	a_{n+1}^2
151	881	1069	776161
152	883	947	779689
153	887	1439	786769
154	907	1753	822649
155	911	1231	829921
156	919	1223	844561
157	929	1013	863041
158	937	1237	877969
159	941	1153	885481
160	947	1489	896809
161	953	1321	908209
162	967	1181	935089
163	971	1987	942841
164	977	1697	954529
165	983	2243	966289
166	991	1867	982081
167	997	1193	994009
168	1009	1097	1018081
169	1013	1289	1026169
170	1019	1999	1038361
171	1021	1103	1042441
172	1031	1601	1062961
173	1033	1453	1067089
174	1039	2131	1079521
175	1049	1231	1100401
176	1051	1163	1104601
177	1061	1063	1125721
178	1063	1163	1129969
179	1069	1453	1142761
180	1087	2357	1181569
181	1091	3559	1190281
182	1093	1429	1194649
183	1097	2689	1203409
184	1103	1597	1216609
185	1109	1381	1229881
186	1117	3089	1247689
187	1123	1669	1261129
188	1129	2099	1274641
189	1151	1831	1324801
190	1153	1327	1329409
191	1163	1867	1352569
192	1171	1759	1371241
193	1181	2351	1394761
194	1187	2287	1408969
195	1193	1607	1423249
196	1201	1429	1442401
197	1213	2239	1471369
198	1217	2381	1481089
199	1223	2011	1495729
200	1229	1619	1510441

Table 3: **The 500 first fortunate numbers "F_n"** $(1 \leqslant n \leqslant 500)$ [17]

Rank	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1 – 20	3	5	7	13	23	17	19	23	37	61	67	61	71	47	107	59	61	109	89	103
21 – 40	79	151	197	101	103	233	223	127	223	191	163	229	643	239	157	167	439	239	199	191
41 – 60	199	383	233	751	313	773	607	313	383	293	443	331	283	277	271	401	307	331	379	491
61 – 80	331	311	397	331	353	419	421	883	547	1381	457	457	373	421	409	1061	523	499	619	727
81 – 100	457	509	439	911	461	823	613	617	1021	523	941	653	601	877	607	631	733	757	877	641
101 – 120	877	1423	929	839	641	839	971	859	1019	643	733	743	653	1031	1069	983	653	769	691	1213
121 – 140	991	1091	2087	733	1307	1481	883	1123	1523	1109	1171	769	1801	1031	1597	829	1201	1453	937	1091
141 – 160	1031	857	1187	863	937	1163	919	911	1187	1153	1069	947	1439	1753	1231	1223	1013	1237	1153	1489
161 – 180	1321	1181	1987	1697	2243	1867	1193	1097	1289	1999	1103	1601	1453	2131	1231	1163	1063	1163	1453	2357
181 – 200	3559	1429	2689	1597	1381	3089	1669	2099	1831	1327	1867	1759	2351	2287	1607	1429	2239	2381	2011	1619
201 – 220	1663	2731	2213	1627	2269	1429	3739	1493	2647	1747	1409	1321	1319	3499	2311	3041	1993	1721	3023	2239
221 – 240	1483	1439	2087	1669	1613	1483	1759	1913	2381	1877	1657	2251	1619	1669	5323	1511	1709	1523	2273	2347
241 – 260	2083	2339	1637	1709	1787	1787	1999	2017	5689	1913	1999	2087	1973	4421	1741	1723	3299	1693	1997	1931
261 – 280	1933	3253	3449	1721	2267	3359	2699	3413	3067	2503	1759	1871	2707	3079	3547	1879	2309	3637	4799	2087
281 – 300	1889	2671	2309	2393	2297	2143	3187	6271	2411	2089	2441	2099	2473	4649	2239	2557	1987	5231	2971	5641
301 – 320	2251	2389	2017	2081	3803	3019	2411	3319	2243	7187	2621	2683	4831	2861	2837	2243	3373	3167	4201	5021
321 – 340	3391	3343	3347	3037	3373	2239	6301	4457	3061	2969	2861	2677	5023	3461	3767	2381	3559	2549	2837	5647
341 – 360	4597	3779	2671	3067	2689	3049	3833	2833	4729	3313	2837	2837	4363	4493	2851	2609	2543	3469	3803	5009
361 – 380	3931	3331	2861	3931	2609	2719	2677	4933	2633	3947	3467	2909	2731	2939	2677	3163	7309	5087	3793	2657
381 – 400	4153	3767	2789	2837	3919	4903	3853	4993	4217	6197	2843	5227	2729	2953	3547	8713	3119	3067	4019	5051
401 – 420	3089	3347	3659	5693	3089	3343	6029	2909	3889	3433	4547	3467	11069	5003	3343	4339	9241	6007	3083	4957
421 – 440	3041	9643	4127	10009	3209	9067	3257	6337	3271	4231	4691	3331	5413	3121	3089	3727	5443	3967	4441	9601
441 – 460	5839	8627	5417	8089	3359	5557	3643	5557	7229	3929	3643	5639	3793	3271	4783	3529	4463	6547	3371	3931
461 – 480	5077	3557	4621	3631	4663	6917	5981	3803	3499	9743	5333	6761	11411	4547	6329	6121	4201	3719	10039	3719
481 – 500	4673	3607	4649	4463	6701	4721	5669	9137	4937	6361	3631	6277	3727	7151	6247	5987	3769	3911	5839	5167

4. Some notes

Corollary 4.1. *Let* $n \in \mathbb{N}^*$. *Since* $(P_n + F_n)$ *is a prime number, thus the two numbers* F_n *and* P_n *are coprime or mutually prime* $\big($*their greatest common divisor (gcd) being* $1\big)$.

