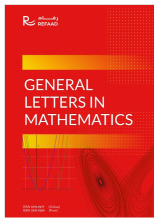
Available online at www.refaad.com
Gen.Lett. Math., 12(4) (2022), 164–168
Research Article

General Letters in Mathematics (GLM)

Journal Homepage: https://www.refaad.com/Journal/Index/1

ISSN: 2519-9277 (Online) 2519-9269 (Print)



When Doesn't the Interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[\ (n, k \in \mathbb{N}^*)$ Contain any Primes? Theorem and Counterexamples (a_n is the n^{th} Prime Number & P_n is the n^{th} Prime Factorial)

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Abstract

The objective of this short paper is to give and prove a main theorem confirming that any interval of the special form:

$$]a_{n+k} + P_n, a_{n+k+1} + P_n[\ (n \in \mathbb{N}^*)$$

does not contain any primes, for all $k \in \mathbb{N}^*$ such that $a_{n+k+1} < a_{n+1}^2$ (a_n is the n^{th} prime number & P_n is the n^{th} prime factorial). Then we give several counterexamples of such intervals, which contain primes, when the condition ($a_{n+k+1} < a_{n+1}^2$) is not satisfied.

Keywords: Prime numbers, prime factorials, *Bertrand*'s postulate, the Fundamental theorem of arithmetic, *Fortune*'s conjecture.
2020 MSC: 11-XX, 11A41, 11A51.

Notations

Let $n \in \mathbb{N}^*$.
In all what follows, we will denote by:
a_n the n^{th} prime number and $P_n = a_1.a_2.a_3...a_n$ the n^{th} prime factorial (product of the first n prime numbers).

1. Introduction

There are several works, in the literature, dealing with the problem of the existence or non-existence of prime numbers in certain intervals (subsets of the set of real numbers $\mathbb{R}$) [2, 5, 6, 7, 8, 9].
In the present manuscript, we will deal with this same problem, but for a specific form of open intervals of $\mathbb{R}$. This work is original, and can help a lot in solving *Fortune*'s conjecture which was recently addressed in various works, as examples, we cite [10, 11, 12, 13].

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doi:10.31559/glm2022.12.4.2

Received 6 Nov 2022 : Revised : 19 Dec 2022 Accepted: 27 Dec 2022

2. Reminder

We recall *Bertrand*'s postulate: [1, 3, 14, 16]

Theorem 2.1. $\forall n \in \mathbb{N}^* : a_{n+1} < 2.a_n$.

3. Main theorem

Proposition 3.1. $\forall n \in \mathbb{N}^*$*: The interval* $]a_{n+1} + P_n, a_{n+2} + P_n[$ *does not contain any prime number.*

Proof. Let $n \in \mathbb{N}^*$ and $m \in]a_{n+1} + P_n, a_{n+2} + P_n[$. Then: $m = m' + P_n$ such that: $a_{n+1} < m' < a_{n+2}$, therefore m' is not prime.
We assume (by the absurd) that m is prime.
Since m' is not prime, it follows from the Fundamental theorem of arithmetic [4, 15] that m' may be written in the following form:

$$m' = a_1^{\beta_1}.a_2^{\beta_2}...a_n^{\beta_n}.a_{n+1}^{\beta_{n+1}} \tag{3.1}$$

such that: $\beta_1, \beta_2, ..., \beta_n, \beta_{n+1}$ are positive integers.
Now, since $m = m' + P_n$ is prime and $P_n = a_1.a_2.a_3...a_n$, then:
$\beta_1 = \beta_2 = ... = \beta_n = 0$, and hence:

$$m' = a_{n+1}^{\beta_{n+1}} \tag{3.2}$$

So: $a_{n+1} < m' < a_{n+2}$ implies that: $a_{n+1} < a_{n+1}^{\beta_{n+1}} < a_{n+2}$. By **Theorem 2.1.**, it follows that:
$a_{n+1} < a_{n+1}^{\beta_{n+1}} < 2.a_{n+1}$ (contradiction, since $\beta_{n+1} \in \mathbb{N}$ and $a_{n+1} \geqslant 2$).
This completes the proof of the proposition. □

We therefore generalize the previous proposition (**Proposition 3.1.**), which will constitute a Theorem from now on:

Theorem 3.2. *(Generalization of* ***Proposition 3.1.****)*
Let n *be a fixed positive integer such that:* $n \geqslant 2$*, then:*
For all $k \in \mathbb{N}^*$ *such that* $a_{n+k+1} < a_{n+1}^2$*:*
The interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[$ *does not contain any prime number.*

Proof. Let n be a fixed positive integer such that: $n \geqslant 2$, and $k \in \mathbb{N}^*$ such that $a_{n+k+1} < a_{n+1}^2$, and let $m \in]a_{n+k} + P_n, a_{n+k+1} + P_n[$. Then:
$m = m' + P_n$, $a_{n+k} < m' < a_{n+k+1}$, so m' is not prime.
We assume (by the absurd) that m is prime.
It follows from the Fundamental theorem of arithmetic [4, 15] that m' may be written in the following form:

$$m' = a_1^{\beta_1}.a_2^{\beta_2}...a_n^{\beta_n}...a_{n+k}^{\beta_{n+k}} \tag{3.3}$$

such that: $\beta_1, \beta_2, ..., \beta_{n+k}$ are positive integers.
Now, since $m = m' + P_n$ is prime and $P_n = a_1.a_2.a_3...a_n$, then $\beta_1 = \beta_2 = ... = \beta_n = 0$; and hence:

$$m' = a_{n+1}^{\beta_{n+1}}.a_{n+2}^{\beta_{n+2}}...a_{n+k}^{\beta_{n+k}} \tag{3.4}$$

So: $a_{n+k} < m' < a_{n+k+1}$ implies that: $a_{n+k} < a_{n+1}^{\beta_{n+1}}.a_{n+2}^{\beta_{n+2}}...a_{n+k}^{\beta_{n+k}} < a_{n+k+1}$. By the hypothesis ($a_{n+k+1} < a_{n+1}^2$), it follows that:
$a_{n+k} < a_{n+1}^{\beta_{n+1}}.a_{n+2}^{\beta_{n+2}}...a_{n+k}^{\beta_{n+k}} < a_{n+k+1} < a_{n+1}^2$, thus: $a_{n+j}^{\beta_{n+j}} < a_{n+1}^2 \leqslant a_{n+j}^2, \forall j \in \mathbb{N} : 1 \leqslant j \leqslant k$, hence: $\beta_{n+j} \in \{0, 1\}, \forall j \in \mathbb{N} : 1 \leqslant j \leqslant k$,
because $a_{n+j} \geqslant 2, \forall j \in \mathbb{N} : 1 \leqslant j \leqslant k$.
On the other hand, it is not possible for two exponents β_{n+j} and $\beta_{n+j'}$ ($j \neq j', 1 \leqslant j, j' \leqslant k$) to equal 1

together, because in this case we get:
$a_{n+1}^2 < a_{n+j}.a_{n+j'} \leqslant a_{n+1}^{\beta_{n+1}}.a_{n+2}^{\beta_{n+2}}...a_{n+j}^{\beta_{n+j}}...a_{n+j'}^{\beta_{n+j'}}...a_{n+k}^{\beta_{n+k}} < a_{n+k+1} < a_{n+1}^2$ (contradiction).
So, only two cases are possible:

1. **First case:** $\beta_{n+j} = 0, \forall j \in \mathbb{N} : 1 \leqslant j \leqslant k$.
   In this case: $m' = 1$ but this is a contradiction because $a_{n+k} < m' < a_{n+k+1}$.
2. **Second case:** $\exists! j' \in \mathbb{N}, 1 \leqslant j' \leqslant k : \beta_{n+j'} = 1$ and $\forall j \in \mathbb{N} : 1 \leqslant j \leqslant k, (j \neq j') : \beta_{n+j} = 0$.
   In this case: $m' = a_{n+j'}$, therefore m' is prime, but this is also rejected because m' is not prime.

Thus, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[$ doesn't contain any primes. This completes the proof of the theorem. □

Remark 3.3. We note that there is a direct relationship between the intervals $]a_{n+k} + P_n, a_{n+k+1} + P_n[\,(n, k \in \mathbb{N}^*)$ and *Fortune*'s conjecture which has been proved to be correct in a particular case in the recent article [10].

