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Linear Stability of Thin Liquid Films Flows Down on an Inclined Plane Using Long-Wave Theory

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Abstract

The Long-Wave Theory is applied to investigate the dynamic stability of free thin fluid films flowing down an inclined plane. We assume that thin supported films have a thickness of $\overline{H}$ and less than or equal to one hundred nm. Equations of Navier and Stokes, continuity-equation, and related boundary conditions are used to represent a two-dimensional stream demonstrated as a continuum. Under long-wave approximation, the governing equations for the film interface have been rescaled and simplified to obtain a highly non-linear condition of development for the film interface. A procedure for evaluating the magnitude of the effects of the high-order effects is also used to formulate simplified governing equations. In the future, we can study this problem by adding heat transfer over the stretching plate. In addition, we can also study the stability analysis to two-dimension flow of a viscous liquid within a horizontal thin liquid film with neglecting the inertia terms of Navier-Stokes equations.

Keywords: Thin Liquid Films, Navier-Stokes equations, continuity equation.

2020 MSC: 76D05, 26A15, 74K35.

1. Introduction

Stability theory is applied to investigations regarding the linear stability of liquid film layers down on an inclined plane. The stability and kinematics of liquid film layers in general are extremely important in science and technology. Integral approximation is used to explore the stability and dynamics of liquid films dripping down on an inclined surface [11]. When a liquid layer on a solid substrate is incredibly thin, it becomes unstable. In [8], it used long-wave-theory, integral-approximation, and numerical solving to explore the stability of liquid film layers on a horizontal surface, then compared the results and found that the advancement equations obtained from integral approximation could adequately describe a thinning liquid layer. As in [3], it investigated the influence of inertia on the tearing of a thinning layer. In [7], it used long-wave theory to construct the non-longitudinal partial differential equation (PDF) on a free thinning liquid layer and discovered that the nonlinear components participate in the speeding up of the tearing occurrence. On the other hand, it applied an integral-approach to construct the highly nonlinear growth equation of thinning liquid layers in [9]. As for in [10], it looked into the influence of an in-soluble surfactant on the dynamics of tearing of a thinning free layer and based on a comparison of their findings to those of [6]. In [5], it investigated the effect of in-soluble surfactant on the stabilization of unlimited liquid layers, taking into consideration the impact of van-der-Waals force and surface-tension. They also

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used long-wave theory to investigate the linear stability of free layers as well as thin liquid films with a horizontal material and compared the results. In [4], it constructed a nonlinear system equation that explains the long-wave expansion of the interface shape to study the dynamic split of a thinning liquid layer on a tube. As indicated in [14], the result of MHD and convective heating on a nonlinear stretching surface is being used to study the boundary layer phenomena for water-based Nano-fluid stagnation point flow. A boundary layer approximation is used to create a basic flow model and represents time-dependent momentum, energy, and concentration expression. As shown in [15], the three-dimensional motion of thin-film nanomaterials over a stretchy turning inclined surface, fundamental systems of equations are converted into a collection of first-order differential equations using similarity variables. This paper is structured as follows: section (1) presents the introduction and literature review. The Navier-Stokes problem for free films is formulated in section (2) while in section (3) the dimensional equations and boundary conditions are transformed into the non-dimensional equations. Section (4) is devoted to the long-wave asymptotic analysis and leads to two coupled equations for the velocity component in the longitudinal direction and the thickness of the film. In section (5), we investigate the linear stability of the resultant equations and determine the effect of inclination angle, surface tension and Hamaker constant. Section (6) presents the results and discussions as conclusion.

2. Formulation and Governing Equations

The structural model of a thin layer is introduced in this section considering a thinning liquid layer going down a surface slanted at an angle θ to the horizontal (figure 1), where the film of primary thickness $\bar{H}$ is limited at the thinning face by a negative gas but still is unbounded sideways. The liquid film is supposed to be thinning sufficient even for van-der Waals influences to work and thick sufficient for a continuous hypothesis of the non-solid to work, and the liquid is supposed to be a Newtonian gooey liquid.

The equations of Navier and Stokes, and the equation of continuity, are used to describe the two-dimensional motions of a liquid film, [17] :

$$\rho\left(\frac{\partial \bar{U}}{\partial \bar{T}}+\bar{U}\,\frac{\partial \bar{U}}{\partial \bar{X}}+\bar{V}\,\frac{\partial \bar{U}}{\partial \bar{Y}}\right)=-\frac{\partial \bar{P}}{\partial \bar{X}}-\frac{\partial \bar{\Phi}}{\partial \bar{X}}+\mu\left(\frac{\partial^2 \bar{U}}{\partial \bar{X}^2}+\frac{\partial^2 \bar{U}}{\partial \bar{Y}^2}\right)+\rho\,\frac{\partial \mathrm{g}}{\partial \bar{X}} \tag{1}$$

$$\rho\left(\frac{\partial \bar{V}}{\partial \bar{T}}+\bar{U}\,\frac{\partial \bar{V}}{\partial \bar{X}}+\bar{V}\,\frac{\partial \bar{V}}{\partial \bar{Y}}\right)=-\frac{\partial \bar{P}}{\partial \bar{Y}}-\frac{\partial \bar{\Phi}}{\partial \bar{Y}}+\mu\left(\frac{\partial^2 \bar{V}}{\partial \bar{X}^2}+\frac{\partial^2 \bar{V}}{\partial \bar{Y}^2}\right)+\rho\,\frac{\partial \mathrm{g}}{\partial \bar{Y}} \tag{2}$$

$$\frac{\partial \bar{U}}{\partial \bar{X}}+\frac{\partial \bar{V}}{\partial \bar{Y}}=0 \tag{3}$$

Where:

$\frac{\partial \mathrm{g}}{\partial \bar{X}}=\mathrm{g}sin\theta$

and

$\frac{\partial \mathrm{g}}{\partial \bar{Y}}=-\mathrm{g}cos\theta$

A Cartesian coordinate system $(\bar{X},\bar{Y})$ is utilized here, with matching velocity components$(\bar{U},\bar{V})$. $\bar{Y}=\bar{H}(\bar{X},\bar{T})$ is the location of the thin films liquid surfaces.

