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Weakly r-Supercontinuous Functions

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Abstract

In this paper, we use r-open sets to introduce the concept of weakly r-supercontinuous functions. Then, basic properties and a number of connections between these functions and other types of generalized continuous functions are examined. A weakly r-supercontinuous function is weaker than θ-continuous but stronger than weakly continuous. Furthermore, we present some characterizations of weakly r-supercontinuous functions. The last section starts with a comparison between weakly r-supercontinuous functions and some other types of functions when the underlying spaces possess particular topological properties. We end this work by analyzing the composition of weakly r-supercontinuous functions.

Keywords: strongly θ-continuous, weakly r-supercontinuous, r-supercontinuous, θ-continuous, continuous, almost continuous, weakly continuous.
2010 MSC: 54C08, 54C10.

1. Introduction

The concept of continuity is one of the most significant tools in all of mathematics, especially in topology and analysis. In the beginning stages of modern mathematics, many classes of generalized continuity were introduced. In 1932, Banach [8] considered almost continuity during the proof of the Closed Graph Theorem. For the same reason, Husain [10] defined another class of almost continuity in 1966, which is now familiar as precontinuity [18]. Signal & Signal [23] provided another sort of almost continuity for analyzing Alexandroff's almost compact spaces and Urysohn's nearly compact spaces in 1968. Then, in this view, distinct kinds of almost continuity were defined, (see [4, 7, 12, 14]). It is familiar that each generalized almost continuity is weaker than its (original) continuity. In 1961, Levine [16] presented a new type of continuity that is weaker than almost continuity by using the closure operator and named it a "weakly continuous function." Following Levine's idea, different classes of weakly continuous functions have appeared for different purposes, (see [6, 13, 19, 20]). In this paper, we introduce a class of weakly r-supercontinuous functions and study its main properties. We see that weakly r-supercontinuity is placed between θ-continuity and weak continuity. We characterize weakly r-supercontinuous functions in terms of various topological operators and different types of open sets.

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2. Preliminaries

From here on, the set of rational and real numbers are respectively denoted by $\mathbb{Q}$ and $\mathbb{R}$. The word space is referred to arbitrary topological space. The closure and interior of $A \subseteq X$ are named by $Cl(A)$ and $Int(A)$.

Definition 2.1. A subset A of a space (X, τ) is said to be

(1) clopen if A is open and closed,

(2) regular open if $A = Int(Cl(A))$,

(3) preopen [18] if $A \subseteq Int(Cl(A))$,

(4) semiopen [17] if $A \subseteq Cl(Int(A))$,

(5) β-open [1] if $A \subseteq Cl(Int(Cl(A)))$,

(6) sc-open [2, 11] if A is semiopen and a union of closed sets,

(7) pc-open [3] if A is preopen and a union of closed sets,

(8) r-open [15] if A is open and a union of closed sets,

(9) θ-open [24] if for each $x \in A$, there exists $U \in \tau$ such that $x \in U \subseteq Cl(U) \subseteq A$.

The family of all r-open, θ-open sets in (X, τ) are respectively denoted by $rO(X)$, $\theta O(X)$. We shall remark that both $rO(X)$, $\theta O(X)$ form topologies on X, we may denote them by τ_r, τ_θ if no confusion arises. A regular closed (resp. preclosed, semi-closed, sc-closed, pc-closed, β-closed, θ-closed) set is the complement of regular open (resp. preopen, semiopen, sc-open, pc-open, β-open, θ-open). The union of all r-open (resp. θ-open) sets in (X, τ) included in A is the r-interior (resp. θ-interior) of A, and is denoted by $rInt(A)$ (resp. $Int_\theta(A)$). The intersection of all r-closed (resp. θ-closed) sets in (X, τ) including A is the r-closure (resp. θ-closure) of A, and is symbolized by $rCl(A)$ (resp. $Cl_\theta(A)$). A point $x \in X$ is in the $r$closure (resp. θ-closure) of $A \subseteq X$ if $U \cap A \neq \phi$ (resp. $Cl(U) \cap A \neq \phi$) for every r-open (resp. open) set U containing x.

