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Set-fuzzy-set-norm Variation of Set-fuzzy-set Multifunctions

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Abstract

In this paper, we introduce $\mathfrak{F}(\mathbb{R}^p)$, the collection of all finite sets of fuzzy subsets in $\mathcal{F}_{bc}(\mathbb{R}^p)$, which is the class of the upper semicontinuous fuzzy subsets of $\mathbb{R}^p$ with bounded convex compact closure of the supports. Then we define set-fuzzy-set multifunctions and discuss various kinds of them such as monotone, fuzzy measure, subadditive, null-additive, null-null-additive, multisubmeasure and null-continuous. We introduce the set-fuzzy-set-norm variation of a set-fuzzy-set multifunction and present some properties between a set-fuzzy-set multifunction and its variations such as fuzzyness, continuity from below, null-additivity, and null-null-additivity.

Keywords: Fuzzy-set-norm, Fuzzy measure, Subadditive, Set-fuzzy-set multifunctions, Set-fuzzy-set-norm variation.
2010 MSC: MSC code1, MSC code2, more.

1. Introduction and preliminaries

Set-valued analysis and non-additive measures have been intensively studied by many authors such as Aletti, Bongiorno and Capasso [3], Apreutesei [4], Aumann and Shapley [6], Chitescu [7], Choquet [8], Croitoru et al. [9, 10, 11, 13, 14], Dempster [16], Dubois and Prade [19], Gavrilut et al. [22, 21, 24] Pap [28], Papageorgiou [29], Precupanu et al. [30, 31], Shafer [33], Stamate [35], Sugeno [34], and Wen et al. [37].

Non-additive set multifunctions and fuzzy sets have been discussed by many authors due to its applications in statistics, economy, theory of games, and human decision making. For example Asahina [5], Choquet [8], Daneshgar and Hashemi [15], Denneberg [17], Drewnowski [18], Dubois and Prade [19], Funiokova [20], Li [26], Merghadi and Aliouche [27], Pap [28], Precupanu [30], Shafer [33], Sugeno [35], Suzuki [36], Zadeh [23], Vaezpour and Karimi [37], Wen, Shi and Li [38], Wu and Bo [39], Gavrilut et al. [21, 24] extended the concepts of pseudo-atom, Darboux property, continuity, exhaustivity, and regularity, to the set-valued case. In [23], properties of set-norm exhaustive set multifunctions which are used in control, robotics, decision theory or in statistical inference are studied (Dempster [16]). The theory of set multifunctions implies serious and delicate problems, and it can not be reduced to the case of set functions (see Remark 2.7 of [12]).

As a recent research, Akram [1] introduced the notion of m-polar fuzzy matroids and discussed certain applications of m-polar fuzzy matroids in decision support systems, ordering of machines and network analysis. He presented in [2] certain metrics in m-polar fuzzy graphs including antipodal m-polar fuzzy graphs and self median m-polar fuzzy graph of given m-polar fuzzy graphs. Sarwar and Akram[35]

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introduced the concept of bipolar fuzzy matroids and applied it to graph theory and linear algebra. They also described certain applications of bipolar fuzzy matroids in decision support system and network analysis.

While all of the researches about the set multifunctions focused on ordinary sets, in this paper, we demonstrate the possibility of improving current definitions using the finite sets of fuzzy sets and introduce a set-fuzzy-set multifunction and its set-fuzzy-set-norm variations.

This paper is organized as follows: in Section 2, we define fuzzy-set-norm on $\mathcal{F}(\mathbb{R}^p)$, the family of all fuzzy subsete of $\mathbb{R}^p$. According to Due Gil et. al. [25], we consider the class of the upper semicontinuous fuzzy subsets of $\mathbb{R}^p$ with bounded convex compact closure of the supports denoted by $\mathcal{F}_{bc}(\mathbb{R}^p)$. As an useful example, we introduce the fuzzy-set norm $|.|_a$ on $\mathcal{F}_{bc}(\mathbb{R}^2)$. In section 3, we deal with $\mathfrak{F}(\mathbb{R}^p)$ which denotes the collection of all finite sets of fuzzy subsets in $\mathcal{F}_{bc}(\mathbb{R}^p)$. We prove any fuzzy-set-norm on $\mathcal{F}_{bc}(\mathbb{R}^p)$ induces a set-fuzzy-set-norm on $\mathfrak{F}(\mathbb{R}^p)$. In Section 4, we consider a nontrivial fuzzy subset T of M and a ring $\mathfrak{C}$ of fuzzy subsets less than or equal to T. We define various kinds of set-fuzzy-set multifunction $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ such as monotone, fuzzy measure, subadditive, null-additive, null-null-additive, multisubmeasure and null-continuous. We prove four theorems about the relations between them and basic results. In section 5, we install the set-fuzzy-set-norm variation of a set-fuzzy-set multifunction and present some properties between a set-fuzzy-set multifunction and its variation such as fuzzyness, continuity from below, nulladditivity, and null-null-additivity.

