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Stability for Pantograph Fractional Differential Equations

Hamid Boulares[a,*]

[a]*Laboratory of Analysis and Control of Differential Equations "ACED", Faculty MISM, Department of Mathematics, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria*

Abstract

In this manuscript, we studied some sufficient condition for the asymptotically stable of the zero solution of pantograph Caputo fractional differential equations of order $\alpha (1 < \alpha < 2)$. In a weighted Banach space, we used Krasnoselskii's fixed point theorem to derive new results based on the asymptotically stable zero solution provided that $g(t, 0) = f(t, 0, 0) = 0$, which incorporates and modifies several previous results. Give an example that reflects our discovery.

Keywords: Caputo fractional derivative, Pantograph fractional differential equations, Krasnoselskii's fixed point theorem, Stability.
2010 MSC: 34K20, 34K30, 34K40, 37C25, 54H25, 55M20.

1. Introduction

Our focus in our study of this paper is the study of the asymmetrical stability of the following nonlinear fractional differential equations (FDEs)

$$ {}^{C}D^{\alpha}_{0+}\varkappa(t) = \mathcal{F}(t, \varkappa(\lambda t)) \ \ t \geqslant 0, \tag{1.1} $$

$$ \varkappa(0) = \theta_0, \ \varkappa'(0) = \theta_1, \tag{1.2} $$

where $\lambda \in (0, 1), 1 < \alpha < 2$, $\theta_0, \theta_1 \in \mathbb{R}$, $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathcal{F}(t, 0) \equiv 0$, ${}^{C}D^{\alpha}_{0+}$ is Caputo fractional derivative of the system (1.1) and (1.2).

For the need for today's modern and large-scale applications of FDEs, the topic of partial calculus has emerged as a new branch of applied mathematics, and has also been applied to a large number in a variety of mathematical models in science and engineering in the past three decades. The theory of FDEs has been extensively studied by many authors [1, 2, 3, 4, 13]. However, to the best of our knowledge, the investigation on stability theory of nonlinear FDEs is still in the initial stage and there is a great deal of work that needs to be done. Recently, several methods have been studied to study the stability of nonlinear FDEs. For example, generalized inequality Gronwall-Bellman is used in [5] to study the stability of nonlinear FDEs and the principle of comparison in partial order in [6] is used. And in [7, 8], the Mittag–Leffler stabilities of nonlinear FDEs of first order $(0 < \alpha < 1)$ are introduced. Moreover, by utilizing the

*Corresponding author
Email address: boulareshamid@gmail.com (Hamid Boulares)

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generalized singular Gronwall inequalities, the Ulam stability for impulsive FDEs is discussed in [9, 11]. However, those methods introduced above are mostly implemented to the stability analysis of FDEs and it is not easy to use them to discuss the stability of nonlinear FDEs of order $(1 < \alpha < 2)$. For this reason, as a meaningful attempt, the purpose of this paper is to find another effective method to study the stability of nonlinear FDEs of order $(1 < \alpha < 2)$.

As is known, fixed point theories have been used to study the stability of differential systems in the right order by many authors, notably Burton [14, 15, 16, 17, 18] and Becker [19, 20]. In addition, there are some contributions to the study of the stability of FDEs by fixed point theories. In [14] the author derived the stability criterion for the Volterra equation which is based on the principle of shrinkage mapping and Baker's theory of solvent shape. Burton and Zhang consider the stability of Caputo-type FDEs of the system at [12, 18] where the stability of FDEs in Banach-weighted space is studied through solvent theory and fixed point theories.

In this paper, driven by those valuable contributions mentioned above, we mainly discuss the stability of nonlinear FDEs for order $(1 < \alpha < 2)$. To achieve this, we first follow the conversion of the fractional differential equation into an ordinary first-order differential equation with an integrative fractional disorder, from which we obtain the equivalent integrative equations (1.1) and (1.2) by different formula constants and some analytical skills. Moreover, by Krasnoselskii's fixed point theorem we investigate the stability of the two problems (1.1) and (1.2) without considering solvent theory and the results are given in a simplified way.

This article is organized as follows: in the next section we present some preliminaries and lemmas that will be used to prove our main results. In Section 3 we give and prove our main results. Finally, an application of the main results is presented.

2. Preliminaries

In this part we present some of the necessary definitions and lemmas that will be used in this paper. For more details, see [1, 2, 3, 16, 21].