Lemma 2.2. *Let A be a subset (X, τ). If $A \in \tau$, then $Cl(A) = Cl_\theta(A)$.*

At this point, perhaps a relationship between the above mentioned generalized open sets related to r-open sets is required.

$$
\begin{array}{ccccccccc}
 & & & & \theta\text{-open} & & & & \\
 & & & & \Downarrow & & & & \\
\text{preopen} & \Longleftarrow & pc\text{-open} & \Longleftarrow & r\text{-open} & \Longrightarrow & sc\text{-open} & \Longrightarrow & \text{semiopen} \\
 & & & & \Uparrow & & & & \\
 & & & & \text{open} & & & &
\end{array}
$$

Diagram 1: Relationships between generalized open sets

In general, none of the preceding arrows are reversible, we only provide counterexamples for the connections of r-open sets. The others can be found in [3, 5].

Example 2.3. *[15] Consider the right order topology $\tau = \{\phi, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ on $\mathbb{R}$. Then any ray (a, ∞) is an open set in but not r-open.*

Example 2.4. *Consider the cofinite topology $\tau = \{A \subseteq \mathbb{R} : A^c$ is finite$\}$ on $\mathbb{R}$. Then any open set is r-open but not θ-open.*

Example 2.5. *Consider the standard topology τ on $\mathbb{R}$, i.e., the topology generated by $\{(a, b) : a, b \in \mathbb{R}\}$. Then the set $(a, b]$ is sc-open but not r-open. And the set $\mathbb{Q}$ is pc-open but not r-open.*

A space (X, τ) is said to be R_0 [22] if for each $x \in X$ and each $U \in \tau$ with $x \in U$, $Cl(\{x\}) \subseteq U$.

Lemma 2.6. *[15] If a space (X, τ) is either R_0 or T_1, then $\tau = rO(X)$,*

Lemma 2.7. *[24] If a space (X, τ) is regular, then $\tau = \theta O(X)$,*

From the above lemma and Diagram 1, we conclude

Lemma 2.8. *If a space (X, τ) is regular, then $\tau = rO(X) = \theta O(X)$,*

Lemma 2.9. *Let A be a subset of (X, τ). If $A \in \tau$, then $Cl(A) = rCl(A) = Cl_\theta(A)$.*

Proof. It follows from Lemma 2.2 and Diagram 1. □

A space (X, τ) is called Alenxandroff if τ is closed under arbitrary intersections.

Lemma 2.10. *If a topological space (X, τ) is Alenxandroff, then $Clop(X) = rO(X)$, where $Clop(X)$ is the family of all clopen sets in X.*

Proof. The first inclusion is evident, i.e., $Clop(X) \subseteq rO(X)$. We now show that $rO(X) \subseteq Clop(X)$. Let $A \in rO(X)$. If $A = \phi$, then it is clear that $A \in Clop(X)$. If $A \neq \phi$, then $A = \bigcup_{i \in I} F_i$, where $A \in \tau$, $F_i^c \in \tau$, and I an index set. Since X is an Alenxandroff space, so $\bigcup F_i$ is closed. Therefore, A is both open and closed. Hence $A \in Clop(X)$ and so $Clop(X) = rO(x)$. □

Corollary 2.11. *If a topological space (X, τ) is finite, then $Clop(X) = rO(X)$.*

Lemma 2.12. *Let (X, τ) be a space and Y be a subspace of X. If $A \subseteq Y$ and $A \in rO(X)$, then $A \in rO(Y)$.*

Proof. If $A \in rO(X)$, then $A \in \tau$ and for each $x \in A$, there exists a closed set F in X such that $x \in F \subseteq A$. Therefore, $x \in F \cap Y \subseteq A \cap Y$. Evidently, $F \cap Y$ and $A \cap Y = A$ are respectively closed and open sets in Y. Thus, $A \in rO(Y)$. □

Lemma 2.13. *Let Y be an r-open subspace of a space X, if $A \in rO(Y)$, then $A \in rO(X)$.*

Proof. Let $A \in rO(Y)$. Then $A \in \tau_Y$ and for each $x \in A$, there exists a closed set E in Y such that $x \in E \subseteq A$. Since Y is an r-open set in X, then $Y \in \tau$ and there exists a closed set F in X such that $x \in F \subseteq Y$. Therefore, $x \in E \cap F \subseteq A \cap Y = A$. Since $A \in \tau_Y$ and $Y \in \tau$, so $A \in \tau$. By the same reasoning, one can see that closed set E in X as F is closed in X. This implies that $A \in O(X)$. □

Definition 2.14. A function $f : X \to Y$ is called

(1) θ-continuous [9] at $x \in X$ if for each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Cl(U)) \subseteq Cl(V)$.