Corollary 4.2. *For every* $n \in \mathbb{N}^*$, F_n *is an odd natural number.*

Corollary 4.3. *Let* $n \in \mathbb{N}^*$. F_n *is coprime with all the prime numbers* $a_1, a_2, a_3, ..., a_n$.

5. Proving that: $\forall n \in \mathbb{N}^* : a_{n+1} \leq F_n$

Proposition 5.1. $\forall n \in \mathbb{N}^* : a_n < F_n$.

Proof. By contradiction, we assume that: $\exists n_0 \in \mathbb{N}^* : F_{n_0} \leqslant a_{n_0}$.
We have two cases:

- **First case:** F_{n_0} is a prime number:
  Here, $F_{n_0} \in \{a_1, a_2, ..., a_{n_0}\}$, thus the great common divisor
  $$\gcd(F_{n_0}, P_{n_0}) = F_{n_0} \geqslant 2 \tag{5.1}$$
  but this is a contradiction with **Corollary 4.1.**

- **Second case:** F_{n_0} is not a prime number:
  It follows from the fundamental theorem of arithmetic [4, 12, 15, 26] (also called the unique factorization theorem) that F_{n_0} may be represented uniquely as a product of prime numbers, that is to say:
  $$F_{n_0} = a_1^{k_1}.a_2^{k_2}...a_{n_0}^{k_{n_0}} \tag{5.2}$$
  where the exponents $k_1, k_2, ..., k_{n_0}$ are natural numbers.
  Since $F_{n_0} \geqslant 2$ then $\exists m \ (1 \leqslant m \leqslant n_0) : k_m \neq 0$, thus:
  $$\gcd(F_{n_0}, a_m) = a_m \neq 1 \tag{5.3}$$
  which contradicts **Corollary 4.3.**

We conclude that it's impossible to have that: $\exists n_0 \in \mathbb{N}^* : F_{n_0} \leqslant a_{n_0}$.
This completes the proof of the proposition. □

Proposition 5.2. $\forall n \in \mathbb{N}^* : a_{n+1} \leqslant F_n$.

Proof. By contradiction, we assume that: $\exists n_0 \in \mathbb{N}^* : F_{n_0} < a_{n_0+1}$.
By **Proposition 5.1.**: $a_{n_0} < F_{n_0}$, hence: $a_{n_0} < F_{n_0} < a_{n_0+1}$, so F_{n_0} isn't a prime number, therefore:
It follows from the fundamental theorem of arithmetic [4, 12, 15, 26] that F_{n_0} may be represented in its factored form:
$$F_{n_0} = a_1^{k_1}.a_2^{k_2}...a_{n_0}^{k_{n_0}} \tag{5.4}$$
where the exponents $k_1, k_2, ..., k_{n_0}$ are natural numbers.
Since $F_{n_0} \geqslant 2$ then $\exists m \ (1 \leqslant m \leqslant n_0) : k_m \neq 0$, thus:
$$\gcd(F_{n_0}, a_m) = a_m \neq 1 \tag{5.5}$$
but, this is a contradiction with **Corollary 4.3.**
This completes the proof of the proposition. □

6. Proving the rightness of *Fortune*'s conjecture in a particular case

R. F. Fortune conjectured that all fortunate numbers are primes. In this section, we will prove, by two different methods, that if the n^{th} fortunate number F_n $(n \in \mathbb{N}^*)$ verifies the condition: $F_n < a_{n+1}^2$, then F_n is indeed prime.

6.1. First method

We give a short and beautiful proof using nothing more than the fundamental theorem of arithmetic and the reasoning by contraposition.

Proposition 6.1. $\forall n \in \mathbb{N}^*$:
The fortunate number F_n *is composite* $\Longrightarrow F_n \geqslant a_{n+1}^2$.

Proof. Let $n \in \mathbb{N}^*$. Suppose that F_n is composite.
Since $gcd(F_n, a_l) = 1, \forall l : 1 \leqslant l \leqslant n$ and $a_{n+1} \leqslant F_n$, it follows from the fundamental theorem of arithmetic that:

$$F_n = a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+m}^{k_{n+m}} \tag{6.1}$$

where:

$$a_{n+1} \leqslant a_{n+m} \leqslant F_n < a_{n+m+1}, m \in \mathbb{N}^* \tag{6.2}$$

and the exponents $k_{n+1}, k_{n+2}, ..., k_{n+m}$ are natural numbers.
But F_n is composite, therefore it has at least two prime divisors (from the set $\{a_{n+1}, a_{n+2}, ..., a_{n+m}\}$), and so, $F_n \geqslant a_{n+1}^2$. □

Corollary 6.2. *The preceding proposition (**Proposition 6.1**) is equivalent to saying:*
$\forall n \in \mathbb{N}^*$: $\quad (F_n < a_{n+1}^2 \Longrightarrow F_n$ *is prime*$)$.

6.2. Second method

In this section, we will take advantage of the properties of the intervals $I_{n,r}, n, r \in \mathbb{N}^*$ to show the veracity of ***Fortune***'s conjecture in a particular case.