Definition 2.1. [1, 2] The fractional integral of order $\alpha > 0$ of a function $\varkappa : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}\varkappa(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varkappa(s)ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$. Where the Euler-Gamma function is defined as flows:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\alpha}s^{\alpha-1}ds;\ \alpha > 0.$$

Definition 2.2. [1, 2] The Caputo fractional derivative of order $\alpha > 0$ of a function $\varkappa : \mathbb{R}^+ \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha}\varkappa(t) = I_{0+}^{n-\alpha}\varkappa^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\varkappa^{(n)}(s)ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3. *[1, 2] Let $\Re(\alpha) > 0$. Suppose $\varkappa(t) \in C^{n-1}[0, +\infty)$ and $\varkappa^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}\varkappa\right)(t) = \varkappa(t) - \sum_{k=0}^{n-1}\frac{\varkappa^{(k)}(0)}{k!}t^k.$$

In particular, when $0 < \Re(\alpha) < 1$, $\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}\varkappa\right)(t) = \varkappa(t) - \varkappa(0)$.

Remark 2.4. From Definition 2.1, 2.2 and Lemma 2.3, it is easy to see that

(1) Let $\Re(\alpha)>0$. If $\varkappa(t)$ is continuous on $\mathbb{R}^+$, then $D^{\alpha}_{0+}I^{\alpha}_{0+}\varkappa(t)=\varkappa(t)$ holds for all $t\in\mathbb{R}^+$.

(2) The Caputo derivative of a constant is equal to zero.

The following Banach space plays a fundamental role in our discussion.

Let $\overline{\omega}:\mathbb{R}^+\to[1,+\infty)$ be a strictly increasing continuous function with

$$\overline{\omega}(0)=1,\ \overline{\omega}(t)\to\infty \text{ as } t\to\infty,\ \overline{\omega}(s)\overline{\omega}(t-s)\leqslant\overline{\omega}(t)$$

for all $0\leqslant s\leqslant t\leqslant\infty$. Let

$$E=\left\{\varkappa(t)\in C\left[0,+\infty\right):\sup_{t\geqslant 0}\left|\varkappa(t)\right|/\overline{\omega}(t)<\infty\right\}.$$

Then E is a Banach space equipped with the norm $\|\varkappa\|=\sup_{t\geqslant 0}\frac{|\varkappa(t)|}{\overline{\omega}(t)}$. For more properties of this Banach space, see [3, 16]. Moreover, let

$$\|\varphi\|_t=\max\{|\varphi(s)|:0\leqslant s\leqslant t\},$$

for any $t\geqslant 0$, any given $\varphi\in C(\mathbb{R}^+)$ and let $\mathfrak{I}(\varepsilon)=\{\varkappa:\varkappa\in E,\|\varkappa\|\leqslant\varepsilon\}$ for any $\varepsilon>0$.

Lemma 2.5. *Let $\mathcal{W}(t)\in C\left[0,+\infty\right)$. Then $\varkappa(t)\in C\left[0,+\infty\right)$ is a solution of the Cauchy type problem*

$$\begin{cases} {}^{C}D^{\alpha}_{0+}\varkappa(t)=\mathcal{W}(t),\ t\in\mathbb{R}^+,\ 1<\alpha<2,\\ \varkappa(0)=\theta_0,\ \varkappa'(0)=\theta_1,\end{cases}\tag{2.1}$$

if and if $\varkappa(t)$ is a solution of the Cauchy type problem

$$\begin{cases} \varkappa(t)=I^{\alpha-1}_{0+}\mathcal{W}(t)+\theta_1\\ \varkappa(0)=\theta_0.\end{cases}\tag{2.2}$$

Proof. To begin with, we claim that for any $0<\gamma<1$, if $\psi\in C\left[0,+\infty\right)$, then $\left(I^{\gamma}_{0+}\psi\right)(0)=0$. In fact, since

$$I^{\gamma}_{0+}\psi(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\psi(s)ds,$$

we can conclude that

$$\left|I^{\gamma}_{0+}\psi(t)\right|=\frac{1}{\Gamma(\gamma)}\left|\int_0^t(t-s)^{\gamma-1}\psi(s)ds\right|\leqslant\frac{\|\psi\|_t}{\Gamma(\gamma+1)}t^k\to 0 \text{ as } t\to 0.$$

(1) Let $\varkappa\in C\left[0,+\infty\right)$ be a solution of the problem (2.1).

For any $t\in\mathbb{R}^+$, Definition 2.2 shows that

$${}^{C}D^{\alpha}_{0+}\varkappa(t)=\left({}^{C}D^{\alpha-1}_{0+}D^1\varkappa\right)(t)=\mathcal{W}(t).$$

According to Lemma 2.3, we have

$$\varkappa'(t)=\varkappa'(0)+I^{\alpha-1}_{0+}\mathcal{W}(t)=I^{\alpha-1}_{0+}\mathcal{W}(t)+\theta_1,$$

which means that $\varkappa(t)$ is a solution of the problem (2.2).

