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A Novel View of Regular and Normal Spaces via Nano S_β-Open Sets in Nano Topological Spaces

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Abstract

In this present study, we shed light on some separation axioms via nano S_β-open sets including nano S^*_β-regular and nano S^*_β-normal spaces (axioms) in nano topological spaces where nano S_β-open set is defined and related to nano semi-open and nano β-closed sets. Here, we implement each axiom on the family of all nano S_β-open sets according to upper and lower approximations in which there exist exactly six families of nano S_β-open sets. In addition, the relationship among those axioms is also considered with the other known axioms.

Keywords: Nano S_β-open sets, nano S^*_β-regular, nano S^*_β-normal.
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1. Introduction

Regarding to the rough set theory [10], the concept of nano topological space is introduced by Thivagar and Richard [1] with respect to a subset X of U as the universe. Then, some types of nano open sets are defined and introduced such as nano semi-open sets, α-open sets and nano pre-open sets by [1] and nano rare sets by [6]. After that, nano β-open sets are introduced by Revathy and Ilango [7]. By using nano semi-open sets, nano S_β-open sets are introduced by Pirbal and Ahmed [2]. Moreover, regarding the structure of nano S_β-open sets, nano S_C-open sets are defined by Pirbal and Ahmed [5]. The authors in [2], studied connectedness by using nano S_β-open sets in [8]. In addition, some separation axioms via nano S_β-open sets are studied by [4]. The authors in[4] studied nano S_β-regular and nano S_β-normal spaces. Also, nano S_β-operators and nano S_β-continuity in nano topological spaces have been studied in [9]. Since separation axioms are the main tool to distinguish two points, two sets or a point with a set topologically, it was such an inspiration for the authors to introduce two new spaces such as nano S^*_β-regular and nano S^*_β-normal spaces in nano topological spaces. Finally, we study the relationship among those separation axioms (spaces).

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2. Preliminaries

Definition 2.1. [10] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U called the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U,\ R)$ is said to be the approximation space. Let $X \subseteq U$:

1. The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U} \{R(x) ; R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U} \{R(x) ; R(x) \bigcap X \neq \phi\}$.
3. The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [1] Let U be the universe and R be an equivalence relation on U and $\tau_R(X) = \{\phi,\ U,\ L_R(X),\ U_R(X),\ B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the followings axioms:

1. U and $\phi \in \tau_R(X)$
2. The union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$.
3. The intersection of elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on U and called the nano topology on U with respect to X. We call $(U,\ \tau_R(X))$ af nano topological space.

Proposition 2.3. *[1] Let U be a non-empty finite universe and $X \subseteq U$, then:*

1. *If $L_R(X) = \phi$ and $U_R(X) = U$, then $\tau_R(X) = \{\phi, U\}$ and called nano indiscrete space.*
2. *If $L_R(X) = U_R(X) \neq U$, then, $\tau_R(X) = \{\phi, U, L_R(X)\}$.*
3. *If $L_R(X) = \phi$ and $U_R(X) \neq U$, then $\tau_R(X) = \{\phi, U, U_R(X)\}$.*
4. *If $L_R(X) \neq \phi$ and $U_R(X) = U$, then $\tau_R(X) = \{\phi, U, L_R(X), B_R(X)\}$.*
5. *If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \phi$ and $U_R(X) \neq U$, then $\tau_R(X) = \{\phi, U, U_R(X), L_R(X), B_R(X)\}$ and called nano discrete space.*

Definition 2.4. [2] A nano semi-open set A of a nano topological space $(U, \tau_R(X))$ is said to be nano S_β-open set if for each $x\in A$, there exist a nano β-closed set F such that $x\in F\subseteq A$. The set of all nano S_β-open sets denoted by $nS_\beta O(U, X)$.

Definition 2.5. [3] Let $(U, \tau_R(X))$ be a nano topological space, then:

1. $nS_\beta int(A) = \cup\{G : G$ is nS_β-open and $G\subseteq A\}$.
2. $nS_\beta cl(A) = \cap\{F : F$ is nS_β-closed and $A\subseteq F\}$.