2. Fuzzy-set-norms

Consider the p-dimensional Euclidean space $\mathbb{R}^p$ and $|\,.\,|$ as the usual associated norm. Zadeh [40] denoted by $\mathcal{K}(\mathbb{R}^p)$, the class of the nonempty compact subsets of $\mathbb{R}^p$ and by $\mathcal{K}_{bc}(\mathbb{R}^p)$, the subclass of bounded convex sets in $\mathcal{K}(\mathbb{R}^p)$. The space $\mathcal{K}(\mathbb{R}^p)$ can be represented by a semilinear structure using the Minkowski sum and the product by a scalar $K + K' = \{k + k \mid k \in K, k' \in K'\}$, $\alpha K = \{\alpha k \mid k \in K\}$, for $K, K' \in \mathcal{K}(\mathbb{R}^p)$ and $\alpha \in \mathbb{R}$.

We denote by $\mathcal{F}_{bc}(\mathbb{R}^p)$ Due Gil et. al. [25], the class of the upper semicontinuous fuzzy subsets of $\mathbb{R}^p$ with bounded convex compact closure of the support.

$$\mathcal{F}_{bc}(\mathbb{R}^p) = \{E : \mathbb{R}^p \to [0,1] \mid E_\alpha \in \mathcal{K}_{bc}(\mathbb{R}^p), \forall \alpha \in [0,1]\}$$

The space $\mathcal{F}_{bc}(\mathbb{R}^p)$ can be endowed with a semilinear structure by means of the sum and the product by a scalar based on Zadeh's extension principle [40]. Applying this principle for elements in $\mathcal{F}_{bc}(\mathbb{R}^p)$ states that

$$(E + F)(x) = \sup_{y+z=x} \min\{E(y), F(z)\}, \tag{2.1}$$

$$(\alpha E)(x) = \sup_{\alpha y = x} E(y) = \begin{cases} E(\frac{1}{\alpha}x), & \text{if } \alpha \neq 0 \\ 1_{\{0\}}(x), & \text{if } \alpha = 0 \end{cases} \tag{2.2}$$

for $E, F \in \mathcal{F}(\mathbb{R}^p)$ and $\alpha \in \mathbb{R}$.

Definition 2.1. A function $|\,.\,| : \mathcal{F}(\mathbb{R}^p) \to [0, +\infty]$ is called a fuzzy-set-norm on $\mathcal{F}(\mathbb{R}^p)$ if it satisfies the conditions:

i) $|E| = 0 \Leftrightarrow E = \tilde{0}$, for $E \in \mathcal{F}(\mathbb{R}^p)$;

ii) $|\alpha E| = |\alpha||E|$, for all $\alpha \in \mathbb{R},\ E \in \mathcal{F}(\mathbb{R}^p)$;

iii) $|E + F| < |E| + |F|,\ \forall E, F \in \mathcal{F}(\mathbb{R}^p)$.

Example 2.2. If $(\mathbb{R}^p, \|\ \ \|)$ is a normed space, then the function

$$|E|_s = \sup\{\|x\| : \ x \in \operatorname{supp} E\}$$

is a fuzzy-set-norm on $(\mathcal{F}(\mathbb{R}^p), \leqslant)$, called the supremum fuzzy-set-norm on $\mathcal{F}(\mathbb{R}^p)$.

Definition 2.3. A fuzzy-set-norm $|\ .\ |$ on $\mathcal{F}(\mathbb{R}^p)$ is called monotone if for every sets $E, F \in \mathcal{F}(\mathbb{R}^p)$, we have $E \leqslant F \ \Rightarrow\ |E| \leqslant |F|$.
We denote $(\mathcal{F}(\mathbb{R}^p), \leqslant, |\ .\ |)$ when $(\mathcal{F}(\mathbb{R}^p), \leqslant)$ is endowed with a monotone fuzzy-set-norm $|\ .\ |$.

Example 2.4. We define the fuzzy-set-norm $|\ .\ |_a$ on $\mathcal{F}_{bc}(\mathbb{R}^2)$ by

$$|E|_a = \iint_{\operatorname{supp} E} E(x, y)\ dxdy.$$

Since $E(x, y) \in [0, 1]$ and $\operatorname{supp} E$ is a bounded convex compact subset of $\mathbb{R}^2$, this integral equals to a nonnegative real number. We show that $|\ .\ |_a$ satisfies three conditions of Def. 2.2:
$i)$ It is clear that $|E|_a = 0$ if and only if $\operatorname{supp} E = \phi$ and or $E = \tilde{0}$.
$ii), iii)$ Since for each $\alpha \in \mathbb{R}$ and $E, F \in \mathcal{F}_{bc}(\mathbb{R}^2)$, we define $(\alpha E)(x) = \alpha E(x)$ and $(E + F)(x) = E(x) + F(x)$, hence $\operatorname{supp}(\alpha E)$ and $\operatorname{supp}(E + F)$ are equal to the bounded convex compact subsets $\operatorname{supp} E$ and $\operatorname{supp} E \cup \operatorname{supp} F$ respectively. By using linearity of integration, we see (ii) and (iii) hold. Also for every fuzzy subsets $E, F \in \mathcal{F}(\mathbb{R}^2)$, we have

$$E \leqslant F \quad \Rightarrow \quad \operatorname{supp} E \subseteq \operatorname{supp} F \quad \Rightarrow \quad |E|_a \leqslant |F|_a.$$

Therefore, $|\ .\ |_a$ is a monotone fuzzy-set-norm on $(\mathcal{F}(\mathbb{R}^2), \leqslant)$.