(2) Let $\varkappa(t)$ be a solution of the problem (2.2).

For any $t\in\mathbb{R}^+$, by Remark 2.4, it is easy to see that

$${}^{C}D^{\alpha}_{0+}\varkappa(t)={}^{C}D^{\alpha-1}_{0+}\varkappa'(t)=\left({}^{C}D^{\alpha-1}_{0+}I^{\alpha-1}_{0+}\mathcal{W}\right)(t)+{}^{C}D^{\alpha-1}_{0+}\theta_1=\mathcal{W}(t).$$

Besides, note that $\mathcal{W}(t)\in C\left[0,+\infty\right)$, we have $\varkappa'(0)=I^{\alpha-1}_{0+}\mathcal{W}(0)+\theta_1=\theta_1$. □

Lemma 2.6. *Shows that the system (1.1) and (1.2) is equivalent to the system*

$$\varkappa'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\mathcal{F}(s,\varkappa(\lambda s))ds + \theta_1, \qquad (2.3)$$
$$\varkappa(0) = \theta_0.$$

For convenience, for $0 \leqslant s \leqslant t < +\infty$, *where* $k \in \mathbb{R}$ *satisfies that there exists a constant* $\beta_1 \in (0,1)$ *such that*

$$e^{-kt}/\varpi(t) \in BC\,[0,+\infty) \cap L^1\,[0,+\infty)\,,\ |k|\int_0^t e^{-ks}/\varpi(s)ds \leqslant \beta_1 < 1. \qquad (2.4)$$

Then (2.3) can be equivalently written as

$$\varkappa'(t) = -k\varkappa(t) + k\varkappa(t) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\mathcal{F}(s,\varkappa(\lambda s))ds + \theta_1, \qquad (2.5)$$
$$\varkappa(0) = \theta_0.$$

By the variation of constants formula, we have

$$\varkappa'(t) + k\varkappa(t) = k\varkappa(t) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\mathcal{F}(s,\varkappa(\lambda s))ds + \theta_1,$$

equivalently

$$\left(\varkappa'(t) + k\varkappa(t)\right)e^{kt} = ke^{kt}\varkappa(t) + \frac{e^{kt}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\mathcal{F}(s,\varkappa(\lambda s))ds + e^{kt}\theta_1,$$

and

$$\begin{aligned}\int_0^t \left(e^{ks}\varkappa(s)\right)' ds &= \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^s e^{ks}(s-u)^{\alpha-2}\mathcal{F}(u,\varkappa(\lambda u))duds + \int_0^t e^{ks}\theta_1 ds\\ &= \frac{e^{kt}-1}{k}\theta_1 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{ks}(s-u)^{\alpha-2}ds\mathcal{F}(u,\varkappa(\lambda u))du.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varkappa(t) &= \theta_0 e^{-kt} + \frac{1-e^{-kt}}{k}\theta_1 + k\int_0^t e^{-k(t-s)}\varkappa(\lambda u)du\\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}ds\mathcal{F}(u,\varkappa(\lambda u))du, \qquad (2.6)\end{aligned}$$

holds we have $\varkappa(0) = \theta_0$. Based on the above argument, we get that the system (1.1) and (1.2) can be equivalently written as (2.6). Then our following study will focus on the system (2.6).

Definition 2.7. The trivial solution $\varkappa \equiv 0$ of (1.1) and (1.2) is said to be

(i) stable in Banach space E if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|\theta_0| + |\theta_1| \leqslant \delta$ implies that the solution $\varkappa(t) = \varkappa(t,\theta_0,\theta_1)$ exists for all $t \geqslant 0$ and satisfies $\|\varkappa\| \leqslant \varepsilon$.

(ii) asymptotically stable if it is stable in Banach space E and there exists a number $\sigma > 0$ such that $|\theta_0| + |\theta_1| \leqslant \sigma$ implies $\lim_{t\to\infty}\|\varkappa(t)\| = 0$.

Theorem 2.8. *Krasnoselskii [21]. Let $\cdot$ be a non-empty closed convex subset of a Banach space* $(S, \|.\|)$. *Suppose that* $\mathcal{P}$ *and* $\mathcal{Q}$ *map* $\cdot$ *into* S *such that*

Lemma 2.9. **i** $\mathcal{P}\varkappa + \mathcal{Q}y \in \cdot$ *for all* $\varkappa, y \in \cdot$.

ii $\mathcal{P}$ *is continuous and* $\mathcal{P}\cdot$ *is contained in a compact set of* S,

iii $\mathcal{Q}$ *is a contraction with constant* $l < 1$.
Then there is a $\varkappa \in \cdot$ *with* $\mathcal{P}\varkappa + \mathcal{Q}\varkappa = \varkappa$.