(2) strongly θ-continuous [21] at $x \in X$ if for each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Cl(U)) \subseteq V$.

(3) weakly continuous [16] at $x \in X$ if for each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq Cl(V)$.

(4) r-supercontinuous [15] at $x \in X$ if for each open set V in Y containing $f(x)$, there exists an r-open set U in X containing x such that $f(U) \subseteq V$.

3. Basic properties and connections

Definition 3.1. A function $f : X \to Y$ is called weakly r-supercontinuous at a point $x \in X$, if for each open set V of Y containing $f(x)$, there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(V)$. If f is weakly r-supercontinuous at every point x of X, then it is called weakly r-supercontinuous.

Evidently, each r-supercontinuous function is weakly r-supercontinuous.

Proposition 3.2. *Let $f : X \to Y$ be a function. Then f is weakly r-supercontinuous at $x \in X$ if and only if $x \in rInt(f^{-1}(Cl(V)))$ for each open set V containing $f(x)$.*

Proof. Let V be any open set containing $f(x)$. There exists an r-open set U containing x such that $f(U) \subseteq Cl(V)$. We have $U \subseteq f^{-1}(Cl(V))$. Since U is r-open, then $x \in U = rInt(U) \subseteq rInt(f^{-1}(Cl(V)))$.

Conversely, let $x \in rInt(f^{-1}(Cl(V)))$ for each open set V containing $f(x)$. Take $U = rInt(f^{-1}(Cl(V)))$. Then, $f(U) \subseteq Cl(V)$. Thus, f is weakly r-supercontinuous at $x \in X$ as U is r-open. □

Proposition 3.3. *Let $f : X \to Y$ be a function. Then f is weakly r-supercontinuous if and only if $f^{-1}(Int(F)))$ is r-closed for each closed set F.*

Proof. It follows from Proposition 3.2. □

Proposition 3.4. *Let $f : X \to Y$ be a function. Then the following statements are true:*

(1) If f is weakly r-supercontinuous and if A is a subspace of X, then the restriction function $f|A : A \to Y$ is weakly r-supercontinuous.

(2) Let $X = \bigcup U_\alpha, \alpha \in \Lambda$, where each U_α is an r-open subset of X. If for each $\alpha \in \Lambda$, $f_\alpha = f|U_\alpha$ is weakly r-supercontinuous, then weakly f is r-supercontinuous.

(3) Let $X = \bigcup_{i=1}^{n} F_i$, where each F_i is an r-closed subset of X. If for each $i = 1, ..., n$, $f|_{F_i}$ is weakly r-supercontinuous, then f is weakly r-supercontinuous.

Proof. (1) Let $x \in X$ and let V be an open set in Y containing $f(x)$. Since f is weakly r-supercontinuous, then there exists an r-open set U containing x such that $f(U) \subseteq Cl(V)$. By Lemma 2.12, $A \cap U$ is r-open in A, and so $(f|_A)(A \cap U) = f(A) \cap f(U) \subseteq f(U) \subseteq Cl(V)$. This shows that $f|_A$ is weakly r-supercontinuous.

(2) Let $x \in X$ and let V be an open set in Y containing $f(x)$. Since $\{U_\alpha : \alpha \in \Lambda\}$ covers X, then $x \in U_\alpha$ for some α. By hypothesis, there exists an r-open set G in U_α containing x such that $(f|_{U_\alpha})(G) \subseteq Cl(V)$. Since all U_α are r-open sets, by Lemma 2.13, G is an r-open set in X and hence $f(G) \subseteq Cl(V)$. Thus, f is weakly r-supercontinuous.