3. Set-fuzzy-set-norms

The largest set class of set of fuzzy set values, we will deal with in this paper, is $\mathfrak{F}(\mathbb{R}^p)$, which denotes the collection of all finite sets of fuzzy subsets

$$\tilde{E} = \left\{E_i \mid E_i \in \mathcal{F}_{bc}(\mathbb{R}^p), \quad 1 \leqslant i \leqslant k\right\}. \tag{3.1}$$

We denote $(\mathfrak{F}(\mathbb{R}^p), \leqslant)$ since we consider the usual fuzzy set inclusion $\subseteq$ as an order $\leqslant$ on $\mathfrak{F}(\mathbb{R}^p)$. The space $\mathfrak{F}(\mathbb{R}^p)$ can be endowed with a semilinear structure by the following defined sum and the product by a scalar:

1) Using (2.1), we define every $\alpha \in \mathbb{R}$ and each $\tilde{E} \in \mathfrak{F}(\mathbb{R}^p)$ in the form (3.1),

$$\alpha\tilde{E} = \left\{\alpha E_i \mid E_i \in \mathcal{F}_{bc}(\mathbb{R}^p), \quad 1 \leqslant i \leqslant k\right\}.$$

2) Using (2.2), we define each $\tilde{E} \in \mathfrak{F}(\mathbb{R}^p)$ in the form (3.1) and each ,$\tilde{F}$ in the form

$$\tilde{F} = \left\{F_i \mid F_i \in \mathcal{F}_{bc}(\mathbb{R}^p), \quad 1 \leqslant i \leqslant k'\right\}.$$

Without loss of generality, assume that $k \leqslant k'$

$$\tilde{E} + \tilde{F} = \left\{E_i + F_i,\ \tilde{0} + F_j \mid E_i \in \tilde{E},\ F_i, F_j \in \tilde{F},\ 1 \leqslant i \leqslant k,\ \ k + 1 \leqslant j \leqslant k'\right\}.$$

Definition 3.1. A function $|\ .\ | : \mathfrak{F}(\mathbb{R}^p) \to [0, +\infty]$ is called a set-fuzzy-set-norm on $\mathfrak{F}(\mathbb{R}^p)$ if it satisfies the conditions:

i) $|\tilde{E}| = 0 \Leftrightarrow \tilde{E} = \{\tilde{0}\}$, for $\tilde{E} \in \mathfrak{F}(\mathbb{R}^p)$;

ii) $|\alpha\tilde{E}| = |\alpha||\tilde{E}|$, for all $\alpha \in \mathbb{R},\ \tilde{E} \in \mathfrak{F}(\mathbb{R}^p)$;

iii) $|\tilde{E}+\tilde{F}| < |\tilde{E}| + |\tilde{F}|, \ \forall \tilde{E}, \tilde{F} \in \mathfrak{F}(\mathbb{R}^p)$.

Lemma 3.2. *Let $| \,.\, |$ be a fuzzy-set-norm on $\mathcal{F}_{bc}(\mathbb{R}^p)$. We have an induced set-fuzzy-set-norm on $\mathfrak{F}(\mathbb{R}^p)$ defined by*

$$|\tilde{E}| = \sum_{i=1}^{k} |E_i|$$

for each $\tilde{E} = \big\{E_i \mid E_i \in \mathcal{F}_{bc}(\mathbb{R}^p), \quad 1 \leqslant i \leqslant k\big\}$.

Proof. Since the finite sum of several non-negative real numbers is a non-negative real number, this norm is well defined. Also we show that it satisfies the conditions of Def. 2.2:
i)

$$\begin{aligned} |\tilde{E}| = 0 \ &\Leftrightarrow \ |E_i| = 0, \ \ 1 \leqslant i \leqslant k \\ &\Leftrightarrow \ E_i = \tilde{0}, \ \ 1 \leqslant i \leqslant k \\ &\Leftrightarrow \ E = \{\tilde{0}\} \end{aligned}$$

ii) Since $|\alpha E_i| = |\alpha||E_i|$, for all $1 \leqslant i \leqslant k$, we have

$$|\alpha\tilde{E}| = \sum_{i=1}^{k} |\alpha E_i| = \sum_{i=1}^{k} |\alpha||E_i| = |\alpha| \sum_{i=1}^{k} |E_i| = |\alpha||\tilde{E}|$$

iii)

$$\begin{aligned} |\tilde{E}+\tilde{F}| &= \textstyle\sum_{i=1}^{k} |E_i + F_i| + \sum_{j=k+1}^{k'} |\tilde{0} + F_j| \\ &\leqslant \textstyle\sum_{i=1}^{k} |E_i| + |F_i| + \sum_{j=k+1}^{k'} |\tilde{0}| + |F_j| \\ &= \textstyle\sum_{i=1}^{k} |E_i| + |F_i| + \sum_{j=k+1}^{k'} 0 + |F_j| \\ &= \textstyle\sum_{i=1}^{k} |E_i| + \sum_{i=1}^{k'} |F_i| \\ &= |\tilde{E}| + |\tilde{F}| \end{aligned}$$

□

4. Set-fuzzy-set multifunctions

Let T be a nontrivial fuzzy subset of M, then any fuzzy subset of M, which is less than or equal to T, is called an fuzzy subset of T. Let $\mathfrak{C}$ be a ring of fuzzy subsets of T and F(T) be the family of all fuzzy subsets of T.