4. Counterexamples showing that the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[\,(n, k \in \mathbb{N}^*)$ may contain prime numbers when $a_{n+k} > a_{n+1}^2$

In this section, we will present 10 counterexamples showing that it is very likely that the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[$ may contain prime numbers when $a_{n+k} > a_{n+1}^2$ (so the condition $(a_{n+k+1} < a_{n+1}^2)$ is not satisfied).

4.1. *First counterexample*

For $n = 1$ and $k = 8$:
We have indeed $a_{n+k} = a_9 = 23 > 9 = a_2^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_9 + P_1, a_{10} + P_1\right[= \left]23 + 2, 29 + 2\right[= \left]25, 31\right[$ contains the prime number 29.

4.2. *Second counterexample*

For $n = 2$ and $k = 14$:
We have indeed $a_{n+k} = a_{16} = 53 > 25 = a_3^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_{16} + P_2, a_{17} + P_2\right[= \left]53 + 6, 59 + 6\right[= \left]59, 65\right[$ contains the prime number 61.

4.3. *Third counterexample*

For $n = 3$ and $k = 58$:
We have indeed $a_{n+k} = a_{61} = 283 > 49 = a_4^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_{61} + P_3, a_{62} + P_3\right[= \left]283 + 30, 293 + 30\right[= \left]313, 323\right[$ contains the prime number 317.

4.4. *Fourth counterexample*

For $n = 4$ and $k = 30$:
We have indeed $a_{n+k} = a_{34} = 139 > 121 = a_5^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_{34} + P_4, a_{35} + P_4\right[= \left]139 + 210, 149 + 210\right[= \left]349, 359\right[$ contains the prime number 353.

4.5. *Fifth counterexample*

For $n = 5$ and $k = 216$:
We have indeed $a_{n+k} = a_{221} = 1381 > 169 = a_6^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_{221} + P_5, a_{222} + P_5\right[= \left]1381 + 2310, 1399 + 2310\right[= \left]3691, 3709\right[$ contains the prime number 3701.

4.6. *Sixth counterexample*

For $n = 6$ and $k = 85$:
We have indeed $a_{n+k} = a_{91} = 467 > 289 = a_7^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[= \left]a_{91} + P_6, a_{92} + P_6\right[= \left]467 + 30030, 479 + 30030\right[= \left]30497, 30529\right[$ contains the prime number 30509.

4.7. *Seventh counterexample*

For $n = 7$ and $k = 173$:
We have indeed $a_{n+k} = a_{180} = 1069 > 361 = a_8^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[=]a_{180} + P_7, a_{181} + P_7[=]1069 + 510510, 1087 + 510510[=]511579, 511597[$ contains the prime number 511583.

4.8. *Eighth counterexample*

For $n = 8$ and $k = 113$:
We have indeed $a_{n+k} = a_{121} = 661 > 529 = a_9^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[=]a_{121} + P_8, a_{122} + P_8[=]661 + 9699690, 673 + 9699690[=]9700351, 9700363[$ contains the prime number 9700357.

4.9. *Ninth counterexample*

For $n = 9$ and $k = 318$:
We have indeed $a_{n+k} = a_{327} = 2179 > 841 = a_{10}^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[=]a_{327} + P_9, a_{328} + P_9[=]2179 + 223092870, 2203 + 223092870[=]223095049, 223095073[$ contains the prime number 223095071.

4.10. *Tenth counterexample*

For $n = 10$ and $k = 580$:
We have indeed $a_{n+k} = a_{590} = 4297 > 961 = a_{11}^2 = a_{n+1}^2$ and also, the interval $]a_{n+k} + P_n, a_{n+k+1} + P_n[=]a_{590} + P_{10}, a_{591} + P_{10}[=]4297 + 6469693230, 4327 + 6469693230[=]6469697527, 6469697557[$ contains the prime number 6469697537.

5. Conclusion

In this manuscript, we have shown that any interval of the form $]a_{n+k} + P_n, a_{n+k+1} + P_n[\,(n \in \mathbb{N}^*)$ such that: $k \in \mathbb{N}^*$, does not contain any prime number, provided that the condition $a_{n+k+1} < a_{n+1}^2$ is satisfied. Moreover, we have given several counterexamples proving that these same intervals may contain prime numbers, when the condition $a_{n+k+1} < a_{n+1}^2$ is not satisfied.
In conclusion, the content of this article can help a lot in solving *Fortune*'s conjecture which is still an open problem in number theory.

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