(3) Let E be any closed subset of Y. Since $f|_{F_i}$ is weakly r-supercontinuous for $i = 1, ..., n$, by Proposition 3.3, $(f|_{F_i})^{-1}(Int(E))$ is r-closed in F_i. Since each F_i is r-closed in X, by Lemma 2.13, $(f|_{F_i})^{-1}(Int(E))$ is r-closed in X. But,

$$f^{-1}(Int(E)) = (f|_{F_1})^{-1}(Int(E)) \bigcup (f|_{F_2})^{-1}(Int(E)) \bigcup \cdots \bigcup (f|_{F_n})^{-1}(Int(E)),$$

is r-closed as it is a finite union of r-closed sets. Thus, f is weakly r-supercontinuous. □

Proposition 3.5. *If a function $f : X \to Y$ is weakly r-supercontinuous, then for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists an r-open set U in X containing x such that $f(U) \subseteq V$.*

Proof. Let $x \in X$ and let V be any θ-open set of Y containing $f(x)$. Then for each $f(x) \in V$, there exists an open set G containing $f(x)$ such that $G \subseteq Cl(G) \subseteq V$. Since f is weakly r-supercontinuous, there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(G) \subseteq V$. This completes the proof. □

Proposition 3.6. *If a function $f : X \to Y$ is weakly r-supercontinuous, then for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists an closed set F in X containing x such that $f(F) \subseteq V$.*

Proof. Let $x \in X$ and let V be any θ-open set of Y containing $f(x)$. Since f is weakly r-supercontinuous, Then by Proposition 3.5, there exists an r-open set $U \in X$ containing x such that $f(U) \subseteq V$. Since U is an r-open set in X, then there exists a closed set F of X such that $x \in F \subseteq U$. Therefore, we obtain $f(F) \subseteq f(U) \subseteq V$. Hence $f(F) \subseteq V$. □

Proposition 3.7. *Let $f : X \to Y$ be a function. If $x \in X$ and each regular closed set R of Y containing $f(x)$ there exists an r-open set $U \in X$ containing x such that $f(U) \subseteq R$, then f is weakly r-supercontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then put $R = Cl(V)$ is regular closed set of Y containing $f(x)$. By hypothesis, there exists an r-open set in X such that $f(U) \subseteq R$. Hence f is weakly r-supercontinuous. □

Proposition 3.8. *If a function $f : X \to Y$ is weakly r-supercontinuous, then the inverse image of each θ-open set of Y is r-open in X.*

Proof. Let V be any θ-open set in Y. We have to show that $f^{-1}(V)$ is an r-open set in X. Since f is weakly r-supercontinuous, by Proposition 3.5, there exists an r-open set U of X containing x such that $f(U) \subseteq V$. This implies that $x \in U \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is an r-open set in X. □

Corollary 3.9. *If a function $f : X \to Y$ is weakly r-supercontinuous, then the inverse image of each θ-closed set of Y is r-closed in X.*

Proposition 3.10. *If a function $f : (X, \tau) \to (Y, \sigma)$ is weakly r-supercontinuous, then $f : (X, \tau) \to (Y, \sigma_\theta)$ is r-supercontinuous.*

Proof. Let $H \in \sigma_\theta$. Then H is a θ-open set in (Y, σ). Since $f : (X, \tau) \to (Y, \sigma)$ is weakly r-supercontinuous, by Proposition 3.8, $f^{-1}(H)$ is an r-open set in X. Therefore, $f : (X, \tau) \to (Y, \sigma_\theta)$ is r-supercontinuous. □

Proposition 3.11. *Let $f : X \to Y$ be a function. If the inverse image of each regular closed set of Y is r-open in X, then f is weakly r-supercontinuous.*

Proof. Let V be any open set in Y. Then $Cl(V)$ is an regular closed set in Y, and by Proposition 3.2, $f^{-1}(Cl(V))$ is r-open in X. Therefore, f is weakly r-supercontinuous. □

Corollary 3.12. *Let $f : X \to Y$ be a function. If the inverse image of each regular open set of Y is r-closed in X, then f is weakly r-supercontinuous.*

Proposition 3.13. *If a function $f : X \to Y$ is weakly r-supercontinuous, then for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a closed set $F \in X$ containing x such that $f(F) \subseteq Cl(V)$.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since f is weakly r-supercontinuous, then there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Since U is an r-open set, then there exists a closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq Cl(V)$. □