Definition 4.1. A set-fuzzy-set multifunction $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ is called:

(i) monotone if $\mu(A) \leqslant \mu(B)$ for every $A, B \in \mathfrak{C}$ with $A \leqslant B$.

(ii) fuzzy measure if μ is monotone and $\mu(\tilde{0}) = \{\tilde{0}\}$.

(iii) subadditive if $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for every $A, B \in \mathfrak{C}$.

(iv) null-additive if for every $A, B \in \mathfrak{C}$, $\mu(B) = \mu(\tilde{0}) \ \Rightarrow \ \mu(A \cup B) = \mu(A)$.

(v) null-null-additive if for every $A, B \in \mathfrak{C}$, $\mu(A) = \mu(B) = \mu(\tilde{0}) \ \Rightarrow \ \mu(A \cup B) = \mu(\tilde{0})$.

(vi) multisubmeasure if μ is fuzzy and subadditive.

(vii) null-continuous if $\mu(\bigcup_{n=0}^{\infty} A_n) = \{\tilde{0}\}$ for every sequence $(A_n)_{n\in N^*} \subset \mathfrak{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A_n) = \{\tilde{0}\}$ for every $n \in N^*$.

Theorem 4.2. *The following concepts are true:*

(i) If μ is a multisubmeasure, then μ is null-additive.

(ii) If μ is null-additive, then μ is null-null-additive.

Proof. (i) Assume that μ is multisubmeasure. Let $A, B \in \mathfrak{C}$ and $\mu(B) = \mu(\tilde{0})$ then

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B) = \mu(A) + \mu(\tilde{0}) = \mu(A) + \tilde{0} = \mu(A)$$

Since μ is monotone, $\mu(A) \leqslant \mu(A \cup B)$. Therefore $\mu(A \cup B) = \mu(A)$. So μ is null-additive.

(ii) Assume that μ is null-additive. Let $A, B \in \mathfrak{C}$ and $\mu(A) = \mu(B) = \mu(\tilde{0})$, then

$$\mu(A \cup B) = \mu(A) = \mu(\tilde{0})$$

Therefore μ is null-null-additive. □

Definition 4.3. A set-fuzzy-set multifunction $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ is called:

(i) fuzzy-set-norm exhaustive (shortly, fsn-exhaustive) if $\lim_{n\to\infty} |\mu(A_n)| = 0$ for every mutually disjoint sequence of sets $(A_n)_{n\in N^*} \subset \mathfrak{C}$.

(ii) fuzzy-set-norm continuous (shortly, fsn-continuous) if $\lim_{n\to\infty} |\mu(A_n)| = 0$ for every $(A_n)_{n\in N^*} \subset \mathfrak{C}$ such that $A_n \searrow \tilde{0}$ (i.e. $A_n \supseteq A_{n+1}$ for all $n \in N^*$ and $\bigcap_{n=0}^{\infty} A_n = \tilde{0}$).

(iii) fsn-continuous from below if $|\mu(\bigcup_{n=0}^{\infty} A_n)| = \lim_{n\to\infty} |\mu(A_n)|$ for every $(A_n)_{n\in N^*} \subset \mathfrak{C}$ with $A_n \subseteq A_{n+1}$ for all $n \in N^*$ (denoted by $A_n \nearrow A$) where $A = \bigcup_{n=0}^{\infty} A_n$.

(iv) strongly fsn-continuous if $\lim_{n\to\infty} |\mu(A_n)| = 0$ for every $(A_n)_{n\in N^*} \subset \mathfrak{C}$ such that $A_n \supseteq A_{n+1}$ for all $n \in N^*$ and $\mu(\bigcap_{n=0}^{\infty} A_n) = \{\tilde{0}\}$.

Definition 4.4. $\mathfrak{C}$ is a σ-ring if the following conditions are met:

(i) $A \backslash B \in \mathfrak{C}$ for every $A, B \in \mathfrak{C}$,

(ii) $\bigcup_{n=0}^{\infty} A_n \in \mathfrak{C}$ for every $(A_n)_{n\in N^*} \subset \mathfrak{C}$.

Theorem 4.5. *If $\mathfrak{C}$ is a σ-ring and $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ is fuzzy measure and fsn-continuous, then μ is fsn-exhaustive.*

Proof. Let $(A_n)_{n\in N^*}$ be a sequence of pairwise disjoint sets of $\mathfrak{C}$ and let $B_n = \bigcup_{k=n}^{\infty} A_k$ for all $n \in N^*$. Then $B_n \in \mathfrak{C}$ for every $n \in N^*$ and $B_n \searrow \tilde{0}$. Since μ is fsn-continuous, then $\lim_{n\to\infty} |\mu(B_n)| = 0$, thus $\lim_{n\to\infty} |\mu(A_n)| = 0$. So μ is fsn-exhaustive. □

Theorem 4.6. *Let $\mathfrak{C}$ be a σ-ring and $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ be fuzzy measure. If μ is null-null-additive and strongly fsn-continuous, then μ is null-continuous.*

Proof. Let $(A_n)_{n\in N^*} \subset \mathfrak{C}$ such that $A_n \subseteq A_{n+1}$ and $\mu(A) = \{\tilde{0}\}$ for all $n \in N^*$. Denote $A = \bigcup_{n=0}^{\infty} A_n$. We define a subsequence (A_{n_k}) of (A_n) as follows. Let $n_1 = 1$. For every $k \in N^*$ since $\mu(A_{n_k}) = \{\tilde{0}\}$ and $A_{n_k} \cup (A \backslash A_n) \searrow A_{n_k}$, when $n \to \infty$, by the fact that μ is strongly fsn-continuous, we can choose n_{k+1} so that $n_{k+1} > n_k$ and

$$|\,\mu\big(A_{n_k} \cup (A \backslash A_{n_{k+1}})\big)\,| < \frac{1}{k}$$

Denote $B = \bigcup_{k=1}^{\infty}(A_{n_{2k}} \backslash A_{n_{2k-1}})$ and $C = A\backslash B = A_{n_1} \cup \bigcup_{k=1}^{\infty}(A_{n_{2k+1}} \backslash (A_{n_{2k}})$.