The following boundary conditions exists at the thin liquid film's interface. [13] The kinematic boundary conditions are as follows:

$$\bar{V}=\bar{H}\bar{V}=\frac{\partial \bar{Y}}{\partial \bar{T}}+\bar{U}\,\frac{\partial \bar{Y}}{\partial \bar{X}} \qquad \text{at } \bar{Y}=\bar{H} \tag{4}$$

The Shear Stress conditions [13] on the interfacing have the shape:

$$2\,\frac{\partial \bar{H}}{\partial \bar{X}}\left(\frac{\partial \bar{U}}{\partial \bar{X}}-\frac{\partial \bar{V}}{\partial \bar{Y}}\right)+\left(\left(\frac{\partial \bar{H}}{\partial \bar{X}}\right)^2-1\right)\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)=0 \tag{5}$$

The Normal Stress conditions [13] on the interfacing become:

$$\bar{P}-2\mu\left[\left(\frac{\partial \bar{H}}{\partial \bar{X}}\right)^2\frac{\partial \bar{U}}{\partial \bar{X}}-\frac{\partial \bar{H}}{\partial \bar{X}}\left(\frac{\partial \bar{U}}{\partial \bar{Y}}+\frac{\partial \bar{V}}{\partial \bar{X}}\right)+\frac{\partial \bar{V}}{\partial \bar{Y}}\right]\left(1+\left(\frac{\partial \bar{H}}{\partial \bar{X}}\right)^2\right)^{-1}=P_a-\frac{\sigma}{R} \quad \text{at} \quad \bar{Y}=\bar{H} \tag{6}$$

where P_a means air pressure, σ is the dimensional coefficient of surface-tension and the radius of Curvature $\frac{1}{R}$ has the form:

$$\frac{1}{R}=\frac{\frac{\partial^2\bar{H}}{\partial\bar{X}^2}}{\left(1+\left(\frac{\partial\bar{H}}{\partial\bar{X}}\right)^2\right)^{\frac{3}{2}}}$$

The conditions for the thin liquid film are:

$$\bar{U}=\bar{V}=0 \tag{7}$$

3. Nondimensional Analysis

We defined dimensionless amounts to refer to the Navier-Stokes equations, the continuity equation, and the related boundary conditions in non-dimensional form[1]:

$$\bar{H}=H\,h_0\,,\qquad \bar{Y}=Y\,h_0\,,\qquad \bar{X}=X\,h_0\,,$$
$$\bar{U}=\frac{U\eta}{h_0}\,,\qquad \bar{V}=\frac{V\eta}{h_0}\,,\qquad \bar{P}=\frac{P\eta^2\rho}{h_o^2}$$
$$\bar{T}=\frac{Hh_0^2}{\eta}\,,\qquad \varphi=\frac{\bar{\Phi}h_0^2}{\eta^2\rho}\,, \tag{8}$$

where h_0 is the liquid's mean thickness and $\eta=\frac{\mu}{\rho}$ is the film fluid's kinematic viscosity. The dashed letters denote dimensional amounts, and $\bar{\Phi}$ is the dimensional analog of φ, with the g (dimensional) defined as:

$$g=\frac{G_a\eta^2}{h_0^3} \tag{9}$$

here G_a is the Galileo number.

The fixed A (nondimensional) is regarding to the (dimensional) Hamaker fixed A' by the following:

$$A'=A6\pi h_0\rho\eta^2 \tag{10}$$

and

$$\sigma=\frac{3\rho\eta^2 S}{h_0} \tag{11}$$

The ratio of the van der Waals potential to viscous dissipation is denoted by A, while the ratio of surface tension to viscous dissipation is denoted by S.

By using the non dimensional factors and parameters that assumed in equations (8) -(11), into equations (1) – (9) excluding the equation (4), we have the Navier-Stoke equations in x-direction:

$$\rho\left[\frac{\eta}{h_0^2}\frac{\partial}{\partial T}\left(\frac{U\eta}{h_0}\right)+\left(\frac{U\eta}{h_0}\right)\frac{1}{h_0}\frac{\partial}{\partial X}\left(\frac{U\eta}{h_0}\right)+\left(\frac{V\eta}{h_0}\right)\frac{1}{h_0}\frac{\partial}{\partial Y}\left(\frac{U\eta}{h_0}\right)\right]=-\rho\frac{1}{h_0}\frac{\partial}{\partial X}\left(\frac{\rho\eta^2 P}{h_0^2}\right)-\rho\frac{1}{h_0}\frac{\partial}{\partial X}\left(\frac{\eta^2\rho\varphi}{h_o^2}\right)+\mu\left(\frac{\partial}{\partial X}\frac{\partial X}{\partial\bar{X}}\frac{\eta}{h_0^2}\frac{\partial U}{\partial X}+\frac{\partial}{\partial Y}\frac{\partial Y}{\partial\bar{Y}}\frac{\eta}{h_0^2}\frac{\partial U}{\partial Y}\right)+\rho g\sin\theta \tag{12}$$