We end this section with the diagram below, which displays the connections between weakly r-supercontinuous and other generalized continuous functions.

$$
\begin{array}{ccc}
\text{strongly } \theta\text{-continuous} & \Longrightarrow & \theta\text{-continuous} \\
\Downarrow & & \Downarrow \\
r\text{-supercontinuous} & \Longrightarrow & \text{weakly } r\text{-supercontinuous} \\
\Downarrow & & \Downarrow \\
\text{continuous} & \Longrightarrow & \text{weakly continuous}
\end{array}
$$

Diagram 2: Relationships between generalized continuous functions

As demonstrated in the following examples, none of the preceding arrows involving weakly r-supercontinuity are reversible in general:

Example 3.14. *Let* $X = \{a, b\}$ *be a set and let* $\tau = \{\emptyset, \{a\}, X\}$ *be a topology on* X*. The identity function* $f : (X, \tau) \to (X, \tau)$ *is weakly* r*-supercontinuous but not* r*-supercontinuous.*

Example 3.15. *Let* $X = \{a, b, c, d\}$ *be a set, and let* $\tau = \{\emptyset, \{c\}, \{c, d\}, \{a, b, c\}, X\}$ *and* $\sigma = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b\}, \{a, b, d\}, \{b, c, d\}, X\}$ *be topologies on* X*. The identity function* $f : (X, \tau) \to (X, \sigma)$ *is weakly continuous but not weakly* r*-supercontinuous.*

Example 3.16. *Let* τ *be the cofinite topology on* $X = \mathbb{R}$ *and let* $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ *be a topology on* $Y = \{a, b, c\}$*. The function* $f : (X, \tau) \to (X, \sigma)$ *defined by*

$$f(x) = \begin{cases} a, & x = 1; \\ b, & x = 0; \\ c, & x \neq \{0, 1\}, \end{cases}$$

is weakly r*-supercontinuous but not* θ*-continuous.*

4. Characterizations

Theorem 4.1. *For a function* $f : X \to Y$*, the following statements are equivalent:*

(1) f *is weakly* r*-supercontinuous.*

(2) $rCl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$ *for each subset* $B \subseteq Y$*.*

(3) $f^{-1}(Int(B)) \subseteq rInt(f^{-1}(Cl(Int(B))))$ *for each subset* $B \subseteq Y$*.*

(4) $f^{-1}(Int(Cl(V))) \subseteq rInt(f^{-1}(Cl(V)))$ *for each open set* V *of* Y*.*

(5) $f^{-1}(V) \subseteq rInt(f^{-1}(Cl(V)))$ *for each regular open set* V *of* Y*.*

(6) $rCl(f^{-1}(Int(F)) \subseteq f^{-1}(F)$ *for each regular closed set* F *of* Y*.*

(7) $rCl(f^{-1}(Int(F)) \subseteq f^{-1}Cl(Int(F))$ *for each closed set* F *of* Y*.*

(8) $rCl(f^{-1}(V) \subseteq f^{-1}(Cl(V))$ *for each open set* V *of* Y*.*

(9) $f^{-1}(Int(F)) \subseteq rInt(f^{-1}(F))$ *for each closed set* F *of* Y*.*

Proof. (1) $\Longrightarrow$ (2). Let B be any subset of Y. Assume that $x \notin f^{-1}Cl(B)$. Then, $f(x) \notin Cl(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \phi$, hence $Cl(V) \cap Int(Cl(B)) = \phi$. By (1), there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Therefore, we have $f(U) \cap Int(Cl(B)) = \phi$, which implies that $U \cap f^{-1}(Int(Cl(B))) = \phi$ and hence $x \notin rCl(f^{-1}(Int(Cl(B))))$. Thus, we obtain $rCl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(Cl(B))$.