Theorem 2.6. *[2] Let $(U, \tau_R(X))$ be a nano topological space, then:*

1. *If $U_R(X) = U$ and $L_R(X) = \phi$, then $nS_\beta O(U, X) = \{U, \phi\}$.*
2. *If $U_R(X) = U$ and $L_R(X) \neq \phi$, then $\tau_R(X) = nS_\beta O(U, X)$.*
3. *If $U_R(X) = L_R(X) = \{x\}, x\in U$, then $nS_\beta O(U, X) = \{\phi, U\}$.*
4. *If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.*
5. *If $U_R(X) \neq U, L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$*

6. *If* $U_R(X) \neq L_R(X)$ *where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$*, then* $\phi, L_R(X), B_R(X), L_R(X) \cup B, B_R(X) \cup B$ *and any set containing* $U_R(X)$ *where* $B \subseteq [U_R(X)]^c$ *are the only* nS_β*-open sets in* U*.*

Theorem 2.7. *[4] Let* $(U, \tau_R(X))$ *be a nano topological space. If* $U_R(X) \neq L_R(X)$ *where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$*, then* $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$, $B_R(X) \cup B$ *where* $B \subseteq [U_R(X)]^c$ *are non-empty proper* nS_β*-clopen sets in* U*.*

Definition 2.8. [4] A nano topological space $(U, \tau_R(X))$ is said to be nS_β-regular if for each $x \in U$ and each nano closed set A such that $x \notin A$, there exist two nS_β-open sets G and H such that $x \in G, A \subseteq H$ and $G \cap H = \phi$.

Definition 2.9. [4] A nano topological space U is said to be nS_β-normal if for any disjoint nano closed sets A, B of U, there exist nS_β-open sets G and H such that $A \subseteq G, B \subseteq H$ and $G \cap H = \phi$.

Theorem 2.10. *[4] Let* $(U, \tau_R(X))$ *be a nano topological space, then* U *is* nS_β*-regular if:*

1. $U_R(X) = U$ *and* $L_R(X) = \phi$.
2. $U_R(X) = U$ *and* $L_R(X) \neq \phi$.
3. $U_R(X) \neq L_R(X)$*, where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$

Remark 2.11. [4] Let $(U, \tau_R(X))$ be a nano topological space, then U is not nS_β-regular if:

1. $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.
2. $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.
3. $U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$.

Theorem 2.12. *[4] Let* $(U, \tau_R(X))$ *be a nano topological space, then* U *is* nS_β*- normal if:*

1. $U_R(X) = U$ *and* $L_R(X) = \phi$.
2. $U_R(X) = U$ *and* $L_R(X) \neq \phi$.
3. $U_R(X) = L_R(X) = \{x\}$, $x \in U$.
4. $U_R(X) = L_R(X) \neq U$ *and* $U_R(X)$ *contains more than one element of* U.
5. $U_R(X) \neq U$, $L_R(X) = \phi$ *and* $U_R(X)$ *contains more than one element of* U.
6. $U_R(X) \neq L_R(X)$*, where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$

3. Nano S^*_β-Regular Spaces

Let $(U, \tau_R(X))$ be a nano topological space. The authors of [4], defined nS_β-regular spaces as like: "if for each $x \in U$ and each nano closed set A such that $x \notin A$, there exist two nS_β-open sets G and H such that $x \in G, A \subseteq H$ and $G \cap H = \phi$". As being clear that A is nano closed set, but we are going to replace nS_β-closed set instead of nano closed set. So, in this section, we define nS^*_β-regular space as follow:

Definition 3.1. A nano topological space $(U, \tau_R(X))$ is said to be nS^*_β-regular if for each $x \in U$ and each nS_β-closed set A such that $x \notin A$, there exist two nS_β-open sets G and H such that $x \in G, A \subseteq H$ and $G \cap H = \phi$.

Example 3.2. *Let* $U = \{a, b, c, d\}$ *with* $U/R = \{\{a, b\}, \{c, d\}\}$ *and* $X = \{a, b, c\}$*. Then* $\tau_R(X) = \{\phi, U, \{a, b\}, \{c, d\}\} = nS_\beta O(U, X)$*. Then,* U *is* nS^*_β*-regular space.*

Remark 3.3. Nano indiscrete topological space is nS^*_β-regular space.

Theorem 3.4. *Let* $(U, \tau_R(X))$ *be a nano topological space. Then* U *is a* nS^*_β*-regular space if and only if for each* $x \in U$ *and each* nS_β*-open set* G *containing* x*, there exists a* nS_β*-open set* V *containing* x *such that* $x \in V \subseteq nS_\beta cl(V) \subseteq G$.