and Navier-Stoke equations in Y-direction

$$\rho\left[\frac{\eta}{h_0^2}\frac{\partial}{\partial T}\left(\frac{V\eta}{h_0}\right)+\left(\frac{U\eta}{h_0}\right)\frac{1}{h_0}\frac{\partial}{\partial X}\left(\frac{V\eta}{h_0}\right)+\left(\frac{V\eta}{h_0}\right)\frac{1}{h_0}\frac{\partial}{\partial Y}\left(\frac{V\eta}{h_0}\right)\right]=-\rho\frac{1}{h_0}\frac{\partial}{\partial Y}\left(\frac{\rho\eta^2 P}{h_0^2}\right)-\rho\frac{1}{h_0}\frac{\partial}{\partial Y}\left(\frac{\eta^2\rho\varphi}{h_o^2}\right)+\mu\left(\frac{\partial}{\partial X}\frac{\partial X}{\partial\bar{X}}\frac{\eta}{h_0^2}\frac{\partial V}{\partial X}+\frac{\partial}{\partial Y}\frac{\partial Y}{\partial\bar{Y}}\frac{\eta}{h_0^2}\frac{\partial V}{\partial Y}\right)-\rho g\cos\theta. \tag{13}$$

The continuity equation gives the form:

$$\frac{1}{h_0}\frac{\partial}{\partial X}\left(\frac{U\eta}{h_0}\right)+\frac{1}{h_0}\frac{\partial}{\partial Y}\left(\frac{V\eta}{h_0}\right)=0. \tag{14}$$

The kinematic boundary condition at the interfaces has the form:

$$\frac{\eta V}{h_0}=\frac{\eta}{h_0}\frac{\partial H}{\partial T}+\frac{\eta U}{h_0}\frac{\partial H}{\partial X}\qquad \text{at}\qquad h_0Y=h_0H \tag{15}$$

Because the surface tension σ is supposed to be fixed, the shear-stresses disappear at the interfacing, and boundary (6) has the following form:

$$2\frac{\partial H}{\partial X}\left(\frac{\eta}{h_0^2}\frac{\partial U}{\partial X}-\frac{\eta}{h_0^2}\frac{\partial V}{\partial Y}\right)+\left(\left(\frac{\partial H_{\pm}}{\partial X}\right)^2-1\right)\left(\frac{\eta}{h_0^2}\frac{\partial U}{\partial Y}+\frac{\eta}{h_0^2}\frac{\partial V}{\partial X}\right)=0\qquad \text{at}\quad h_0Y=h_0H \tag{16}$$

The normal-stress border terms (7) equilibrium the normal stress with the item of surface tension times interfacing curvature, and take the following form:

$$\frac{\rho\eta^2 P}{h_0^2} - 2\mu\left[\left(\frac{\partial H}{\partial X}\right)^2 \frac{\eta}{h_0^2}\frac{\partial U}{\partial X} - \frac{\partial H_\pm}{\partial X}\left(\frac{\eta}{h_0^2}\frac{\partial U}{\partial Y} + \frac{\eta}{h_0^2}\frac{\partial V}{\partial X}\right) + \frac{\eta}{h_0^2}\frac{\partial V}{\partial Y}\right]\left(1 + \left(\frac{\partial H}{\partial X}\right)^2\right)^{-1} = -\frac{3S\rho\eta^2}{h_0^2}\frac{\frac{\partial^2 H}{\partial X^2}}{\left(1+\left(\frac{\partial H}{\partial X}\right)^2\right)^{\frac{3}{2}}} \quad \text{at} \quad h_0 Y = h_0 H \quad (17)$$

where S is the non-dimensional surface tension parameter. In formulating equation (17), we have supposed that the compression of the gas is nil [16].

Thus, the condition (8) has the form:

$$\frac{\partial}{\partial X}\left(\frac{U\eta}{h_o}\right) = \frac{V\eta}{h_o} = 0 \qquad \text{at} \qquad h_0 Y = 0. \quad (18)$$

Furthermore, we can express the van der Waals potential as:

$$\frac{\varphi\eta^2\rho}{h_0^2} = \frac{6\pi h_0 \rho\eta^2 A}{6\pi}(2Hh_0)^{-3}. \quad (19)$$

After simplifying equations (12) - (19), the equations of Navier & Stokes equations and the continuity equation have the form:

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} - \frac{\partial \varphi}{\partial X} + \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + G_a \sin\theta \quad (20)$$

$$\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} - \frac{\partial \varphi}{\partial Y} + \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} - G_a \cos\theta \quad (21)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \quad (22)$$

and the boundary conditions are determined to donate:

$$V = \frac{\partial H}{\partial T} + U\frac{\partial H}{\partial X} \qquad \text{at } Y = H \quad (23)$$

$$2\frac{\partial H}{\partial X}\left(\frac{\partial U}{\partial X} - \frac{\partial V}{\partial Y}\right) + \left(\left(\frac{\partial H}{\partial X}\right)^2 - 1\right)\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) = 0 \qquad \text{at } Y = H \quad (24)$$

$$P - 2\frac{\left[\frac{\partial U}{\partial X}\left(\left(\frac{\partial H}{\partial X}\right)^2 - 1\right) - \frac{\partial H}{\partial X}\left(\frac{\partial U}{\partial Y} + \frac{\partial U}{\partial X}\right)\right]}{N^2} = -\frac{3S\frac{\partial^2 H}{\partial X^2}}{N^3}$$