(2) $\Longrightarrow$ (3). Let B be any subset of Y. By applying (2) to $Y - B$, we obtain

$$\begin{aligned} & rCl(f^{-1}(Int(Cl(Y - B)))) \subseteq f^{-1}(Cl(Y - B)) \\ \Longleftrightarrow\ & rCl(f^{-1}(Int(Y - Int(B))) \subseteq f^{-1}(Y - Int(B)) \\ \Longleftrightarrow\ & rClf^{-1}(Y - ClInt(B)) \subseteq f^{-1}(Y - Int(B) \\ \Longleftrightarrow\ & X - rInt(f^{-1}(Cl(Int(B)) \subseteq X - f^{-1}(Int(B)) \\ \Longleftrightarrow\ & f^{-1}(Int(B)) \subseteq rInt(f^{-1}(Cl(Int(B)). \end{aligned}$$

Therefore, we obtain $f^{-1}(Int(B)) \subseteq rInt(f^{-1}(Cl(Int(B)).$

(3) $\Longrightarrow$ (4). Let V be an open subset of Y. Then, by applying (3) to $Cl(V)$, we obtain

$$f^{-1}(Int(Cl(V))) \subseteq rInt(f^{-1}(Cl(Int(Cl(V))) = rInt(f^{-1}((Cl(V))).$$

Therefore, we obtain $f^{-1}(Int(Cl(V)) \subseteq rInt(f^{-1}((Cl(V)))$.

(4) $\Longrightarrow$ (5). Let V be any regular open set of Y. Then V is an open set of Y. By (4), we have $f^{-1}(V) = f^{-1}(Int(Cl(V)) \subseteq rInt(f^{-1}((Cl(V)))$. Therefore, $f^{-1}(V) \subseteq rInt(f^{-1}((Cl(V)))$.

(5) $\Longrightarrow$ (6). Let F be any regular closed set of Y. Then $Y - F$ is a regular open set of Y. By (5), we have $f^{-1}(Y-F) \subseteq rInt(f^{-1}(Cl(Y-F)) \Leftrightarrow X - f^{-1}(F) \subseteq rInt(f^{-1}(Y - Int(F)) \Leftrightarrow X - f^{-1}(F) \subseteq rInt(f^{-1}(X - f^{-1}(Int(F))) \Leftrightarrow X - f^{-1}(F) \subseteq X - rCl(f^{-1}(Int(F))) \Leftrightarrow rCl(f^{-1}(Int(F))) \subseteq f^{-1}(F)$. Hence $rCl(f^{-1}(Int(F))) \subseteq f^{-1}(F)$.

(6) $\Longrightarrow$ (7). Let F be any closed set of Y. Then $Cl(Int(F))$ is regular closed set of Y. By(6), we have $rCl(f^{-1}(Int(Cl(Int(F))))) = rCl(f^{-1}(Int(F))) \subseteq f^{-1}(Cl(Int(F)))$. Therefore, we obtain $rCl(f^{-1}(Int(F))) \subseteq f^{-1}(Cl(Int(F)))$.

(7) $\Longrightarrow$ (8). Let V be any open set of Y. Then by (7), we have

$$rCl(f^{-1}(V)) \subseteq rCl(f^{-1}(Int(Cl(V)))) \subseteq f^{-1}(Cl(Int(Cl(V))) = f^{-1}(Cl(V)).$$

Therefore, $rCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$.

(8) $\Longrightarrow$ (9). Let F be any closed set of Y. Then $Y - F$ is an open set of Y. By (8), we have $rCl(f^{-1}(Y-F))) \subseteq f^{-1}(Cl(Y-F)) \Leftrightarrow rCl(X - f^{-1}(F)) \subseteq f^{-1}(Y - Int(F)) \Leftrightarrow X - rInt(f^{-1}(F) \subseteq X - f^{-1}Int(F) \Leftrightarrow f^{-1}Int(F) \subseteq rInt(f^{-1}(F)$. Therefore, $f^{-1}Int(F) \subseteq rInt(f^{-1}(F)$.

(9) $\Longrightarrow$ (1). Let x be any point of X and let V any open set of Y containing $f(x)$. Then, $x \in f^{-1}(V)$ and $Cl(V)$ is closed set in Y. By (9), we have $x \in f^{-1}(V) \subseteq f^{-1}(Int(Cl(V))) \subseteq rInt(f^{-1}(Cl(V)))$. Set $U = rInt(f^{-1}(Cl(V)))$. Then we obtain $x \in U \in rO(X)$ and $f(U) \subseteq Cl(V)$. Therefore, f is weakly r-supercontinuous. □

Proposition 4.2. *For a function* $f : X \to Y$, *the following statements are equivalent:*

(1) f *is weakly* r*-supercontinuous.*

(2) $f(rCl(A)) \subseteq Cl_\theta(f(A))$ *for each subset* A *of* X.