For each $k \in N^*$ since $B \subseteq A_{n_{2k}} \cup (A\backslash(A_{n_{2k+1}})$, so

$$|\mu(B)| \leqslant \mu\big(A_{n_{2k}} \cup (A\backslash(A_{n_{2k+1}}))\big) < \frac{1}{2k}$$

It follows $\mu(B) = \{\tilde{0}\}$. For every $k \in N^*$ since $C \subseteq \big(A_{n_{2k-1}} \cup (A\backslash A_{n_{2k}})\big)$, thus

$$\mu(C) \leqslant \mu\big(A_{n_{2k-1}} \cup (A\backslash A_{n_{2k}})\big) < \frac{1}{2k-1}$$

which implies that $\mu(C) = \{\tilde{0}\}$. Since μ is null-null-additive, we obtain $\mu(A) = \mu(B \cup C) = \{\tilde{0}\}$. Hence μ is null-continuous. □

Definition 4.7. Let $A \in F(T)$. An $\mathfrak{C}$-partition of A is set $\{A_i\}_{i=1}^{n}$ of fuzzy subsets of T that satisfies three conditions:

i) $\tilde{0} \neq A_i \in \mathfrak{C}$,

ii) $A_i \cap A_j = \tilde{0}, \;\; i \neq j, \;\; i, j \in \{1, \ldots, n\}$

iii) $A = \sum_{i=1}^{n} A_i$

Remark 4.8. When we have an ordinary partition $\{D_i\}_{i=1}^{n}$ of the set $suppA$, we can set $A_i = A|_{D_i}$ (i.e. $A_i(x) = A(x)$ for all $x \in D$). If $A_i \in \mathfrak{C}$, $\forall i \in \{1, \ldots, n\}$, then we have an $\mathfrak{C}$-partition of A. Conversly if we have an $\mathfrak{C}$-partition $\{A_i\}_{i=1}^{n}$ of A, then show in Example 4.9.

Example 4.9. Let $M = \{1, 2, 3\}$ and $T : M \to [0, 1]$, $T(1) = T(2) = T(3) = 0.8$, $\mathfrak{C} = F(T)$. We define the fuzzy set $A : M \to [0, 1]$ by:

$$A(x) = \begin{cases} 0.3 & x = 1 \\ 0.7 & x = 2 \\ 1 & x = 3 \end{cases}$$

and the fuzzy sets $A_1, \; A_2 : M \to [0, 1]$ by:

$$A_1(x) = \begin{cases} 0.3 & x = 1 \\ 0 & x = 2 \\ 0 & x = 3, \end{cases} \qquad A_2(x) = \begin{cases} 0 & x = 1 \\ 0.7 & x = 2 \\ 0 & x = 3. \end{cases}$$

It is seen that

$$A_1(1) + A_2(1) = 0.3 = A(1), \quad A_1(2) + A_2(2) = 0.7 = A(2), \quad A_1(1) + A_2(2) = 0.3 + 0.7 = 1 = A(3).$$

Hence $A_1 + A_2 = A$. Since $A_1 \cap A_2 = \tilde{0}$; therefore $\{A_1, A_2\}$ is an $\mathfrak{C}$-partition of A. But $\{suppA_1, \; suppA_2\} = \big\{\{1\}, \; \{2\}\big\}$ is not a partition of $suppA = \{1, 2, 3\}$.

5. Set-fuzzy-set-norm variation

Definition 5.1. For a set-fuzzy-set multifunction $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, | \; . \; |)$, the following function is introduced as: $\bar{\mu} : F(T) \to [0, \infty]$,

$$\bar{\mu}(A) = \sup\left\{ \sum_{i=1}^{n} |\mu(A_i)| \;\middle|\; \{A_i\}_{i=1}^{n} \text{ is an } \mathfrak{C}-\text{partition of } A \right\} \tag{5.1}$$

for every $A \in F(T)$. $\bar{\mu}$ is called the set-fuzzy-set-norm variation (shortly variation) of μ.