6.2.1. Relations between Fortune's conjecture and the intervals $I_{n,r}, n, r \in \mathbb{N}^*$

Proposition 6.3. $\forall n \in \mathbb{N}^*$: *The interval* $I_{n,1}$ *does not contain any primes.*

Proof. Let $n \in \mathbb{N}^*$. Suppose (by contradiction), that there is at least a prime number $x \in I_{n,1}$. Then: $x = x' + P_n$, such that: $a_{n+1} < x' < a_{n+2}, x' \in \mathbb{N}^*$, thus x' isn't a prime number because it is completely situated between two consecutive prime numbers, and it follows from the fundamental theorem of arithmetic that x' may be represented in its factored form:

$$x' = a_1^{k_1}.a_2^{k_2}...a_{n+1}^{k_{n+1}} \tag{6.3}$$

where the exponents $k_1, k_2, ..., k_{n+1}$ are natural numbers.
However, since $x = x' + P_n$ is a prime number and $P_n = a_1.a_2.a_3...a_n$, it follows that:
$k_1 = k_2 = ... = k_n = 0$, thus: $x' = a_{n+1}^{k_{n+1}}$. It results from the famous *Bertrand*'s postulate [2, 19, 25] that: $x' = a_{n+1}^{k_{n+1}} < a_{n+2} < 2.a_{n+1}$, therefore: $k_{n+1} \in \{0, 1\}$.
<u>In the case $k_{n+1} = 1$:</u>
We have: $x' = a_{n+1}$, which means that x' is a prime number, and this is a contradiction.
<u>In the case $k_{n+1} = 0$:</u>
We have: $x' = 1$, but this contradicts our assumption too.
So, we came up with a contradiction, thus the interval $I_{n,1}$ does not contain any prime number. □

We now prove the following theorem:

Theorem 6.4. *(Generalization of the previous Proposition)*
Let $n \in \mathbb{N}^*$, hence: For every $r \in \mathbb{N}^*$, such that: $a_{n+r+1} < a_{n+1}^2$, the interval $I_{n,r}$ does not contain any prime number.

Proof. Let $n \in \mathbb{N}^*$ be fixed and $r \in \mathbb{N}^*$ such that: $a_{n+r+1} < a_{n+1}^2$. Suppose (by contradiction), that there is at least a prime number $x \in I_{n,r}$. Then: $x = x' + P_n$, such that: $a_{n+r} < x' < a_{n+r+1}$, thus x' isn't a prime number because it is completely situated between two consecutive prime numbers, and it follows from the fundamental theorem of arithmetic that x' may be represented in its factored form:

$$x' = a_1^{k_1}.a_2^{k_2}...a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}} \tag{6.4}$$

where the exponents $k_1, k_2, ..., k_{n+1}, ..., k_{n+r}$ are natural numbers.
However, since $x = x' + P_n$ is a prime number and $P_n = a_1.a_2.a_3...a_n$, it follows that: $k_1 = k_2 = ... = k_n = 0$, thus: $x' = a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}}$.
The natural numbers $k_{n+1}, k_{n+2}, ..., k_{n+r}$ fulfill: $k_i \in \{0,1\}, \forall i \in \{n+1, n+2, ..., n+r\}$, because:
$\forall i \in \{n+1, n+2, ..., n+r\} : a_i^{k_i} \leqslant x' = a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}} < a_{n+r+1} < a_{n+1}^2 \leqslant a_i^2$.
On the other hand, two numbers k_i and k_j $(n+1 \leqslant i, j \leqslant n+r, i \neq j)$ can't be equal to 1 together, otherwise we would have:
$a_{n+1}^2 < a_i.a_j \leqslant x' < a_{n+1}^2$, but this is a contradiction.
Therefore, only one of the following two cases can be achieved:

- $\forall i : n+1 \leqslant i \leqslant n+r : k_i = 0$, but here, $x' = 1$ which is impossible.
- $\exists i : n+1 \leqslant i \leqslant n+r$: such that: $k_i = 1$ and $k_j = 0, \forall j : n+1 \leqslant j \leqslant n+r, j \neq i$, but here, $x' = a_i$ which means that x' is prime (impossible).

So, we came up with a contradiction, thus the interval $I_{n,r}$ does not contain any prime number. □

6.2.2. A direct result of ***Theorem 6.4***

We will prove again that: $\forall n \in \mathbb{N}^*$: $\left(F_n < a_{n+1}^2 \Longrightarrow F_n \text{ is prime}\right)$, but this time, by another method.

Proof. Consider $n \in \mathbb{N}^*$ such that: $F_n < a_{n+1}^2$. Since the natural number a_{n+1}^2 is not prime, we deduce two cases:

- **First case.** $\exists r \in \mathbb{N}^*$ such that: $a_{n+1} \leqslant a_{n+r} \leqslant F_n \leqslant a_{n+r+1} < a_{n+1}^2$ (see Figure 3).

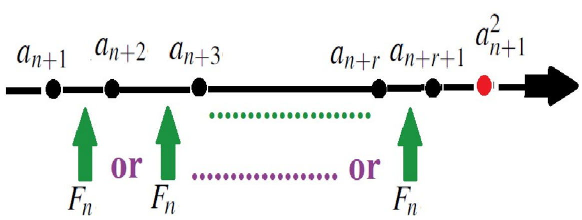

Figure 3: **First case:** $\exists r \in \mathbb{N}^* : a_{n+r} \leqslant F_n \leqslant a_{n+r+1} < a_{n+1}^2$

Since $(F_n + P_n)$ is a prime number and
$a_{n+1} + P_n \leqslant a_{n+r} + P_n \leqslant F_n + P_n \leqslant a_{n+r+1} + P_n < a_{n+1}^2 + P_n$, i.e.
$(F_n + P_n) \in \left[a_{n+r} + P_n, a_{n+r+1} + P_n\right]$, hence it is **Theorem 6.4** that makes allowable only the following two possibilities: $F_n + P_n = a_{n+r} + P_n$ or $F_n + P_n = a_{n+r+1} + P_n$, namely: $F_n = a_{n+r}$ or $F_n = a_{n+r+1}$. Hence, the fortunate number F_n is prime.

- **Second case.** $\exists r \in \mathbb{N}^*$ such that: $a_{n+1} \leqslant a_{n+r} \leqslant F_n < a_{n+1}^2 < a_{n+r+1}$ (see Figure 4).