Where $N = \left(1 + \left(\frac{\partial H}{\partial X}\right)^2\right)^{\frac{1}{2}}$

$$P - 2\left[\left(\frac{\partial H}{\partial X}\right)^2 \frac{\partial U}{\partial X} - \frac{\partial H}{\partial X}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) + \frac{\partial V}{\partial Y}\right]\left(1 + \left(\frac{\partial H}{\partial X}\right)^2\right)^{-1} = -\frac{3S\frac{\partial^2 H}{\partial X^2}}{\left(1+\left(\frac{\partial H}{\partial X}\right)^2\right)^{\frac{3}{2}}} \quad at \quad Y = H \quad (25)$$

$$U = V = 0 \qquad at\ Y = 0. \quad (26)$$

Furthermore, the van der Waals potential in dimensionless form is given by:

$$\varphi = A(2H)^{-3}. \quad (27)$$

4. Long-Wave Expansion Theory

The non-linear evolution equation of layer thickness can be revealed using long-wave expansion theory. Long-wave disturbance, according to [18], is suitable for layer flow with a low Reynolds number. The perturbed variable can be used to describe the wave number k. Turning to [7] it described a deviation technique, which we now provide. The system's new dependent and independent variables are as follows:

$$X = k^{-1}\alpha, \qquad Y = \beta, \qquad T = k^{-2}\tau,$$
$$H = \bar{h}, \qquad U = k\bar{u}, \qquad V = k^2\bar{v}, \qquad P = k^2\bar{p}. \quad (28)$$

and rescale the parameters as follows:

$$S = \bar{S}, \qquad A = k^{-2}\bar{A}, \qquad G_a = k\bar{G}_a,$$
$$\sin(\theta) = O(k^2), \qquad \cos(\theta) = O(k). \quad (29)$$

Now, the capitalized non-constants and differentiation with regard to parameters (factors) are unit order as $k \to 0$. The final shape suggests that $\phi = k^{-2}\varphi = O(1)$ so that:

$$\phi = \bar{A}(2\bar{h})^{-3}. \quad (30)$$

We presently look for arrangements of the shape:

$$(\bar{u},\bar{v},\bar{p}) = (\bar{u}_0,\bar{v}_0,\bar{p}_0) + k^2(\bar{u}_1,\bar{v}_1,\bar{p}_1) + k^4(\bar{u}_2,\bar{v}_2,\bar{p}_2) + \cdots \tag{31}$$

Where $\bar{h} = O(1)$. We get a series of issues by equating like powers of k^2 to zero coefficients. We get the following results by substituting equations (28)-(31) into equations (20) and (21):

$$\frac{\partial(k\bar{u})}{\partial(k^{-2}\tau)} + k\bar{u}\frac{\partial(k\bar{u})}{\partial(k^{-1}\alpha)} + k^2\bar{v}\frac{\partial(k\bar{u})}{\partial(\beta)} = -\frac{\partial(k^2\bar{p})}{\partial(k^{-1}\alpha)} - \frac{\partial(k^2\phi)}{\partial(k^{-1}\alpha)} + \frac{\partial}{\partial(k^{-1}\alpha)}\frac{\partial(k\bar{u})}{\partial(k^{-1}\alpha)} + \frac{\partial}{\partial(\beta)}\frac{\partial(k\bar{u})}{\partial(\beta)} + k\bar{G}_a k^2 \sin\theta \tag{32}$$

and

$$\frac{\partial(k^2\bar{v})}{\partial(k^{-2}\tau)} + k\bar{u}\frac{\partial(k^2\bar{v})}{\partial(k^{-1}\alpha)} + k^2\bar{v}\frac{\partial(k^2\bar{v})}{\partial(\beta)} = -\frac{\partial(k^2\bar{p})}{\partial(\beta)} - \frac{\partial(k^2\phi)}{\partial(\beta)} + \frac{\partial}{\partial(k^{-1}\alpha)}\frac{\partial(k^2\bar{v})}{\partial(k^{-1}\alpha)} + \frac{\partial}{\partial(\beta)}\frac{\partial(k^2\bar{v})}{\partial(\beta)} - k\bar{G}_a k \cos\theta. \tag{33}$$

From equation (22), we get:

$$\frac{\partial(k\bar{u})}{\partial(k^{-1}\alpha)} + \frac{\partial(k^{-2}\bar{v})}{\partial\beta} = 0. \tag{34}$$