In order to prove our main results, the following modified compactness criterion is needed.

Lemma 2.10. *[3]. Let* $\aleph$ *be a subset of the Banach space* E. *Then* $\aleph$ *is relatively compact in* E *if the following conditions are satisfied :*
(i) $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \aleph\}$ *is uniformly bounded,*
(ii) $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \aleph\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$;
(iii) $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \aleph\}$ *is equiconvergent at infinity.i.e. for any given* $\varepsilon > 0$, *there exists a* $T_0 > 0$ *such that for all* $\varkappa \in \aleph$ *and* $t_1, t_2 > T_0$, *if holds*

$$|\varkappa(t_2)/\varpi(t_2) - \varkappa(t_1)/\varpi(t_1)| < \varepsilon.$$

3. Main results

In this section, we care about and prove our main results by assuming certain conditions.

Theorem 3.1. *Suppose that (2.4) holds and there exists constants* $\eta > 0$, $\beta_2 \in (0, 1-\beta_1)$ *and a continuous function* $\mathcal{F} : \mathbb{R}^+ \times (0, \eta] \to \mathbb{R}^+$ *such that*

$$\frac{|\mathcal{F}(t, \upsilon\varpi(t))|}{\varpi(t)} \leqslant \mathcal{F}(t, |\upsilon|), \tag{3.1}$$

holds for all $t \geqslant 0$, $0 < |\upsilon| \leqslant \eta$ *and*

$$\sup_{t\geqslant 0} \int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)} \frac{\mathcal{F}(t,r)}{\eta} du \leqslant \beta_2 < 1 - \beta_1, \tag{3.2}$$

holds for every $0 < r \leqslant \eta$, *where* β_1 *is defined in (2.4),* $\mathcal{F}(t,r)$ *is nondecreasing in* r *for fixed* t *and* $\mathcal{F}(t,r) \in L^1[0,+\infty)$ *in* t *for fixed* r *and*

$$\mathcal{N}(t-u) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds, & t-u \geqslant 0, \\ 0, & t-u < 0. \end{cases} \tag{3.3}$$

Then the trivial solution $\varkappa \equiv 0$ *of (1.1) and (1.2) is stable in Banach space* E.

Proof. For any given $\varepsilon > 0$, we first prove the existence of $\delta > 0$ such that

$$|\theta_0| + |\theta_1| < \delta \text{ implies } \|\varkappa\| \leqslant \varepsilon.$$

In fact, according to (2.4), there exists a constant $M_1 > 0$ such that

$$\frac{e^{-kt}}{\varpi(t)} \leqslant M_1. \tag{3.4}$$

Let $0 < \delta \leqslant \frac{(1-\beta_1-\beta_2)|k|}{|k|M_1+1+M_1}\varepsilon$. Consider the non-empty closed convex subset $\mathfrak{I}(\varepsilon) \subseteq E$, for $t \geqslant 0$, we denote two mapping $\mathcal{P}, \mathcal{Q}$ on $\mathfrak{I}(\varepsilon)$ as follows:

$$\begin{aligned} \mathcal{P}\varkappa(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds \mathcal{F}(u, \varkappa(\lambda u)) du \\ &= \int_0^t \mathcal{N}(t-u)\mathcal{F}(u, \varkappa(\lambda u)) du, \end{aligned} \tag{3.5}$$

$$Q\varkappa(t) = e^{-kt}\theta_0 + \frac{1-e^{-kt}}{k}\theta_1 + k\int_0^t e^{-k(t-s)}\varkappa(u)du \tag{3.6}$$

Obviously, for $\varkappa \in \mathfrak{I}(\varepsilon)$, both $\mathcal{P}\varkappa$ and $Q\varkappa$ are continuous functions on $\mathbb{R}^+$. Furthermore, for $\varkappa \in \mathfrak{I}(\varepsilon)$, by (2.4-3.1-3.2) for any $t \geqslant 0$, we have

$$\begin{aligned}
\frac{|\mathcal{P}\varkappa(t)|}{\varpi(t)} &\leqslant \int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\left|\mathcal{F}(\frac{u,\varkappa(\lambda u)}{\varpi(u)})\right| du \\
&\leqslant \int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\mathcal{F}\left(u,\frac{|\varkappa(\lambda u)|}{\varpi(u)}\right) du \\
&\leqslant \beta_2\|\varkappa\| \leqslant \beta_2\varepsilon < \infty,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\frac{|Q\varkappa(t)|}{\varpi(t)} &= \left|\frac{e^{-kt}}{\varpi(t)}\theta_0 + \frac{1-e^{-kt}}{k\varpi(t)}\theta_1 + k\int_0^t \frac{e^{-k(t-s)}}{\varpi(t)}\varkappa(s)ds\right| \\
&\leqslant M_1|\theta_0| + \frac{1+M_1}{|k|}|\theta_1| + |k|\int_0^\infty \frac{e^{-ku}}{\varpi(u)}du\,\|\varkappa\| \\
&\leqslant M_1|\theta_0| + \frac{1+M_1}{|k|}|\theta_1| + \beta_1\varepsilon < \infty.
\end{aligned} \tag{3.8}$$