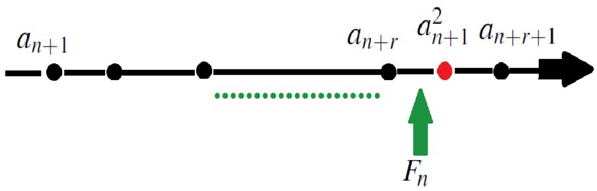

Figure 4: **Second case:** $\exists r \in \mathbb{N}^* : a_{n+r} \leqslant F_n < a_{n+1}^2 < a_{n+r+1}$

By a similar argument as in the proof of **Theorem 6.4**:
From the fundamental theorem of arithmetic, we deduce that:

$$F_n = a_1^{k_1}.a_2^{k_2}...a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}} \tag{6.5}$$

where the exponents $k_1, k_2, ..., k_{n+1}, ..., k_{n+r}$ are natural numbers.
But, since $(F_n + P_n)$ is a prime number, $k_1 = k_2 = ... = k_n = 0$, thus: $F_n = a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}}$.
The natural numbers $k_{n+1}, k_{n+2}, ..., k_{n+r}$ fulfill: $k_i \in \{0, 1\}, \forall i \in \{n+1, n+2, ..., n+r\}$, because:
$\forall i \in \{n+1, n+2, ..., n+r\} : a_i^{k_i} \leqslant F_n = a_{n+1}^{k_{n+1}}.a_{n+2}^{k_{n+2}}...a_{n+r}^{k_{n+r}} < a_{n+1}^2 \leqslant a_i^2$.
On the other hand, two numbers k_i and k_j $(n+1 \leqslant i, j \leqslant n+r, i \neq j)$ can't be equal to 1 together, otherwise we would have:
$a_{n+1}^2 < a_i.a_j \leqslant F_n < a_{n+1}^2$, but this is a contradiction.
Therefore, only one of the following two cases can be achieved:

- $\forall i : n+1 \leqslant i \leqslant n+r : k_i = 0$, but here, $F_n = 1$ which is impossible.
- $\exists i : n+1 \leqslant i \leqslant n+r$: such that: $k_i = 1$ and $k_j = 0, \forall j : n+1 \leqslant j \leqslant n+r, j \neq i$, but here, $F_n = a_i$ which means that F_n is prime.

Thus, from the first case and the second one, we deduce that F_n is prime.

□

7. Proving the inequality: $\forall n \in \mathbb{N}^*$: $F_n \leq 1 + P_n$

Proposition 7.1. *$\forall n \in \mathbb{N}^*$, the interval K_n doesn't contain any prime number.*

Proof. Let $n \in \mathbb{N}^*$. Reasoning by the absurd: Suppose that the interval K_n contains at least a prime number x, such that: $x = x' + P_n$, where: $1 < x' < a_{n+1}$,$x' \in \mathbb{N}^*$. By the fundamental theorem of arithmetic, x' may be represented in its factored form: $x' = a_1^{k_1}.a_2^{k_2}...a_n^{k_n}$, where the exponents $k_1, k_2, ..., k_n$ are natural numbers. But, $x = x' + P_n$ is a prime number, therefore: $k_1 = k_2 = ... = k_n = 0$, that is to say $x' = 1$ (contradiction). □

Proposition 7.2. $\forall n \in \mathbb{N}^*$: $F_n \in [a_{n+1}, 1 + P_n]$.

Proof. Let $n \in \mathbb{N}^*$. We recall that *Bertrand*'s postulate [2, 19, 25] states that there exists at least one prime number p, such that: $1 + P_n < p < 2 + 2P_n$.
We note that:

$$\begin{aligned}]1+P_n, 2+2P_n[&=]1+P_n, a_{n+1}+P_n[\bigcup[a_{n+1}+P_n, a_{n+1}^2+P_n[\bigcup \\ &\quad \bigcup[a_{n+1}^2+P_n, 2+2.P_n[\\ &= K_n \bigcup J_n \bigcup[a_{n+1}^2+P_n, 2+2.P_n[. \end{aligned}$$

On the other hand, by the definition of $F_n > 1$, it follows that $(P_n + F_n)$ is the prime number closest to $(1 + P_n)$.

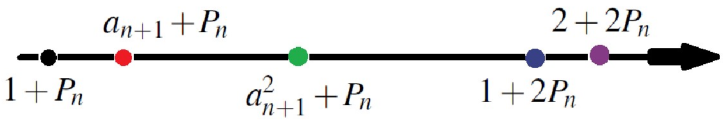

Figure 5: Clarification

We deduce from *Bertrand*'s postulate, **Proposition 7.1** and Figure 5, that: $(P_n + F_n) \in [a_{n+1} + P_n, 1 + 2.P_n]$, thus: $F_n \in [a_{n+1}, 1 + P_n]$. □

Remark 1. *The proof of* ***Proposition 7.2*** *can also be used to prove* ***Proposition 5.2****.*

Remark 2. *By* ***Proposition 7.2****, if someone intends to deny* ***Fortune's conjecture****, he must find a counterexample of a fortunate number* F_n *belonging to the interval* $[a_{n+1}^2, 1 + P_n]$*, because this is the only chance that the number* F_n *is not prime. In fact, so far, there is no "known" fortunate number that satisfies this belonging, making* ***Fortune's conjecture*** *correct until now.*

8. An upper bound (other than a_{n+1}^2) on F_n, for every $n \leq 3000$

Based on [17], the graphs in Figure 6, were plotted using the software *Graph 4.4.2* (an open source application which can be easily download from [10] or [29]) and 3000 points on each graph.
Now, look at the following Figure (Figure 6.):