Now by substituting equations (28) -(31) into the boundary conditions (23) -(25), we obtain:

$$k^2\bar{v} = \frac{\partial\bar{h}}{\partial(k^{-2}\tau)} + k\bar{u}\frac{\partial\bar{h}}{\partial(k^{-2}\alpha)} \tag{35}$$

$$2\frac{\partial\bar{h}}{\partial(k^{-1}\alpha)}\left(\frac{\partial(k\bar{u})}{\partial(k^{-1}\alpha)} - \frac{\partial(k^2\bar{v})}{\partial(\beta)}\right) + \left(\left(\frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\right)^2 - 1\right)\left(\frac{\partial(k\bar{u})}{\partial(\beta)} + \frac{\partial(k^2\bar{v})}{\partial(k^{-1}\alpha)}\right) = 0 \tag{36}$$

$$k^2\bar{p} - 2\left[\left(\frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\right)^2\frac{\partial(k\bar{u})}{\partial(k^{-1}\alpha)} - \frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\left(\frac{\partial(k\bar{u})}{\partial(\beta)} + \frac{\partial(k^2\bar{v})}{\partial(k\alpha)}\right) + \frac{\partial(k^2\bar{v})}{\partial(\beta)}\right] \times \left(1 + \left(\frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\right)^2\right)^{-1} =$$
$$-3\bar{S}\frac{\partial}{\partial(k^{-1}\alpha)}\frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\left(1 + \left(\frac{\partial(\bar{h})}{\partial(k^{-1}\alpha)}\right)^2\right)^{-\frac{3}{2}} \tag{37}$$

and the boundary condition (26) at $Y = 0$, gives the form:

$$k\bar{u} = 0, \qquad k^2\bar{v} = 0. \tag{38}$$

We may express the Navier-Stokes equations in the following form after simplifying the previous equations:

$$k^3\frac{\partial\bar{u}}{\partial\tau} + k^3\bar{u}\frac{\partial\bar{u}}{\partial\alpha} + +k^3\bar{v}\frac{\partial\bar{u}}{\partial\beta} = -k^3\frac{\partial\bar{p}}{\partial\alpha} - k^3\frac{\partial\phi}{\partial\alpha} + k^3\frac{\partial^2\bar{u}}{\partial\alpha^2} + k\frac{\partial^2\bar{u}}{\partial\beta^2} + k^3\bar{G}_a\sin\theta \tag{39}$$

and

$$k^4\frac{\partial\bar{v}}{\partial\tau} + k^4\bar{u}\frac{\partial\bar{v}}{\partial\alpha} + k^4\bar{v}\frac{\partial\bar{v}}{\partial\beta} = -k^2\frac{\partial\bar{p}}{\partial\beta} - k^2\frac{\partial\phi}{\partial\beta} + k^4\frac{\partial^2\bar{v}}{\partial\alpha^2} + k^2\frac{\partial^2\bar{v}}{\partial\beta^2} - k^2\bar{G}_a\cos\theta. \tag{40}$$

The continuity equation (34) is expressed as:

$$k^2\frac{\partial\bar{u}}{\partial\alpha} + k^2\frac{\partial\bar{v}}{\partial\beta} = 0. \tag{41}$$

At the interfaces, the kinematic boundary conditions and other boundary conditions are determined by:

$$k^2\bar{v} = k^2\frac{\partial\bar{h}}{\partial\tau} + k^2\bar{u}\frac{\partial\bar{h}}{\partial\alpha} \qquad \text{at} \qquad Y = H \tag{42}$$

$$2k\frac{\partial\bar{h}}{\partial\alpha}\left(k^2\frac{\partial\bar{u}}{\partial\alpha} - k^2\frac{\partial\bar{v}}{\partial\beta}\right) + \left(k^2\left(\frac{\partial\bar{h}}{\partial\alpha}\right)^2 - 1\right)\left(k\frac{\partial\bar{u}}{\partial\beta} + k^3\frac{\partial\bar{v}}{\partial\alpha}\right) = 0 \quad \text{at} \quad Y = H \tag{43}$$

$$k^2\bar{p} - 2\left(k^3\left(\frac{\partial\bar{h}}{\partial\alpha}\right)^2\frac{\partial\bar{u}}{\partial\alpha} - k^2\frac{\partial\bar{h}}{\partial\alpha}\frac{\partial\bar{u}}{\partial\beta} - k^2\frac{\partial\bar{h}}{\partial\alpha}\frac{\partial\bar{v}}{\partial\alpha} + k^2\frac{\partial\bar{v}}{\partial\beta}\right)\left(1 + k^2\left(\frac{\partial\bar{h}}{\partial\alpha}\right)^2\right)^{-1} = -3\bar{S}k^2\frac{\partial^2\bar{h}}{\partial\alpha^2}\left(1 + k^2\left(\frac{\partial\bar{h}}{\partial\alpha}\right)^2\right)^{-\frac{3}{2}} \quad \text{at } Y = H \tag{44}$$

and on the centerline is given by:

$$k\bar{u} = 0, \qquad k^2\bar{v} = 0 \qquad \text{at } Y = 0. \tag{45}$$

At leading order, we obtain:

$$\frac{\partial^2\bar{u}_0}{\partial\beta^2} = 0 \tag{46}$$

$$\frac{\partial^2\bar{v}_0}{\partial\beta^2} - \frac{\partial(\bar{p}_0+\phi_0)}{\partial\beta} - \bar{G}_a\cos\theta = 0 \tag{47}$$

$$\frac{\partial\bar{u}_0}{\partial\alpha} + \frac{\partial\bar{v}_0}{\partial\beta} = 0 \tag{48}$$

$$\bar{v}_0 = \frac{\partial\bar{h}}{\partial\tau} + \bar{u}_0\frac{\partial\bar{h}}{\partial\alpha} \qquad \text{at} \qquad \beta = \bar{h} \tag{49}$$

$$\frac{\partial \bar{u}_0}{\partial \beta} = 0 \qquad \text{at} \qquad \beta = \bar{h} \tag{50}$$

$$-\bar{p}_0 - 2\frac{\partial \bar{h}}{\partial \alpha}\frac{\partial \bar{u}_0}{\partial \beta} + 2\frac{\partial \bar{v}_0}{\partial \beta} - 3\bar{S}\frac{\partial^2 \bar{h}}{\partial \alpha^2} = 0 \qquad \text{at} \quad \beta = \bar{h} \tag{51}$$

$$\frac{\partial \bar{u}_0}{\partial \alpha} = \bar{v}_0 = 0 \qquad \text{at} \qquad \beta = 0\,. \tag{52}$$