Then $\mathcal{P}\mathfrak{I}(\varepsilon) \subseteq E$,and $Q\mathfrak{I}(\varepsilon) \subseteq E$. Next, we shall use Lemma 2.10 to prove there exists at least one fixed point of the operator $\mathcal{P}+Q$ in $\mathfrak{I}(\varepsilon)$. Here, we divide the proof into three steps.

Step 1. We prove that $\mathcal{P}\varkappa + Qy \in \mathfrak{I}(\varepsilon)$ for all $\varkappa, y \in \mathfrak{I}(\varepsilon)$.

For any $\varkappa, y \in \mathfrak{I}(\varepsilon)$, from (3.7) and (3.8), we obtain that

$$\begin{aligned}
\sup_{t\geqslant 0}\frac{|\mathcal{P}\varkappa + Qy|}{\varpi(t)} &= \sup_{t\geqslant 0}\left\{\left|\frac{e^{-kt}}{\varpi(t)}\theta_0 + \frac{1-e^{-kt}}{k\varpi(t)}\theta_1 + k\int_0^t \frac{e^{-k(t-s)}}{\varpi(t)}y(\lambda s)ds\right.\right. \\
&\quad \left.\left. + \int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t)}\mathcal{F}(u,\varkappa(\lambda u))du\right|\right\} \\
&\leqslant M_1|\theta_0| + \frac{1+M_1}{|k|}|\theta_1| + |k|\int_0^\infty \frac{e^{-ku}}{\varpi(u)}du\,\|y\| + \beta_2\|\varkappa\| \\
&\leqslant \frac{M_1|k| + 1 + M_1}{|k|}\delta + \beta_1\varepsilon + \beta_2\varepsilon \leqslant \varepsilon,
\end{aligned}$$

which implies $\mathcal{P}\varkappa + Qy \in \mathfrak{I}(\varepsilon)$ for all $\varkappa, y \in \mathfrak{I}(\varepsilon)$.

Step 2. It is easy to see that $\mathcal{P}$ is continuous. Now we only prove that $\mathcal{P}\mathfrak{I}(\varepsilon)$ is a relatively compact in E. In fact, from (3.7), we get that $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \mathfrak{I}(\varepsilon)\}$ is uniformly bounded in E. Moreover, a classical theorem states the fact that the convolution of an L^1–function with a function tending to zero. Then we conclude that for $t-u \geqslant 0$, we have

$$\begin{aligned}
0 \leqslant \lim_{t\to\infty}\frac{\mathcal{N}(t-u)}{\varpi(t-u)} &\leqslant \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_u^t \frac{e^{-k(t-s)}}{\varpi(t-u)}\frac{(s-u)^{\alpha-2}}{\varpi(s-u)}ds \\
&= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{e^{-k(t-u-s)}}{\varpi(t-u-s)}\frac{s^{\alpha-2}}{\varpi(s)}ds = 0,
\end{aligned} \tag{3.9}$$

due to the fact $\lim_{t\to\infty}\frac{t^{\alpha-2}}{\varpi(t)} = 0$. Together with the continuity continuity of $\mathcal{N}(t)$ and $\varpi(t)$, we get that there exists a constant $M_2 > 0$ such that

$$\left|\frac{\mathcal{N}(t-u)}{\varpi(t-u)}\right| \leqslant M_2, \tag{3.10}$$

and for any $T_0 \in \mathbb{R}^+$, the function $\mathcal{N}(t-u)\varpi(u)/\varpi(t)$ is uniformly continuous on $\{(t,u) : 0 \leqslant u \leqslant t \leqslant T_0,\}$. For any $t_1, t_2 \in [0, T_0], t_1 < t_2$, we have