Proof. Let G be a nS_β-open set and $x\in G$. Then $U-G$ is a nS_β-closed set such that $x\notin U-G$. By nS^*_β-regularity of U, there are nS_β-open sets M and W such that $x\in M$, $U-G\subseteq W$ and $M\cap W=\phi$. Therefore, $x\in M\subseteq U-W\subseteq G$, Hence $x\in M\subseteq nS_\beta cl(M)\subseteq nS_\beta cl(U-W)=U-W\subseteq G$. Thus, $nS_\beta cl(M)\subseteq U-W\subseteq G$. Conversely, let F be nS_β-closed set in U and let $x\notin F$. Then $U-F$ is an nS_β-open set and $x\in U-F$. By assumption, there exist a nS_β-open set H such that $x\in H$ and $nS_\beta cl(H)\subseteq U-F$. Define $K=U-nS_\beta cl(H)$. Then $K\in nS_\beta O(U,X)$ and $H\subseteq nS_\beta cl(H)$, then $H\cap K=H\cap\left(U-nS_\beta cl(H)\right)=\phi$, since $(U-nS_\beta cl(H)\subseteq U-H)$. Thus for $x\notin F$,$\exists$ disjoint nS_β-open sets H and K such that $x\in H$ and $F\subseteq K$. Hence, U is a nS^*_β-regular space. □

Theorem 3.5. *Let $(U, \tau_R(X))$ be a nano topological space, then U is nS^*_β-regular if:*

1. $U_R(X)=U$ *and* $L_R(X)=\phi$.
2. $U_R(X)=U$ *and* $L_R(X)\neq\phi$.
3. $U_R(X)=L_R(X)\neq U$ *and* $U_R(X)=\{x\}$, $x\in U$
4. $U_R(X)\neq L_R(X)$, *where* $U_R(X)\neq U$ *and* $L_R(X)\neq\phi$

Proof.

1. Since $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U\}$. Hence U is nS^*_β-regular space.
2. Since $\tau_R(X)=nS_\beta O(U,X)$, therefore, U is nS^*_β-regular space.
3. Since $\tau_R(X)=\{\phi, U,\{x\}\}$ and $nS_\beta O(U,X)=\{\phi, U\}$. Therefore, U is nS^*_β-regular space.
4. Let $x\in U_R(X)$, then either $x\in L_R(X)$ or $x\in B_R(X)$. If $x\in L_R(X)$, then the nS_β-open sets which contain x are $U_R(X)$, $L_R(X)$, $L_R(X)\cup B$ and $A\supseteq U_R(X)$ where $B\subseteq[U_R(X)]^c$.
   By Theorem 2.7, $L_R(X)$, $B_R(X)$, $L_R(X)\cup B$ and $B_R(X)\cup B$ are nS_β-clopen in U. Then, there are two cases:

   First, if $x\in L_R(X)$, then $x\in L_R(X)\subseteq nS_\beta cl(L_R(X))\subseteq\begin{cases} L_R(X)\\ U_R(X)\\ L_R(X)\cup B\\ A\supseteq U_R(X)\end{cases}$.

   Second, if $x\in B_R(X)$, then the nS_β-open sets which contain x are $U_R(X)$, $B_R(X)$, $B_R(X)\cup B$ and $A\supseteq U_R(X)$. Then:

$$x\in B_R(X)\subseteq nS_\beta cl(B_R(X))\subseteq\begin{cases} B_R(X)\\ U_R(X)\\ B_R(X)\cup B\\ A\supseteq U_R(X).\end{cases}$$

If $x\notin U_R(X)$, then the nS_β-open sets which contain x are $L_R(X)\cup B$, $B_R(X)\cup B$ and any subsets for which $A\supseteq U_R(X)$. Then:

$$x\in\begin{cases} L_R(X)\cup B\\ B_R(X)\cup B\end{cases}\subseteq nS_\beta cl\left(\begin{cases} L_R(X)\cup B\\ B_R(X)\cup B\end{cases}\right)\subseteq\begin{cases} L_R(X)\cup B\\ B_R(X)\cup B\\ A\supseteq U_R(X)\end{cases}$$

Therefore, U is nS^*_β-regular space. □

Remark 3.6. Let $(U, \tau_R(X))$ be a nano topological space, then U is not nS^*_β-regular if:

1. $U_R(X)=L_R(X)\neq U$ and $U_R(X)$ contain more than one element of U. Since $nS_\beta cl(U_R(X))=U\nsubseteq U_R(X)$, then U is not nS^*_β-regular,
2. $U_R(X)\neq U$, $L_R(X)=\phi$ and $U_R(X)$ contain more than one element of U. Since $nS_\beta cl(U_R(X))=U\nsubseteq U_R(X)$, then U is not nS^*_β-regular,

4. $NanoS^*_\beta$-Normal Spaces

[4] defined nS_β-normal spaces as: if for any disjoint nano closed sets A, B of U, there exist nS_β-open sets G and H such that $A \subseteq G, B \subseteq H$ and $G \cap H = \phi$.". As clear A and B are nano closed sets, but we are going to use any disjoint nS_β-closed set instead of any disjoint nano closed set. So, in this section, we define nS^*_β-normal as follow.

Definition 4.1. A nano topological space U is said to be nS^*_β-normal if for any disjoint nS_β-closed sets A, B of U, there exist nS_β-open sets G and H such that $A \subseteq G, B \subseteq H$ and $G \cap H = \phi$.

Example 4.2. *Let* $U = \{a, b, c, d\}$ *with* $U/R = \{\{a, b\}, \{c, d\}\}$ *and* $X = \{a, b, c\}$. *Then* $\tau_R(X) = \{\phi, U, \{a, b\}, \{c, d\}\} = nS_\beta O(U, X)$. *Then,* U *is* nS^*_β*-normal space.*

Theorem 4.3. *The nano topological space* $(U, \tau_R(X))$ *is* nS^*_β*-normal if and only if for each* nS_β*-closed set* F *in* U *and* nS_β*-open set* G *contains* F*, there is a* nS_β*-open set* H *such that* $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.

Proof. Suppose that G is nS_β-open set containing F, then $U - G$ and F are disjoint nS_β-closed sets in U. Since U is nS^*_β-normal, there exist nS_β-open sets H and V such that $F \subseteq H, U - G \subseteq V$ and $H \cap V = \phi$. Hence $F \subseteq H \subseteq nS_\beta cl(H) \subseteq nS_\beta cl(U - V) = U - V \subseteq G$, or $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.
Conversely, assume that for any nS^*_β-closed F and nano open set G containing F, there exists a nS_β-open set H such that $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$. Let F and K be disjoint nS_β-closed sets in U. So $F \cap K = \phi$ then $F \subseteq U - K$. As F is a nS_β-closed set and $U - K$ is a nS_β-open set, by assumption, $\exists$ nS_β-open sets H in U such that, $F \subseteq H \subseteq nS_\beta cl(H) \subseteq U - K$. We get $K \subseteq U - nS_\beta cl(H)$. Define $G = U - nS_\beta cl(H)$. Thus $\exists$ $G, H \in nS_\beta O(U, X)$ such that $F \subseteq H, K \subseteq G$ and $H \cap G = \phi$. Hence U is a nS^*_β-normal space. □

Theorem 4.4. *Let* $(U, \tau_R(X))$ *be a nano topological space, then* U *is* nS^*_β*-normal if:*

1. $U_R(X) = U$ *and* $L_R(X) = \phi$.
2. $U_R(X) = U$ *and* $L_R(X) \neq \phi$.
3. $U_R(X) = L_R(X) = \{x\}$, $x \in U$.
4. $U_R(X) \neq L_R(X)$, *where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$

Proof.

1. $\tau_R(X) = nS_\beta O(U, X) = \{\phi, U\}$. Hence, U is nS^*_β-normal space.
2. $\tau_R(X) = nS_\beta O(U, X) = \{\phi, U, L_R(X), B_R(X)\}$. Since $L_R(X) \cap B_R(X) = \phi$, $L_R(X) \cup B_R(X) = U$ and $L_R(X) = [B_R(X)]^c$. Hence by Theorem 4.3, U is nS^*_β-normal space.
3. $\tau_R(X) = \{\phi, U, \{x\}\}$ and $nS_\beta O(U, X) = \{\phi, U\}$. Then it is clear that U is nS^*_β-normal.
4. The only non-empty proper nS_β-closed sets are $[U_R(X)]^c$, $[L_R(X)]^c$, $[B_R(X)]^c$, $[L_R(X) \cup B]^c$, $[B_R(X) \cup B]^c$ where $B \subseteq [U_R(X)]^C$ and those subsets A where $A^C \subseteq [U_R(X)]^C$. Then:

$[U_R(X)]^c \subseteq L_R(X) \cup B \subseteq nS_\beta cl(L_R(X) \cup B) \subseteq L_R(X) \cup B$ and

$$[U_R(X)]^c \subseteq B_R(X) \cup B \subseteq nS_\beta cl(B_R(X) \cup B) \subseteq B_R(X) \cup B.$$

Also,
$[L_R(X)]^c \subseteq B_R(X) \cup B \subseteq nS_\beta cl(B_R(X) \cup B) \subseteq B_R(X) \cup B$ and

$$[B_R(X)]^c \subseteq L_R(X) \cup B \subseteq nS_\beta cl(L_R(X) \cup B) \subseteq L_R(X) \cup B.$$

Also, $[L_R(X) \cup B]^c \subseteq B_R(X) \subseteq nS_\beta cl(B_R(X)) \subseteq \begin{cases} U_R(X) \\ B_R(X) \\ A \supseteq U_R(X) \\ B_R(X) \cup B \end{cases}$ and

$$[B_R(X) \cup B]^c \subseteq L_R(X) \subseteq nS_\beta cl(L_R(X)) \subseteq \begin{cases} U_R(X) \\ L_R(X) \\ A \supseteq U_R(X) \\ L_R(X) \cup B \end{cases}$$

Therefore, U is nS^*_β-normal space. □

Remark 4.5. Let $(U,\ \tau_R(X))$ be a nano topological space where $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U (or $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U), then U may not be nS^*_β-normal space in general. Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\phi,\ U,\ \{a, b\}\}$ and $nS_\beta O(U, X) = \{\phi,\ U,\ \{a, b\},\ \{a, b, c\},\ \{a, b, d\}\}$. Now, $\{c\}$ and $\{d\}$ are disjoint nS_β-closed sets in U, but there are no non-empty disjoint nS_β-open sets. Hence, U is not nS^*_β-normal space.

In addition, the next result provide a condition for those two cases, then they are going to be nS^*_β-normal space.

Theorem 4.6. *Let* $(U,\ \tau_R(X))$ *be a nano topological space where* $[U_R(X)]^c = \{x\},\ x \in U$, *then* U *is* nS^*_β*-normal space if:*

1. $U_R(X) = L_R(X) \neq U$ *and* $U_R(X)$ *contain more than one element of* U.
2. $U_R(X) \neq U$, $L_R(X) = \phi$ *and* $U_R(X)$ *contain more than one element of* U.

Proof. Since in both cases $nS_\beta O(U, X) = \{\phi,\ U,\ U_R(X)\}$. Hence U is nS^*_β-normal space. □

Proposition 4.7. *Let* $(U,\ \tau_R(X))$ *be a nano topological space, then the following statements are true:*

1. *Every* nS^*_β*-regular space is* nS^*_β*-normal space.*
2. *Every* nS^*_β*-regular (resp.,* nS^*_β*-normal) space is* nS_β*-normal space.*
3. *Every* nS_β*-Regular space is* nS^*_β*-regular and* nS^*_β*-normal space.*
4. *Every* nS^*_β*-Regular (resp.,* nS^*_β*-normal) space is* nS_β*-normal space.*

Proof. Clear. □

The convers of each part of above theorem may not be true and examples can be constructed easily from the cases.

5. Conclusion

In this paper, we have introduced the concepts of nano S^*_β-regular and S^*_β-normal spaces in nano topological spaces. According to the family of all nano S_β-open sets, the spaces are studied, and the relationship among them is presented in the table below. As an instruction, the "1" means the case holds and "0" means the case does not hold and the result is closed. Additionally, the "0*" means the case holds under a condition and the result is not closed.

Family of nS_β-open sets in term of upper and lower approximations	nS_β-regular	nS^*_β-regular	nS_β-normal	nS^*_β-normal
1)$U_R(X) = U$ and $L_R(X) = \phi$	1	1	1	1
2)$U_R(X) = U$ and $L_R(X) \neq \phi$	1	1	1	1
3)$U_R(X) = L_R(X) = \{x\}$, $x \in U$	0	1	1	1
4)$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contain more than one element of U.	0	0	1	0*
5)$U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contain more than one element of U.	0	0	1	0*
6)$U_R(X) \neq U$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$	1	1	1	1

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