The solution of equations (46) is given by:

$$\frac{\partial \bar{u}_0}{\partial \beta} = E(\alpha, \tau) \tag{53}$$

from the boundary condition (51), we get:
$E(\alpha, \tau) = 0.$

Then by integrating equation (53) with respect toς, we obtain:

$$\bar{u}_0 = C(\alpha, \tau) \tag{54}$$

where C is a function of α and τ that is currently unknown. Equation (48) can be integrated by using the condition (52) and from (54), we have:

$$\int \frac{\partial \bar{v}_0}{\partial \beta}\partial \beta = -\int \frac{\partial C}{\partial \alpha}\partial \beta$$

$$\bar{v}_0 = -\frac{\partial C}{\partial \alpha}\beta + l(\alpha, t)$$

$$\bar{v}_0 = -\frac{\partial C}{\partial \alpha}\beta. \tag{55}$$

Equation (47) can be integrated once with respect to β yield:

$$(\bar{p}_0 + \phi_0) = \frac{\partial \bar{v}_0}{\partial \alpha} - \bar{G}_a \cos\theta\, \beta + D(\alpha, \tau). \tag{56}$$

where D is other obscure mapping of α and τ. Since $\bar{p}_0$ is separate of β, $\bar{p}_0$ is specific through normal-stress condition (51). Then by using equation (51) and from equation (56), we can determine the function D as:

$$D = -\frac{\partial C}{\partial \alpha} + \phi_0 - 3\bar{S}\frac{\partial^2 \bar{h}}{\partial \alpha^2} + \bar{G}_a \cos\theta\, \bar{h}. \tag{57}$$

Finally, the forms for $\bar{u}_0$ and $\bar{v}_0$ are used into the kinematic equation (49) to get:

$$-\frac{\partial C}{\partial \alpha}\beta = \frac{\partial \bar{h}}{\partial \tau} + C\frac{\partial \bar{h}}{\partial \alpha} \qquad \text{at} \qquad \beta = \bar{h} \tag{58}$$

$$\frac{\partial \bar{h}}{\partial \tau} + \frac{\partial(\bar{h}C)}{\partial \alpha} = 0. \tag{59}$$

Because both C and $\bar{h}$ are obscure, a another relationship among the unknown mapping C and $\bar{h}$ is required. For this purpose, we look at the $O(k)$ problem of $\bar{u}_1$ as follows:

$$\frac{\partial^2 \bar{u}_1}{\partial \beta^2} = \left(\frac{\partial \bar{u}_0}{\partial \tau} + \bar{u}_0\frac{\partial \bar{u}_0}{\partial \alpha} + \bar{v}_0\frac{\partial \bar{u}_0}{\partial \alpha} - \frac{\partial^2 \bar{u}_0}{\partial \alpha^2} + \frac{\partial \bar{p}_0}{\partial \alpha} + \frac{\partial \phi}{\partial \alpha} - \bar{G}_a \sin\theta\right) \tag{60}$$

$$\frac{\partial \bar{u}_1}{\partial \beta} = 0 \quad \text{at} \quad \beta = 0 \tag{61}$$

$$\frac{\partial \bar{u}_1}{\partial \beta} + \left[\frac{\partial \bar{v}_0}{\partial \alpha} - \frac{\partial \bar{u}_0}{\partial \beta}\left(\frac{\partial \bar{h}}{\partial \alpha}\right)^2 + 2\frac{\partial \bar{h}}{\partial \alpha}\left(\frac{\partial \bar{v}_0}{\partial \beta} - \frac{\partial \bar{u}_0}{\partial \alpha}\right)\right] = 0 \;\; \text{at} \quad \beta = \bar{h}. \tag{62}$$

By using the expressions for $\bar{u}_0$, $\bar{v}_0$ and $\bar{p}_0 + \phi_0$ given by equations (54) -(57) and (60), we get:

$$\frac{\partial^2 \bar{u}_1}{\partial \beta^2} = \left(\frac{\partial C}{\partial \tau} + C\frac{\partial C}{\partial \alpha} - \frac{\partial^2 C}{\partial \alpha^2} - \frac{\partial^2 C}{\partial \alpha^2} + \frac{\partial D}{\partial \alpha} - \bar{G}_a \sin\theta\right) \tag{63}$$

$$\frac{\partial \bar{u}_1}{\partial \beta} = 0 \qquad \text{at} \;\; \beta = 0 \tag{64}$$

$$\frac{\partial \bar{u}_1}{\partial \beta} + \left(-\frac{\partial^2 C}{\partial \alpha^2}\bar{h} - 4\frac{\partial \bar{h}}{\partial \alpha}\frac{\partial C}{\partial \alpha}\right) = 0. \qquad \text{at} \qquad \beta = H \tag{65}$$

We integrate equation (63) once and use equation (64) to obtain:

$$\frac{\partial \bar{u}_0}{\partial \beta} = \left(\frac{\partial C}{\partial \tau} + C\frac{\partial C}{\partial \alpha} - 2\frac{\partial^2 C}{\partial \alpha^2} + \frac{\partial D}{\partial \alpha} - \bar{G}_a \sin\theta\right)\beta. \tag{66}$$