$$\begin{aligned}
\left|\frac{\mathcal{P}\varkappa(t_2)}{\varpi(t_2)} - \frac{\mathcal{P}\varkappa(t_1)}{\varpi(t_1)}\right| &= \left|\int_0^{t_2} \frac{\mathcal{N}(t_2-u)}{\varpi(t_2)}\mathcal{F}(u,\varkappa(\lambda u))du \right.\\
&\left. - \int_0^{t_1} \frac{\mathcal{N}(t_1-u)}{\varpi(t_1)}\mathcal{F}(u,\varkappa(\lambda u))du\right|\\
&\leqslant \int_0^{t_1} \left|\frac{\mathcal{N}(t_2-u)}{\varpi(t_2)} - \frac{\mathcal{N}(t_1-u)}{\varpi(t_1)}\right| |\mathcal{F}(u,\varkappa(\lambda u))|\, du\\
&+ \int_{t_1}^{t_2} \frac{\mathcal{N}(t_2-u)}{\varpi(t_2-u)}\mathcal{F}(u,\varepsilon)\, du\\
&\leqslant \int_0^{t_1} \left|\frac{\mathcal{N}(t_2-u)\leftrightarrow(u)}{\varpi(t_2)} - \frac{\mathcal{N}(t_1-u)\varpi(u)}{\varpi(t_1)}\right| \mathcal{F}(u,\varepsilon)\, du\\
&+ M_2 \int_{t_1}^{t_2} \mathcal{F}(u,\varepsilon)\, du \to 0,
\end{aligned}$$

as $t_2 \to t_1$, which means that $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \mathfrak{I}(\varepsilon)\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$. By Lemma 2.10, in order to show that $\mathcal{P}\mathfrak{I}(\varepsilon)$ is a relatively compact set of E, we only need to prove that $\{\varkappa(t)/\varpi(t) : \varkappa(t) \in \mathfrak{I}(\varepsilon)\}$ is equiconvergent at infinity. In fact, for any $\varepsilon_1 > 0$, there exists a $L > 0$ such that

$$M_2 \int_L^{\infty} \mathcal{F}(u,\varepsilon)\, du \leqslant \frac{\varepsilon_1}{3}.$$

According to (3.9), we get that

$$\lim_{t\to\infty} \sup_{u\in[0,L]} \frac{\mathcal{N}(t-u)}{\varpi(t-u)} \leqslant \max\left\{\lim_{t\to\infty} \frac{\mathcal{N}(t-L)}{\varpi(t-L)}, \lim_{t\to\infty} \frac{\mathcal{N}(t)}{\varpi(t)}\right\} = 0.$$

Thus, there exists $T < L$ such that $t_1, t_2 \geqslant T$, we have

$$\begin{aligned}
\sup_{s\in[0,L]} \left|\frac{\mathcal{N}(t_2-u)\varpi(u)}{\varpi(t_2)} - \frac{\mathcal{N}(t_1-u)\varpi(u)}{\varpi(t_1)}\right| &\leqslant \sup_{s\in[0,L]} \left|\frac{\mathcal{N}(t_2-u)}{\varpi(t_2-u)}\right| + \sup_{s\in[0,L]} \left|\frac{\mathcal{N}(t_1-u)}{\varpi(t_1-u)}\right|\\
&\leqslant \frac{\varepsilon_1}{3}\left(\int_0^{\infty} \mathcal{F}(u,\varepsilon,\varepsilon)\, du\right)^{-1}.
\end{aligned}$$

Therefore, for $t_1, t_2 \geqslant T$,

$$\begin{aligned}
\left|\frac{\mathcal{P}\varkappa(t_2)}{\varpi(t_2)} - \frac{\mathcal{P}\varkappa(t_1)}{\varpi(t_1)}\right| &= \left|\int_0^{t_2} \frac{\mathcal{N}(t_2-u)}{\varpi(t_2)}\mathcal{F}(u,\varkappa(\lambda u))du \right.\\
&\left. - \int_0^{t_1} \frac{\mathcal{N}(t_1-u)}{\varpi(t_1)}\mathcal{F}(u,\varkappa(\lambda u))du\right|\\
&\leqslant \int_0^{L} \left|\frac{\mathcal{N}(t_2-u)\varpi(u)}{\varpi(t_2)} - \frac{\mathcal{N}(t_1-u)\varpi(u)}{\varpi(t_1)}\right| \mathcal{F}(u,\varepsilon)\, du\\
&+ \int_L^{t_2} \frac{\mathcal{N}(t_2-u)}{\varpi(t_2-u)}\mathcal{F}(u,\varepsilon)\, du + \int_L^{t_1} \frac{\mathcal{N}(t_1-u)}{\varpi(t_1-u)}\mathcal{F}(u,\varepsilon)\, du\\
&\leqslant \frac{\varepsilon_1}{3} + 2M_2 \int_L^{\infty} \mathcal{F}(u,\varepsilon)\, du \leqslant \varepsilon_1.
\end{aligned}$$

Hence the required conclusion is true.