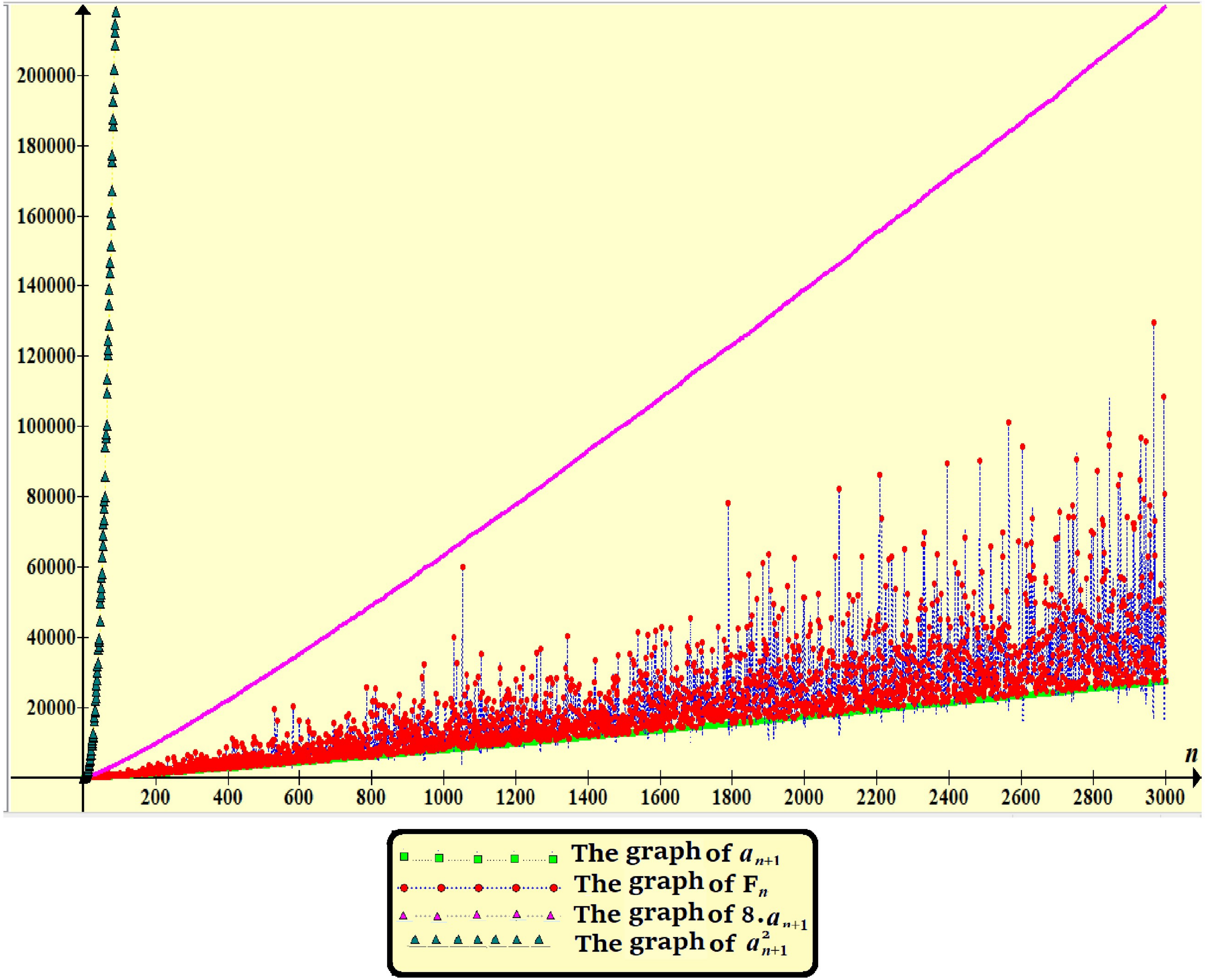

Figure 6: **The graphs of values of** a_{n+1}, F_n, $8 \times a_{n+1}$ **and** $(a_{n+1})^2$, $(1 \leqslant n \leqslant 3000)$ [17]

From Table 1 and Table 2, we notice that: $\forall n \in \{1,2,3,...,200\}$: $a_{n+1} \leqslant F_n < (a_{n+1})^2$.
And from [17], we have noticed this same property even for all values of the integer n such that: $n \leqslant 3000$.
We noticed (from Figure 6.) that the upper bound a^2_{n+1} (on F_n) is too large, that's why we thought of reducing it to another smaller bound which can be $8.a_{n+1}$.
So, from [17] and Figure 6, it follows that: $\forall n \in \{1,2,3,...,3000\}$: $a_{n+1} \leqslant F_n < 8.a_{n+1}$.
Nothing is certain, but if the right inequality: $F_n < 8.a_{n+1}$ holds true for any positive integer n, then ***Fortune*'s conjecture** will be proven once and for all, because: $8.a_{n+1} < a^2_{n+1}$, for every positive integer n such that: $n \geqslant 4$, and thanks to **Corollary 6.2**, it follows that F_n is prime.