Theorem 5.2. *Let* $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\ .\ |)$ *be a set-fuzzy-set multifunction. Then the following statements hold:*

i) $|\mu(A)| \leqslant \bar{\mu}(A)$ *for all* $A \in \mathfrak{C}$.

ii) $\bar{\mu}(A) = 0 \Rightarrow |\mu(A)| = 0$, *for every* $A \in \mathfrak{C}$.

iii) Furthermore, if μ *is monotone, then we also have:*

$$|\mu(A)| = 0 \;\Rightarrow\; \bar{\mu}(A) = 0 \text{ for every } A \in \mathfrak{C}. \tag{5.2}$$

iv) If μ *has a finite variation, then* μ *is fsn-exhaustive.*

v) $\bar{\mu}$ *is monotone.*

vi) If $\mu(\tilde{0}) = \tilde{0}$, *then* $\bar{\mu}(\tilde{0}) = 0$.

vii) If μ *is fuzzy, then* $\bar{\mu}$ *is fuzzy.*

viii) For each disjoint pair fuzzy subsets $A, B \in F(T)$, *we have* $\bar{\mu}(A \cup B) \geqslant \bar{\mu}(A) + \bar{\mu}(B)$. *But in the case* $A \cap B \neq \tilde{0}$, *it may be false. So* $\bar{\mu}$ *is not neceserily superadditive. See Exmaple* 5.3(a).

ix) Suppose $|\ .\ |$ *is a monotone set-fuzzy-set-norm on* $(\mathfrak{F}(\mathbb{R}^p), \leqslant, |\ .\ |)$. *If* $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\ .\ |)$ *is a fuzzy set-fuzzy-set multifunction and* $A \in \mathfrak{C}$ *is an atom of* μ, *then* $\bar{\mu}(A) = |\mu(A)|$

Proof. (i) Since $A \in \mathfrak{C}$, $\{A\}$ is an $\mathfrak{C}$-partition of A. Therefore, (i) holds.

(ii) This part is a direct result of (i).

(iii) Let ${B_i}_{i=1}^n$ be an $\mathfrak{C}$-partition of A. Since μ is monotone, $|\mu(B_i)| \leqslant |\mu(A)| = 0$ for all $i \in \{1, \ldots, n\}$. Hence $\sum_{i=1}^n |\mu(B_i)| = 0$. So $\bar{\mu}(A) = 0$
If μ is not monotone, then (4.1) may be false. See Example 5.3(b).

(iv) Indeed, if μ would not be exhaustive, then there are $\varepsilon > 0$ and $(A_n)_{n \in N^*} \subset \mathfrak{C}$ such that, $A_n \cap A_m = \phi (n \neq m)$ so that $|\mu(An)| > \varepsilon$ for every $n \in N^*$. Then $|\bar{\mu}(An)| > \varepsilon$ for every $n \in N^*$. Since $|\bar{\mu}(A_n)| < \infty$ for every $n \in N^*$, there is $\mathfrak{C}$-partition $\{B_i^n\}_{i=1}^{m_n}$ of A_n, so that $\sum_{i=1}^{m_n} B_i^n = A_n$ and

$$|\bar{\mu}(A_n)| - \frac{\varepsilon}{2} \;<\; \sum_{i=1}^{m_n} |\mu(B_i^n)|$$

Thus $\sum_{i=1}^{m_n} |\mu(B_i^n)| > \frac{\varepsilon}{2}$, which implies that

$$|\bar{\mu}(T)| > \sum_{n=1}^{\infty} \sum_{i=1}^{m_n} |\mu(B_i^n)| > \frac{m\varepsilon}{2}$$

for every $m \in N$. We obtain $|\bar{\mu}(T)| = +\infty$, false.

(v) Let $A, B \in F(T)$ so that $A \leqslant B$ and let $\{A_i\}_{i=1}^n$ be an $\mathfrak{C}$-partition of A. Then $A_i \subseteq B$ for every $i \in \{1, \ldots, n\}$ and so $\sum_{i=1}^n |\mu(A_i)| \leqslant \bar{\mu}(B)$. Taking the supremum over all $\mathfrak{C}$-partitions $\{A_i\}_{i=1}^n$, we obtain $\bar{\mu}(A) \leqslant \bar{\mu}(B)$.

$vi)$ If $\{A_i\}_{i=1}^n$ be an $\mathfrak{C}$-partition of $\tilde{0}$. Then $A_i = \tilde{0}$ and so $|\mu(A_i)| = 0$ for every $i \in \{1, \ldots, n\}$. So $\sum_{i=1}^n |\mu(A_i)| = 0$. Since $\{A_i\}_{i=1}^n$ is arbitrary, it implies that $\bar{\mu}(\tilde{0}) = 0$

(vii) This follows (v) and (vi).

$(viii)$ Let $A, B \in F(T)$. If $\bar{\mu}(A) = +\infty$ or $\bar{\mu}(A) = +\infty$ since $\bar{\mu}$ is monotone, $\bar{\mu}(A \cup B) = +\infty$ and so $\bar{\mu}(A \cup B) \geqslant \bar{\mu}(A) + \bar{\mu}(B)$.
Suppose $\bar{\mu}(A) < +\infty$ and $\bar{\mu}(B) < +\infty$. Let $\{A_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m$ be $\mathfrak{C}$-partitions of A and B respectively. Since $A_i \subseteq A \cup B$ for all $i \in \{1, \ldots, n\}$ and $B_j \subseteq A \cup B$ for every $j \in \{1, \ldots, m\}$ and $A \cap B = \tilde{0}$, hence $\sum_{i=1}^n |\mu(A_i)| + \sum_{j=1}^m |\mu(B_i)| \leqslant \bar{\mu}(A \cup B)$. Taking the supremum firstly over $\{A_i\}_{i=1}^n$ and secondly over $\{B_j\}_{j=1}^m$ we obtain $\bar{\mu}(A) + \bar{\mu}(B) \leqslant \bar{\mu}(A \cup B)$.