Step 3. we claim that $Q : \mathfrak{I}(\varepsilon) \to E$ is a contraction mapping.

In fact, for any $\varkappa, y \in \mathfrak{I}(\varepsilon)$, from (2.4), we obtain that

$$\begin{aligned}\sup_{t\geqslant 0}\left|\frac{Q\varkappa(t)}{\varpi(t)}-\frac{Qy(t)}{\varpi(t)}\right| &= \sup_{t\geqslant 0}\left|k\int_0^t \frac{e^{-k(t-u)}}{\varpi(t)}\varkappa(u)du - k\int_0^t \frac{e^{-k(t-u)}}{\varpi(t)}y(u)du\right| \\ &\leqslant \sup_{t\geqslant 0}|k|\int_0^t \frac{e^{-k(t-u)}}{\varpi(t-u)}\frac{|\varkappa(u)-y(u)|}{\varpi(u)}du \\ &\leqslant |k|\int_0^t \frac{e^{-k(t-u)}}{\varpi(t-u)}du\,\|\varkappa-y\| \\ &\leqslant \beta_1\|\varkappa-y\| < \|\varkappa-y\|.\end{aligned}$$

By lemma 2.10, we know that there exists at least one point of the operator $\mathcal{P}+\mathcal{Q}$ in $\mathfrak{I}(\varepsilon)$. Finally, for any ε_2, if $0<\delta_1\leqslant \frac{(1-\beta_1-\beta_2)|k|}{|k|M_1+1+M_1}\varepsilon_2$, then $|\theta_0|+|\theta_1|\leqslant\delta_1$ implies that

$$\begin{aligned}\|\varkappa\| &= \sup_{t\geqslant 0}\left\{\left|\theta_0\frac{e^{-kt}}{\varpi(t)}+\frac{1-e^{-kt}}{k\varpi(t)}\theta_1 + k\int_0^t \frac{e^{-k(t-s)}}{\varpi(t)}\varkappa(u)du\right.\right. \\ &\left.\left.+\int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t)}\mathcal{F}(u,\varkappa(\lambda u))du\right|\right\} \\ &\leqslant \sup_{t\geqslant 0}\left\{\frac{e^{-kt}}{\varpi(t)}\theta_0+\frac{|1-e^{-kt}|}{|k|\,\varpi(t)}|\theta_1|\right. \\ &\left.+|k|\int_0^t \frac{e^{-k(t-u)}}{\varpi(t-u)}\frac{|\varkappa(u)|}{\varpi(u)}du+\int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\frac{|\mathcal{F}(u,\varkappa(\lambda u)|}{\varpi(u)}du\right\} \\ &\leqslant M_1\delta_1+\frac{1+M_1}{|k|}\delta_1+\beta_1\|\varkappa\|+\beta_2\|\varkappa\| \\ &\leqslant \frac{M_1|k|+1+M_1}{(1-\beta_1-\beta_2)|k|}\delta_1\leqslant\varepsilon_2.\end{aligned}$$

Thus, we know that trivial solution of the system (1.1) and (1.2) is stable in Banach space E. □

Theorem 3.2. *Suppose that all conditions of Theorem 3.1. are satisfied*

$$\lim_{t\to\infty} e^{-kt}/\varpi(t)=0, \tag{3.11}$$

and for any $r>0$, *there exists a function* $\varphi_r(t)\in L^1[0,+\infty)$, $\varphi_r(t)>0$ *such that* $|u|\leqslant r$ *implies*

$$|\mathcal{F}(t,u)|/\varpi(t)\leqslant\varphi_r(t),\ a.e.\, t\in[0,+\infty). \tag{3.12}$$

Then the trivial solution of (1.1) and (1.2) is asymptotically stable.

Proof. First, it follows from Theorem 3.1 that the trivial solution of (1.1) and (1.2) is stable in the Banach space E. Next, we shall show that the trivial solution $\varkappa\equiv 0$ of (1.1) and (1.2) is attractive. For any $r>0$, defining

$$\mathfrak{I}_*(r)=\left\{\varkappa:\varkappa\in\mathfrak{I}(r),\lim_{t\to\infty}\varkappa(t)/\varpi(t)=0\right\}.$$

We only need to prove that $\mathcal{P}\varkappa+\mathcal{Q}y\in\mathfrak{I}_*(r)$ for any $\varkappa,y\in\mathfrak{I}_*(r)$, i.e.

$$\frac{\mathcal{P}\varkappa(t)+\mathcal{Q}y(t)}{\varpi(t)}\to 0 \text{ as } t\to\infty,$$

where

$$\mathcal{P}\varkappa + \mathcal{Q}y = \theta_0 e^{-kt} + \frac{1-e^{-kt}}{k}\theta_1 + k\int_0^t e^{-k(t-s)}y(u)du + \int_0^t \mathcal{N}(t-u)\mathcal{F}(u,\varkappa(\lambda u))du.$$