8.1. The rightness of the double inequality: $a_{n+1} \leqslant F_n < 8.a_{n+1}, \forall n \leqslant 3000$

According to the results obtained in [17], we note that:

$$\forall n \in \mathbb{N}^* \text{ such that } n \leqslant 3000 : a_{n+1} \leqslant F_n < 8.a_{n+1}. \quad (8.1)$$

We quote below, some inequalities deduced from [17]:
$547 = a_{101} \leqslant F_{100} = 641 < 8.a_{101} = 4376$
$1229 = a_{201} \leqslant F_{200} = 1619 < 8.a_{201} = 9832$
$1993 = a_{301} \leqslant F_{300} = 5641 < 8.a_{301} = 15944$
$2749 = a_{401} \leqslant F_{400} = 5051 < 8.a_{401} = 21992$
$3581 = a_{501} \leqslant F_{500} = 5167 < 8.a_{501} = 28648$
$4421 = a_{601} \leqslant F_{600} = 16187 < 8.a_{601} = 35368$
$5281 = a_{701} \leqslant F_{700} = 12853 < 8.a_{701} = 42248$
$6143 = a_{801} \leqslant F_{800} = 6323 < 8.a_{801} = 49144$
$7001 = a_{901} \leqslant F_{900} = 7547 < 8.a_{901} = 56008$
$7927 = a_{1001} \leqslant F_{1000} = 8719 < 8.a_{1001} = 63416$
$8837 = a_{1101} \leqslant F_{1100} = 13879 < 8.a_{1101} = 70696$
$9739 = a_{1201} \leqslant F_{1200} = 28069 < 8.a_{1201} = 77912$
$10663 = a_{1301} \leqslant F_{1300} = 13457 < 8.a_{1301} = 85304$
$11677 = a_{1401} \leqslant F_{1400} = 15161 < 8.a_{1401} = 93416$
$12569 = a_{1501} \leqslant F_{1500} = 14281 < 8.a_{1501} = 100552$
$13513 = a_{1601} \leqslant F_{1600} = 22861 < 8.a_{1601} = 108104$
$14533 = a_{1701} \leqslant F_{1700} = 18793 < 8.a_{1701} = 116264$
$15413 = a_{1801} \leqslant F_{1800} = 16229 < 8.a_{1801} = 123304$
$16411 = a_{1901} \leqslant F_{1900} = 63499 < 8.a_{1901} = 131288$
$17393 = a_{2001} \leqslant F_{2000} = 51137 < 8.a_{2001} = 139144$
$18329 = a_{2101} \leqslant F_{2100} = 19661 < 8.a_{2101} = 146632$
$19427 = a_{2201} \leqslant F_{2200} = 21101 < 8.a_{2201} = 155416$
$20359 = a_{2301} \leqslant F_{2300} = 37021 < 8.a_{2301} = 162872$
$21391 = a_{2401} \leqslant F_{2400} = 23203 < 8.a_{2401} = 171128$
$22343 = a_{2501} \leqslant F_{2500} = 25643 < 8.a_{2501} = 178744$
$23327 = a_{2601} \leqslant F_{2600} = 23909 < 8.a_{2601} = 186616$
$24317 = a_{2701} \leqslant F_{2700} = 44533 < 8.a_{2701} = 194536$
$25409 = a_{2801} \leqslant F_{2800} = 32173 < 8.a_{2801} = 203272$
$26407 = a_{2901} \leqslant F_{2900} = 26573 < 8.a_{2901} = 211256$
$27457 = a_{3001} \leqslant F_{3000} = 27583 < 8.a_{3001} = 219656$.

9. A computing section

A partial summary

By **Theorem 6.4**:
If for every $n \in \mathbb{N}^*$, the interval J_n contains at least a prime number, then ***Fortune*'s conjecture** will be

proven. This result follows from the fact that: $\forall n \in \mathbb{N}^*, \exists r \in \mathbb{N}^*$ such that:
$J_n = [a_{n+1} + P_n, a_{n+2} + P_n] \bigcup [a_{n+2} + P_n, a_{n+3} + P_n] \bigcup ... \bigcup [a_{n+r} + P_n, a_{n+r+1} + P_n] \bigcup [a_{n+r+1} + P_n, a_{n+1}^2 + P_n[$
where:

$$a_{n+r+1} < a_{n+1}^2 < a_{n+r+2}. \tag{9.1}$$

The double inequality (9.1) is justified by the infinitude of primes [3, 5, 13, 27] and the fact that: $\forall n \in \mathbb{N} : a_{n+1}^2 > 2$.

9.1. On the number of primes contained in the interval J_n

In this subsection, we will present 10 examples on the number of primes that exist in the interval J_n, where $n \in \mathbb{N}^*, 1 \leqslant n \leqslant 10$ [21]:
For $n = 1$, J_n=J_1=$[a_2 + P_1, a_2^2 + P_1[$=$[5, 11[$ contains the two primes: 5 and 7.
For $n = 2$, J_n=J_2=$[a_3 + P_2, a_3^2 + P_2[$=$[11, 31[$ contains the 6 following primes: 11,13,17,19,23,29.
For $n = 3$, J_n=J_3=$[a_4 + P_3, a_4^2 + P_3[$=$[37, 79[$ contains the 10 following primes: 37,41,43,47,53,59,61,67,71,73.
For $n = 4$, J_n=J_4=$[a_5 + P_4, a_5^2 + P_4[$=$[221, 331[$ contains the 19 following primes: 223,227,229,233,239,241,251,257, 263,269,271,277,281,283,293,307,311,313,317.
For $n = 5$, J_n=J_5=$[a_6 + P_5, a_6^2 + P_5[$=$[2323, 2479[$ contains the 23 following primes: 2333,2339,2341,2347,2351,2357, 2371,2377,2381,2383,2389,2393,2399,2411,2417,2423,2437,2441,2447,2459,2467,2473,2477.
For $n = 6$, J_n=J_6=$[a_7 + P_6, a_7^2 + P_6[$=$[30047, 30319[$ contains the 29 following primes: 30047,30059,30071,30089, 30091,30097,30103,30109,30113,30119,30133,30137,30139,30161,30169,30181,30187,30197,30203,30211,30223,30241, 30253,30259,30269,30271,30293,30307,30313.
For $n = 7$, J_n=J_7=$[a_8 + P_7, a_8^2 + P_7[$=$[510529, 510871[$ contains the 25 following primes: 510529,510551,510553, 510569,510581,510583,510589,510611,510613,510617,510619,510677,510683,510691,510707,510709,510751,510767, 510773,510793,510803,510817,510823,510827,510847.
For $n = 8$, J_n=J_8=$[a_9 + P_8, a_9^2 + P_8[$=$[9699713, 9700219[$ contains the 38 following primes: 9699713,9699727, 9699731,9699733,9699749,9699763,9699769,9699773,9699799,9699803, 9699817,9699827,9699841,9699853,9699887, 9699889,9699913,9699917,9699919,9699923, 9699929,9699941,9699953,9699959,9699973,9700003,9700027, 9700039,9700063,9700073,9700079,9700099,9700111,9700139,9700147,9700153,9700169,9700213.
For $n = 9$, J_n=J_9=$[a_{10} + P_9, a_{10}^2 + P_9[$=$[223092899, 223093711[$ contains 42 primes.
For $n = 10$, $J_n = J_{10} = [a_{11} + P_{10}, a_{11}^2 + P_{10}[= [6469693, 6469694191[$ contains 39 primes.
The following diagram (Figure 7) clearly shows that the number of these primes is not small. We hope that in the future, we will be able to prove that the interval J_n contains many prime numbers, for each value of the natural number $n \in \mathbb{N}^*$, which means to prove ***Fortune*'s conjecture**.