(ix) Recal that (see [11, 24]) that $A \in \mathfrak{C}$ is called an atom of μ if $\mu(A) > \mu(\phi)$ and for every $B \in \mathfrak{C}$ with $B \subseteq A$, we have $\mu(B) = \mu(\phi)$ or $\mu(A \backslash B) = \mu(\phi)$.
We prove that $\bar{\mu}(A) \leqslant |\mu(A)|$. Let $\{B_i\}_{i=1}^n \subseteq \mathfrak{C}$ be an arbitrary $\mathfrak{C}$-partition of A, where $n \in N^*$. If $\mu(B_i) = \tilde{0}$, for every $i \in \{1, \ldots, n\}$, then $\sum_{i=1}^n |\mu(B_i)| = 0 \leqslant |\mu(A)|$.
Without loss of generality assume that $\mu(B_1) > \tilde{0}$ and $\mu(B_2) > \tilde{0}$. Since A is an atom of μ, $\mu(A \backslash B_1) = \{\tilde{0}\}$. Since $B_2 \subseteq A \backslash B_1$ and μ is fuzzy, it follows $\mu(B_2) = \{\tilde{0}\}$, false. Thus there exists an unique $i_0 \in \{1, \ldots, n\}$ such that $\mu(B_{i_0}) > \{\tilde{0}\}$. Since $B_i \subseteq A \backslash B_{i_0}$ and μ is fuzzy, it results $\mu(B_i) = \{\tilde{0}\}$, for every $i \in \{1, \ldots, n\} \backslash \{i_0\}$. So $\sum_{i=1}^n |\mu(B_i)| = |\mu(B_{i_0})| \leqslant |\mu(A)|$. Since $\{B_i\}_{i=1}^n$ is arbitrary, we conclude that $\mu(A) \leqslant |\mu(A)|$. Using now Theorem 5.2$(i)$, it results $\bar{\mu}(A) = |\mu(A)|$. □

Example 5.3. Let $M = \{1, 2\}$, $T : M \to [0, 1]$, $T(1) = T(2) = 0.9$, $\mathfrak{C} = F(T)$. We define $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}), \leqslant, |\,.\,|_s)$ by

$$\mu(A) = \begin{cases} \tilde{0} & suppA = \phi \text{ or } suppA = M \\ \chi_{[1,3]} & suppA = \{1\} \\ \chi_{[2,4]} & suppA = \{2\}. \end{cases}$$

We have

$$|\mu(A)|_s = \begin{cases} 0 & suppA = \phi \text{ or } suppA = M \\ 3 & suppA = \{1\} \\ 4 & suppA = \{2\} \end{cases}$$

We define the fuzzy set $D, B : M \to [0, 1]$, by:

$$D(x) = \begin{cases} 0.2 & x = 1 \\ 0.7 & x = 2 \end{cases} \qquad B(x) = \begin{cases} 0 & x = 1 \\ 0.5 & x = 2 \end{cases}$$

and the fuzzy sets $A_1,\ A_2 : M \to [0, 1]$ by:

$$A_1(x) = \begin{cases} 0.2 & x = 1 \\ 0 & x = 2 \end{cases} \qquad A_2(x) = \begin{cases} 0 & x = 1 \\ 0.7 & x = 2 \end{cases}$$

$a)$ It is seen that $\{A_1, A_2\}$ is an $\mathfrak{C}$-partition of D and $\{B\}$ is an $\mathfrak{C}$-partition of B . But

$$\bar{\mu}(A) + \bar{\mu}(B) = |\mu(A_1)|_s + |\mu(A_2)|_s + |\mu(B)|_s = 3 + 4 + 4 = 11\ >\ 7 = \bar{\mu}(A \cup B).$$

$b)$ It is seen that μ is not monotone: $A_1 \leqslant A$ but $\mu(A_1) = \chi_{[1,3]} \geqslant \mu(A) = \{\tilde{0}\}$.
$c)$ It is seen that (4.1) is false: $\mu(A) = \{\tilde{0}\}$ but $\bar{\mu}(A) = 7$.

Theorem 5.4. *Let $\mathfrak{C}$ is a σ-ring and $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^p), \leqslant, |\,.\,|)$ a set-fuzzy-set multifunction*

I) If μ is continuous from below, then $\bar{\mu}$ is continuous from below on $\mathfrak{C}$.

II) If μ is a multisubmeasure of finite variation and μ is continuous from below on $\mathfrak{C}$ then $\bar{\mu}$ is continuous from below.