In fact, for $\varkappa, y \in \mathfrak{I}_*(r)$, based on the fact that used in the proof of Theorem 3.1 (Step2), it follows from (2.4) and (3.11) that

$$\int_0^t \frac{e^{-k(t-u)}}{\varpi(t-u)}\frac{y(u)}{\varpi(u)}du \to 0 \text{ and } \frac{\mathcal{N}(t-u)}{\varpi(t-u)} = \frac{\int_u^t \frac{e^{-k(t-s)}}{\varpi(t-u)}(s-u)^{\alpha-2}ds}{\Gamma(\alpha-1)} \to 0,$$

as $t \to \infty$. Together with the hypothesis $\varphi_r(t) \in L^1[0,+\infty)$, we obtain that

$$\int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\frac{|\mathcal{F}(u,\varkappa(\lambda u))|}{\varpi(u)}du \leqslant \int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\varphi_r(u)du \to 0$$

as $t \to \infty$. Thus we get the conclusion. □

4. Example

We make an assumption of theoretical application in the following non-linear fractional initial value problem

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}}\varkappa(t) = \mu\left(\frac{t^2(\lambda\varkappa)^2}{e^{(\vartheta+1)t}} + \frac{(\lambda\varkappa)^{3/2}}{(1+t^2)e^{\vartheta t/2}}\right), \\ \varkappa(0) = \theta_0, \varkappa'(0) = \theta_1, \end{cases} \tag{4.1}$$

where $\vartheta > 1, \mu > 0$. Suppose $0 < |k| \leqslant \frac{\vartheta-1}{2}$, let $\varpi(t) = e^{\vartheta t}, \beta_1 = \frac{|k|}{\vartheta+k}$, then (2.4) holds and the Banach space is

$$E^* = \left\{\varkappa(t) \in C[0,+\infty) : \sup_{t\geqslant 0}|\varkappa(t)|/e^{\vartheta t} < \infty\right\},$$

equipped with the norm $\|\varkappa\| = \sup_{t\geqslant 0}\frac{|\varkappa(t)|}{e^{\vartheta t}}$, and taking $\lambda = 1$. Let $\mathcal{F}(t,r) = \mu(r^2t^2e^{-t} + \frac{r^{3/2}}{1+t^2})$. Then (3.1) holds and $\mathcal{F}(t,r) \in L^1[0,+\infty)$ in t for fixed r. Note that

$$\frac{\mathcal{N}(t-u)}{e^{\vartheta(t-u)}} = \frac{1}{\Gamma(1/2)}\int_u^t \frac{1}{e^{(\vartheta+k)(t-s)}}\frac{(s-u)^{-1/2}}{e^{\vartheta(s-u)}}ds \leqslant \frac{\int_u^t \frac{(s-u)^{-1/2}}{e^{\vartheta(s-u)}}ds}{\Gamma(1/2)} = \frac{\int_0^{t-u}\frac{\omega^{-1/2}}{e^{\vartheta\omega}}ds}{\Gamma(1/2)} \leqslant \vartheta^{1/2},$$

for all $t \geqslant 0$, if there exists $\eta \geqslant 0$ such that

$$\mu \leqslant \frac{1}{2(2\eta + \frac{\pi}{2}\eta^{1/2})(\vartheta+k)\vartheta^{1/2}+1}, \tag{4.2}$$

then

$$\int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}\frac{\mathcal{F}(u,r)}{r}du = \mu\int_0^t \frac{\mathcal{N}(t-u)}{\varpi(t-u)}(rt^2e^{-t} + \frac{r^{1/2}}{1+t^2})du \leqslant \frac{1/2}{\vartheta+k} < 1-\beta_1,$$

for all $t \geqslant 0, 0 \leqslant r \leqslant \eta$. Thus the trivial solution of (4.1) is stable in E^* follows from Theorem 3.1.

Moreover, let $\varphi_r(t) = \mu(\frac{t^2r^2}{e^{(\vartheta+1)t}} + \frac{r^{3/2}}{(1+t^2)e^{\vartheta t/2}}) \in L^1[0,+\infty)$. For any bounded $r > 0$, we get that $|\mathcal{F}(t,u)| \leqslant \varphi_r(t)$ and

$$\lim_{t\to\infty} e^{-kt}/\varpi(t) \leqslant \lim_{t\to\infty} e^{-\frac{\vartheta t}{2}} = 0.$$

Then, by Theorem 3.2, we get that the trivial solution of (4.1) is asymptotically stable.

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