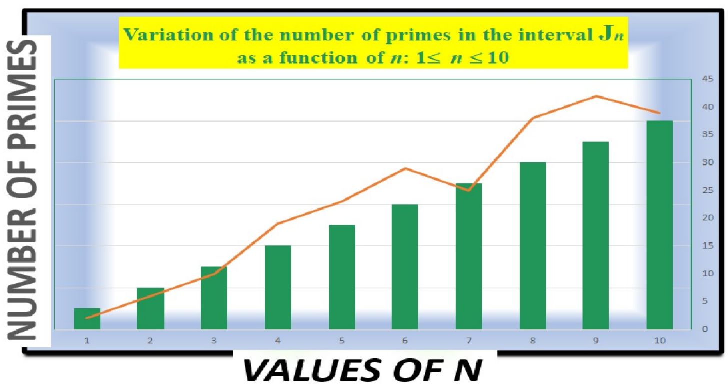

Figure 7: **Variation of the number of primes in the interval** $J_n, 1 \leqslant n \leqslant 10$

9.2. Some counterexamples showing that the interval $I_{n,r}, n, r \in \mathbb{N}^*$ *may contain prime numbers when* $a_{n+r} > a_{n+1}^2$

In this section, we will present 10 counterexamples showing that it is very likely that the interval $I_{n,r}, n, r \in \mathbb{N}^*$ may contain prime numbers when $a_{n+r} > a_{n+1}^2$ [23].

9.2.1. ***First counterexample***

For $n = 1$ and $r = 8$:

We have indeed $a_{n+r} = a_9 = 23 > 9 = a_2^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{1,8} =]a_9 + P_1, a_{10} + P_1[=]23 + 2, 29 + 2[=]25, 31[$ contains the prime number 29.

9.2.2. ***Second counterexample***

For $n = 2$ and $r = 14$:

We have indeed $a_{n+r} = a_{16} = 53 > 25 = a_3^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{2,14} =]a_{16} + P_2, a_{17} + P_2[=]53 + 6, 59 + 6[=]59, 65[$ contains the prime number 61.

9.2.3. ***Third counterexample***

For $n = 3$ and $r = 58$:

We have indeed $a_{n+r} = a_{61} = 283 > 49 = a_4^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{3,58} =]a_{61} + P_3, a_{62} + P_3[=]283 + 30, 293 + 30[=]313, 323[$ contains the prime number 317.

9.2.4. ***Fourth counterexample***

For $n = 4$ and $r = 30$:

We have indeed $a_{n+r} = a_{34} = 139 > 121 = a_5^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{4,30} =]a_{34} + P_4, a_{35} + P_4[=]139 + 210, 149 + 210[=]349, 359[$ contains the prime number 353.

9.2.5. ***Fifth counterexample***

For $n = 5$ and $r = 216$:

We have indeed $a_{n+r} = a_{221} = 1381 > 169 = a_6^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{5,100} =]a_{221} + P_5, a_{222} + P_5[=]1381 + 2310, 1399 + 2310[=]3691, 3709[$ contains the prime number 3701.

9.2.6. ***Sixth counterexample***

For $n = 6$ and $r = 85$:

We have indeed $a_{n+r} = a_{91} = 467 > 289 = a_7^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{6,85} =]a_{91} + P_6, a_{92} + P_6[=]467 + 30030, 479 + 30030[=]30497, 30529[$ contains the prime number 30509.

9.2.7. ***Seventh counterexample***

For $n = 7$ and $r = 173$:

We have indeed $a_{n+r} = a_{180} = 1069 > 361 = a_8^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{7,173} =]a_{180} + P_7, a_{181} + P_7[=]1069 + 510510, 1087 + 510510[=]511579, 511597[$ contains the prime number 511583.

9.2.8. ***Eighth counterexample***

For $n = 8$ and $r = 113$:

We have indeed $a_{n+r} = a_{121} = 661 > 529 = a_9^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{8,113} =]a_{121} + P_8, a_{122} + P_8[=]661 + 9699690, 673 + 9699690[=]9700351, 9700363[$ contains the prime number 9700357.

9.2.9. ***Ninth counterexample***

For $n = 9$ and $r = 318$:

We have indeed $a_{n+r} = a_{327} = 2179 > 841 = a_{10}^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{9,318} =]a_{327} + P_9, a_{328} + P_9[=]2179 + 223092870, 2203 + 223092870[=]223095049, 223095073[$ contains the prime number 223095071.