(3) $Int_\theta(f(A)) \subseteq f(rInt(A))$ *for each subset* A *of* X.

(4) $f^{-1}(Int_\theta(B)) \subseteq rIntf^{-1}(B)$ *for each subset* B *of* Y

(5) $rCl(f^{-1}(B)) \subseteq f^{-1}(Cl_\theta(B))$ *for each subset* B *of* Y.

Proof. (1) $\Longrightarrow$ (2). Let A be any subset of X. Suppose that $f(rCl(A)) \nsubseteq Cl_\theta(f(A))$. Then there exists $y \in f(rCl(A))$ such that $y \notin Cl_\theta(f(A))$, then there exists an open set G in Y containing y such that $Cl(G) \cap f(A) = \phi$. If $f^{-1}(y) = \phi$, then there is nothing to prove. Suppose that x is an arbitrary point of $f^{-1}(y)$, so $f(x) \in G$. Since G is an open set of Y containing $f(x)$ and f is weakly r-supercontinuous, there exists an r-open set $H \subseteq X$ containing x such that $f(H) \subseteq Cl(G)$. Therefore, we have $f(H) \cap f(A) = \phi$ which implies $x \notin rCl(A)$. Thus, $y \notin f(rCl(A))$, a contradiction. Hence, $f(rCl(A)) \subseteq Cl_\theta(f(A))$.

(2) $\Longrightarrow$ (3). Let A be any subset of X. Then apply (2) to $X - A$, we obtain $f(rCl(X - A)) \subseteq Cl_\theta(f(X - A)) \Leftrightarrow f(X - rInt(A)) \subseteq Cl_\theta(Y - f(A)) \Leftrightarrow Y - f(rInt(A)) \subseteq Y - Int_\theta(f(A)) \Leftrightarrow Int_\theta(f(A)) \subseteq f(rInt(A))$. Therefore, $Int_\theta(f(A)) \subseteq f(rInt(A))$.

(3) $\Longrightarrow$ (4). Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (3) we have $Int_\theta(f(f^{-1}(B))) \subseteq f(rInt(f^{-1}(B)))$. Then $Int_\theta(B) \subseteq f(rInt(f^{-1}(B)))$ and hence $f^{-1}(Int_\theta(B)) \subseteq rInt(f^{-1}(B))$.

(4) $\Longrightarrow$ (5). Let B be any subset of Y. By applying (4) to $Y - B$, we obtain $f^{-1}(Int_\theta(Y - B)) \subseteq rInt(f^{-1}(Y - B)) \Leftrightarrow f^{-1}(Y - (Cl_\theta(B)))) \subseteq rInt(X - f^{-1}(B)) \Leftrightarrow X - f^{-1}(Cl_\theta(B)) \subseteq X - rCl(f^{-1}(B)) \Leftrightarrow rClf^{-1}(B) \subseteq f^{-1}(Cl_\theta(B))$. Therefore, we obtain $rClf^{-1}(B) \subseteq f^{-1}(Cl_\theta(B))$.

(5) $\Longrightarrow$ (1). Let V be any open set of Y containing $f(x)$. Since $Cl(V) \cap (Y - Cl(V)) = \phi$, we have $f(x) \notin Cl_\theta(Y - Cl(V))$ and hence $x \notin f^{-1}(Cl_\theta(Y - Cl(V)))$. By (5), $x \notin rCl(f^{-1}(Y - Cl(V)))$. Then there exists $U \in rO(X)$ containing x such that $U \cap f^{-1}(Y - Cl(V)) = \phi$, hence $f(U) \cap (Y - Cl(V)) = \phi$. This shows that $f(U) \subseteq Cl(V)$. Thus, f is weakly r-supercontinuous. □

Theorem 4.3. *A function $f : X \to Y$ is weakly r-supercontinuous if and only if $rClf^{-1}(V) \subseteq f^{-1}(rCl(V))$ for each open subset V of Y.*

Proof. Let V be any open set of Y. Since f is weakly r-supercontinuous, by Proposition 4.1 (8), $rClf^{-1}(V) \subseteq f^{-1}(Cl(V))$. Since V is an open set, by Lemma 2.2, $Cl(V)) = rCl(V)$ and so $rClf^{-1}(V) \subseteq f^{-1}(rCl(V))$.