Proof. I. Let $(A_n)_{n\in N^*} \subset \mathfrak{C}$ so that, $A_n \nearrow A$ where $A = \bigcup_{n=0}^{\infty} A_n$. We show that

$$\lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}(A).$$

Let $\{B_i\}_{i=1}^m$ be an $\mathfrak{C}$-partition of A, we have

$$\begin{aligned} \textstyle\sum_{i=1}^m |\mu(B_i)| &= \textstyle\sum_{i=1}^m \big(|\mu(B_i)| - |\mu(B_i \cap A_n)|\big) + \sum_{i=1}^m |\mu(B_i \cap A_n)| \\ &\leqslant \textstyle\sum_{i=1}^m \big(|\mu(B_i)| - |\mu(B_i \cap A_n)|\big) + \bar{\mu}(A_n) \end{aligned}$$

Since $B_i \cap A_n \nearrow B_i$ and μ is continuous from below for every $i \in \{1, \dots, m\}$, there is $n_i(\varepsilon)$, $n_{i\in\mathbb{N}}$ so that $0 \leqslant |\mu(B_i)| - |\mu(B_i \cap A_n)| < \dfrac{\varepsilon}{2^{i+1}}$ for every $n > n_i$. Let $n_0 = \max_{0\leqslant i\leqslant m}\ n_i$. Then for all $n > n_0$ we have

$$\sum_{i=1}^m |\mu(B_i)| = \sum_{i=1}^m \frac{\varepsilon}{2^i} + \bar{\mu}(A_n) < \varepsilon + \bar{\mu}(A_n)$$

Since B_i is arbitrary, we obtain $\bar{\mu}(A) < \varepsilon + \bar{\mu}(A_n)$ for every $n > n_0$. This shows that

$$\lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}(A)$$

Therefore, $\bar{\mu}$ is continuous from below on $\mathfrak{C}$.
II. Let $(A_n)_{n\in N^*} \subset \mathfrak{C}$ so that, $A_n \nearrow A$. Since $\bar{\mu}$ is superadditive, we have:

$$\bar{\mu}(A) = \bar{\mu}\big((A\backslash A_n) \cup A_n\big) \geqslant \bar{\mu}(A\backslash A_n) + \bar{\mu}(A_n)$$

Since μ is a multisubmeasure, we have:

$$|\mu(A)| \leqslant |\mu(A\backslash A_n)| + |\mu(A_n)|$$

This implies

$$0 \leqslant |\mu(A)| - |\mu(A_n)| \leqslant |\mu(A\backslash A_n)| \leqslant \bar{\mu}(A\backslash A_n) \leqslant \bar{\mu}(A) - \bar{\mu}(A_n)$$

Scince $\bar{\mu}$ is continuous from below, we have $\lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}(A)$.
It results $\lim_{n\to\infty} |\mu(A_n)| = |\mu(A)|$ and so μ is continuous from below. □

Example 5.5. Let $M = \mathbb{R}^{10}$, $T : M \to [0, 1]$,

$$T(x) = \begin{cases} 1 & x = 9999999999 \\ 0 & elsewhere \end{cases}$$

Let $\mathfrak{C} = F(T)$. We know everyone in Iran has a ten-digit number called national code. So we can assign a fuzzy function to each individual, which is actually a fuzzy point using the same code. For example assume that my national code is 0385234673. Then my fuzzy point will be

$$A(x) = \begin{cases} 1 & x = 0385234673 \\ 0 & elsewhere \end{cases}$$

Suppose the tax department wants to determine which apartment or house or land each person owns and how much his share is. Therefore, the function $\mu : \mathfrak{C} \to (\mathfrak{F}(\mathbb{R}^3), \leqslant, |\ .\ |_s)$ assigns to each fuzzy point associated with each person's code, a finite set of fuzzy functions.

$$\mu(A) = \tilde{E} = \big\{E_i \mid E_i \in \mathcal{F}_{bc}(\mathbb{R}^3), \quad 1 \leqslant i \leqslant k\big\},$$

where each $\mathrm{supp}E_i$ denotes an apartment or house or land and E_i shows the share of A. If one person have'nt anything, then $\mu(A) = \{\tilde{0}\}$. For example let I have a 150 meter apartment located on the sixth floor which my share is only 0.6, then the fuzzy set E_1 denotes this:

$$E_1(x,y,z) = \begin{cases} 0.6 & 1 \leqslant x \leqslant 10,\ 15 \leqslant y \leqslant 30,\ z = 18 \\ 0 & elsewhere \end{cases}$$

Also, assume that I have a 200-meter plot of land where my share is the whole land. Then the fuzzy set E_2 denotes this:

$$E_2(x,y,z) = \begin{cases} 1 & 10 \leqslant x \leqslant 30,\ 25 \leqslant y \leqslant 35,\ z = 0 \\ 0 & elsewhere \end{cases}$$

Suppose the function $w = g(x,y,z)$ calculates bramelk taxes in different regions, so $\bar{\mu}$, the set-fuzzy-set-norm variation of μ calculates bramelk taxes of each person. Since each fuzzy point A is an atom of μ, thus by using Theorem 5.2(ix), we have

$$\bar{\mu}(A) = |\mu(A)| \tag{5.3}$$

where

$$|\mu(A)| = \sum_{i=1}^{k} |E_i|_t$$

and

$$|E_i|_t = \iint_{\mathrm{supp}E_i} g(x,y,z)E(x,y,z)\ d\sigma. \tag{5.4}$$

When $S = \mathrm{supp}E_i$ is a piece of a quadratic surface $f(x,y,z) = c$, then we define the fuzzy-set-norm $|\ .\ |_t$ on $\mathcal{F}_{bc}(\mathbb{R}^3)$ by:

$$|E_i|_t = \iint_S g(x,y,z)E(x,y,z)\ d\sigma. \tag{5.5}$$

We emphasize that we calculate this surface integral using shadow of S on xy-plane. We can prove almost similar to the proof of Example 2.4 that $|\ .\ |_t$ in formula (5.4) or (5.5) satisfies three conditions of Definition 2.2.

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