9.2.10. ***Tenth counterexample***

For $n = 10$ and $r = 580$:
We have indeed $a_{n+r} = a_{590} = 4297 > 961 = a_{11}^2 = a_{n+1}^2$ and also, the interval $I_{n,r} = I_{10,580} =]a_{590} + P_{10}, a_{591} + P_{10}[=]4297 + 6469693230, 4327 + 6469693230[=]6469697527, 6469697557[$ contains the prime number 6469697537.

10. Conclusion

Basically, this paper deals with the still open problem: ***Fortune*'s conjecture**, in great detail, especially concerning the fortunate numbers. In particular, we have highlighted a new remarkable double inequality that the fortunate numbers verify.
The novelty of the work presented in this paper lies mainly in the following two points:

- The appearance of the domains $I_{n,r}$ in the partial proof of ***Fortune*'s conjecture**.
- The proof of the double inequality (8.1) for all $n \leqslant 3000$.

The mathematical tools used in this paper vary between *Bertrand*'s postulate, the fundamental theorem of arithmetic and the properties of 3 intervals of special types.
The work is still in progress, we always hope to arrive at other results which are not quoted in this manuscript, by means of various mathematical tools.
We mention that in the past few months, 5 scientific works (concerning *Fortune*'s conjecture and fortunate numbers) were published, these works are: [20, 21, 22, 23, 24].

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Competing Interests

Author declared that no competing interests exist.

Statement

This manuscript has no associate data.

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Appendix

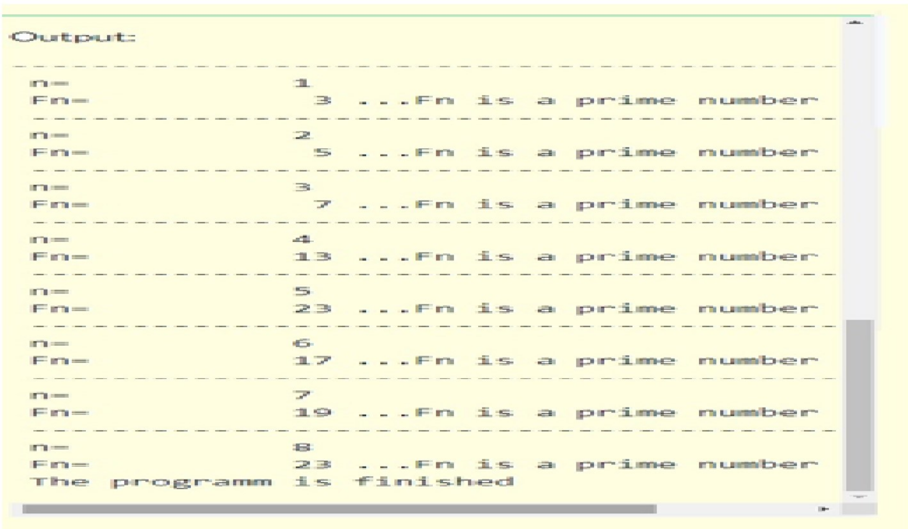
In the appendix, we describe an efficient algorithm (written in *Fortran* language) that allows to determine the fortunate numbers (we can use another programming language, but this program is intended especially for beginners in programming).
The next program calculates the fortunate number, and makes sure it is prime, and for example, we put the condition: "if a_n is less than 20" in the program, so that the program stops as soon as this condition is met, but you can replace the number 20 by any other number.
The results obtained after the execution of the program implemented with the proposed algorithm, show that the obtained fortunate number is always a prime number.


Calculating Fn: Fortune's Numbers

```
program Fortune
Integer Pn,nn,an,counter,Fn
real vk,s
  an=2
  i=2
  s=an/2.
  Pn=1
  counter=1
3 if(an.le.20)then
5     if(i.le.s) then
      vk=an*(1./i)
      k=int(vk)
            if((vk-k).eq.0.) then
            go to 4
            endif
      i=i+1
      go to 5
      endif
  Pn=an*Pn
  Fn=2
22  nn=Pn+Fn
    ii=2
    ss=nn/2.
55  if(ii.le.ss) then
    vk=nn*(1./ii)
    k=int(vk)
            if((vk-k).eq.0.) then

            Fn=1+Fn
            go to 22
            else
            ii=ii+1
            go to 55
            endif
    endif
write(*,*)' ','-----------------------------------------------------------'
write(*,*)'n=',counter
write(*,*)'Fn=',Fn,'...',  'Fn is a prime number'
 counter=counter+1
4 an=1+an
  i=2
  s=an/2.
  go to 3
  endif
  write(*,*)'The programm is finished'
  Stop
end program Fortune
```

Figure 8: Program calculating Fortunate numbers



```
Output:

 -----------------------------------------------------------
 n=              1
 Fn=               3 ...Fn is a prime number
 -----------------------------------------------------------
 n=              2
 Fn=               5 ...Fn is a prime number
 -----------------------------------------------------------
 n=              3
 Fn=               7 ...Fn is a prime number
 -----------------------------------------------------------
 n=              4
 Fn=              13 ...Fn is a prime number
 -----------------------------------------------------------
 n=              5
 Fn=              23 ...Fn is a prime number
 -----------------------------------------------------------
 n=              6
 Fn=              17 ...Fn is a prime number
 -----------------------------------------------------------
 n=              7
 Fn=              19 ...Fn is a prime number
 -----------------------------------------------------------
 n=              8
 Fn=              23 ...Fn is a prime number
 The programm is finished
```

Figure 9: Results of Fortunate Program