Finally, using Equation (65) into equation (66), we find that:

$$\frac{\partial^2 C}{\partial\alpha^2}\bar{h} + 4\frac{\partial\bar{h}}{\partial\alpha}\frac{\partial C}{\partial\alpha} = \left(\frac{\partial C}{\partial\tau} + C\frac{\partial C}{\partial\alpha} - \frac{\partial^2 C}{\partial\alpha^2} - \frac{\partial^2 C}{\partial\alpha^2} + \frac{\partial D}{\partial\alpha} - \bar{G}_a \sin\theta\right)\bar{h}. \tag{67}$$

Using equation (57), we eliminate D from equation (67) and get the taking after equation for$C$ and $\bar{h}$:

$$\left(\frac{\partial C}{\partial\tau} + C\frac{\partial C}{\partial\alpha} + \frac{\partial\phi_0}{\partial\alpha} - 3\bar{S}\frac{\partial^3\bar{h}}{\partial\alpha^3} + \bar{G}_a\cos\theta\frac{\partial\bar{h}}{\partial\alpha} - \bar{G}_a\sin\theta\right)\bar{h} = 4\frac{\partial\left(H\frac{\partial C}{\partial\alpha}\right)}{\partial\alpha}. \tag{68}$$

For the longitudinal portion of speed and the thickness of the film, we discovered two coupled non-linear evolution equations. These equations are rewritten in the original parameters as follows:

$$\alpha = kX, \qquad \tau = k^2 T, \qquad \phi = k^{-2}\varphi,$$
$$C = k^{-1}U, \qquad \bar{h} = H \qquad \bar{G}_a = k^{-1}G_a$$

By substituting the above equations into equations (59) and (68), we obtain:

$$\frac{\partial H}{\partial T} + \frac{\partial(UH)}{\partial X} = 0 \tag{69}$$

$$H\left(\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + \frac{\partial\varphi}{\partial X} - 3S\frac{\partial^3 H}{\partial X^3} + G_a\cos\theta\frac{\partial H}{\partial X} - G_a\sin\theta\right) = 4\frac{\partial\left(H\frac{\partial U}{\partial X}\right)}{\partial X} \tag{70}$$

where $\varphi = A(2H)^{-3}$ and $U = O(k)$.

5. Linear Stability Analysis

Analyzing the stability of the steady-state solution [10] is informative.
$(U, H) = (0,1)$
by introducing the deviation:
$H' = H - 1.$

We obtain from equations (69) and (70) the following linearized problem:

$$\frac{\partial H'}{\partial T} + \frac{\partial U}{\partial X} = 0 \tag{71}$$

And

$$\frac{\partial U}{\partial T} - \frac{3}{8}A\frac{\partial H'}{\partial X} - 3S\frac{\partial^3 H'}{\partial X^3} - 4\frac{\partial^2 U}{\partial X^2} + G_a\cos\theta\frac{\partial H'}{\partial X} - G_a\sin\theta = 0. \tag{72}$$

where H' stands for the initial film thickness disturbance. Additionally, normal mode disturbances could be used to represent the perturbed state of velocity and film thickness[4] and [8]:

$$(H', U) = (H_0, U_0)\exp(VT + ikX) \tag{73}$$

whereH_0 and U_0 define the initial disturbance's amplitude. By putting equation (73) into equations (71) and (72), we get the characteristic equation:

$$H_0 V + U_0 ik = 0 \tag{74}$$

and

$$U_0 V - \frac{3}{8}AH_0 ik + 3SH_0 ik^3 + 4U_0 k^2 + H_o G_a\cos\theta\, ik - G_a\sin\theta = 0. \tag{75}$$

Using equation (74) into equation (75), we get the dispersion relation as:

$$V^2 - \frac{3}{8}Ak^2 + 3Sk^4 + 4k^2 V + k^2 G_a\cos\theta = 0 \tag{76}$$

V and k are the complex increment and the real wave number, respectively. Although the real orders of magnitude of A and S are low [16] (e.g., $S = 0.1,\ A = 0.0001$ for ethanol and, $S = 1\ ,\ A = 0.001$ for water [2]), [7] utilized $A = 1$ and $S = -1$. When we solve the above relation in terms of V, we get:

$$V = -4k^2 + \frac{k\sqrt{16k^2 - 12Sk^2 + \frac{3}{2}A - 4G_a\cos\theta}}{2} \tag{77}$$

where the other solution has been discarded because its real part is always negative. As a result, as it were when $k < k_c$, where k_c is a critical wave-number, does the film becomes unstable [12], viz., $V > 0$.
For impartially stable-wave ($V = 0$), the critical-wave number k_c as the following:

$$k_c = \left(\frac{1}{8S}A - \frac{1}{3S}G_a\cos\theta\right)^{\frac{1}{2}}. \tag{78}$$

The quickest increasing wave number, k_m which is produced by setting $\frac{dV}{dK} = 0$ from equation (76), has the highest growth rate V_m of the linear waves, as follows:

$$k_m{}^2 = \frac{k_c{}^2}{\sqrt{2}} = \frac{1}{S\sqrt{2}}\left(\frac{1}{8}A - \frac{1}{3}G_a \cos\theta\right). \tag{79}$$

When $0 < k < k_c$ is given by equation, the linear theory suggests an unrestricted increase of surface deformations $V > 0$ up to the point of layer rupture (77). The linear theory implies a constant force across all steps of film deformation, however in reality, as deformations increase, thinner and thicker regions of the film experience distinct (non-linear) force fields.

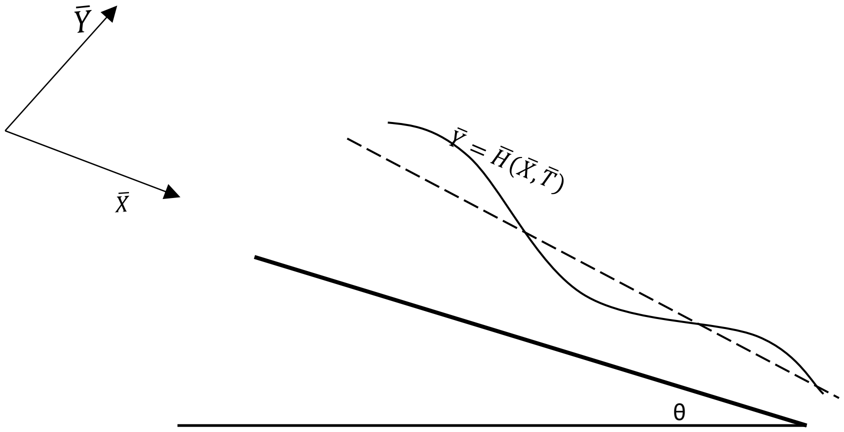

Figure (1): Thin liquid films with substrate

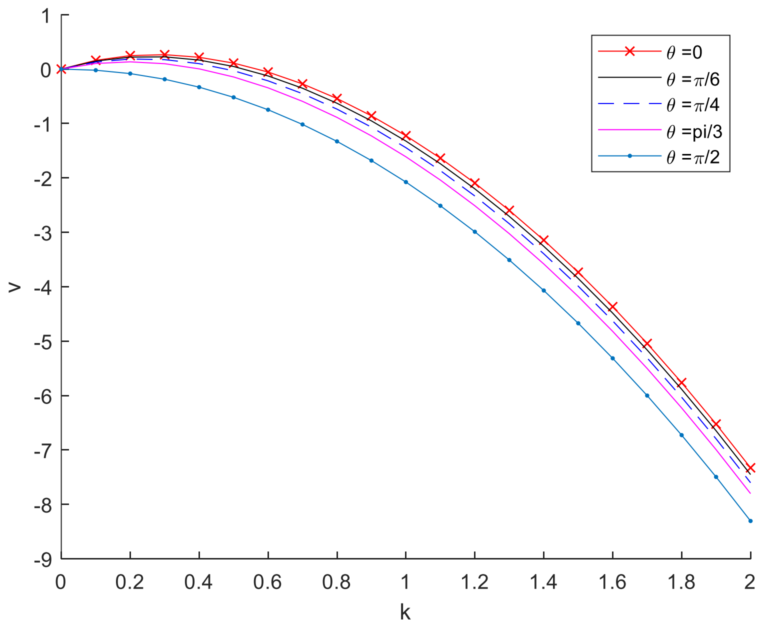

Figure (2): The Growth Rate v vs. Wave Number k plotted after Equation (77) for $S = 0.1$, $A = 0.0001$ and $Ga = -1$ under various θ

Table (1): List of symbols with definitions in article

Symbols	Definitions in article	Symbols	Definitions in article
A	Non-dimensional Hamaker constant	$\bar{T}$	Time
A'	Dimensional Hamaker constant	U	Dimensionless velocity in the x-direction
G_a	Galileo number	$\bar{U}$	Dimensional velocity of x-component
$g_{\bar{x}}$	Acceleration component in x -direction due to gravity	U_0	Amplitude of the initial disturbance of the velocity component
$g_{\bar{y}}$	Acceleration component in y -direction due to gravity	V	Dimensionless velocity in the y-direction
$\bar{H}(\bar{X},\bar{T})$	Dimensionless film thickness	$\bar{V}$	Dimensional velocity of y-component
$H(X,T)$	Dimension film thickness	V_m	Maximum growth rate
H'	Initial disturbance of the film thickness	X, Y	Spatial coordinates
H_0	Amplitude of the initial disturbance of the thickness	ρ	Fluid density
k	Wave number	η	Kinematic viscosity
K_c	Critical wave number	μ	Fluid viscosity
k_m	Dominant wave number	$\bar{\Phi}$	Dimensional van der Waals potential
$\bar{P}$	Dimensional fluid pressure	σ	Dimensional coefficient of surface tension
P	Non-dimensional pressure	φ	Non-dimensional van der Waals potential
S	Dimensionless coefficient of surface tension	θ	Inclined angle

6. Conclusion

The stability of thin liquid films was investigated in this work. The study findings are revealed via linear stability analysis. As appeared in Figure (2), the film gets to be stable to wavelength-short perturbation on the off chance that $k > k_c$, and unstable to longwave perturbation on the off chance that $k < k_c$. Thus, the influence of a slant of thinning liquid layers is a non-stable factor, and an increase in value will clearly result in a decrease in the area of stability. Also, figure (2) illustrates the solutions to the characteristic equation for various inclination θ values. Each curve's peak $(v - k)$ represents the maximum growth rate v_m and the corresponding wave number k_m. It is obvious that an increase in value θ will result in a decrease in the region of stability.

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