Conversely. Let F be any closed set of Y. Then $Int(F)$ is an open set in Y. By hypothesis, we have $rClf^{-1}(Int(F)) \subseteq f^{-1}(rCl(Int(F)))$. Since Int(F) is an open set, by Lemma 2.2, $rClf^{-1}(Int(F)) \subseteq f^{-1}(Cl(Int(F)))$. Therefore, by Proposition 4.1, f is weakly r-supercontinuous. □

From Theorem 4.3, we obtain that:

Corollary 4.4. *A function $f : X \to Y$ is weakly r-supercontinuous if and only if $f^{-1}(rInt(F)) \subseteq rIntf^{-1}(F)$ for each closed set F of Y.*

5. Further properties and operations

In this section, we give some properties of weakly r-supercontinuous functions when the underlying spaces have certain topological properties and study the composition of them.

Proposition 5.1. *If $f : X \to Y$ is weakly r-supercontinuous and Y is regular. Then f is r-supercontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since Y is regular, then there exists an open G of Y such that $f(x) \in G \subseteq Cl(G) \subseteq V$. Since f is weakly r-supercontinuous, there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(G) \subseteq V$. Therefore, f is r-supercontinuous. □

Proposition 5.2. *If $f : X \to Y$ is weakly r-supercontinuous and X is R_0 or T_1. Then f is weakly continuous.*

Proof. It follows from Lemma 2.6 that open and r-open sets are equivalent. □

Proposition 5.3. *If X is a T_1 space and Y is a regular space, the following statements are equivalent for a function $f : X \to Y$:*

1. *f is θ-continuous.*
2. *f is r-supercontinuous.*
3. *f is weakly r-supercontinuous.*
4. *f is continuous.*

Proof. It follows from Propositions 5.2-5.1. □

Proposition 5.4. *If X, Y are regular spaces, the following statements are equivalent for a function $f : X \to Y$:*

1. *f is θ-continuous.*
2. *f is continuous.*
3. *f is r-supercontinuous.*
4. *f is weakly continuous.*

5. *f is weakly r-supercontinuous.*

6. *f is strongly θ-continuous.*

Proof. It follows from Proposition 5.1 and Lemma 2.8. □

Proposition 5.5. *Let $f : X \to Y$ be a weakly r-supercontinuous and for each $x \in X$. If Y is any open subset of a topological space Z containing $f(x)$, then $f : X \to Z$ is weakly r-supercontinuous.*

Proof. Let $x \in X$ and V be an open set in Z containing $f(x)$. Then $V \cap Y$ is open in Y containing $f(x)$. Since f is weakly r-supercontinuous, there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(V \cap Y)$ and hence $f(U) \subseteq Cl(V)$. Therefore $f : X \to Z$ is weakly r-supercontinuous. □

Proposition 5.6. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition function $g \circ f : X \to Z$ is weakly r-supercontinuous if f and g satisfy one of the following condition :*

(1) f is r-supercontinuous and g is weakly continuous.

(2) f is weakly r-supercontinuous and g is θ-continuous.

(3) f is weakly r-supercontinuous and g is continuous.

(4) f is continuous and open and g is weakly r-supercontinuous.

Proof. (1) Let $x \in X$ and let W be an open set of Z containing $g(f(x))$. Since g is weakly continuous, there exists an open set V of Y containing $f(x)$ such that $g(V) \subseteq Cl(W)$. Hence $g^{-1}(Cl(W))$ is open subset in Y containing $f(x)$. Since f is weakly r-supercontinuous, there exists an r-open set U of X containing x such that $f(U) \subseteq (g)^{-1}(W))$. Therefore, we obtain $(g \circ f)(U) = g(f(U)) \subseteq Cl(W)$. Hence $g \circ f : X \to Z$ is weakly r-supercontinuous.

(2) Let $x \in X$ and W be an open subset of Z containing $g(f(x))$. Since g is θ-continuous, there exists an open set V of Y containing $f(x)$ such that $g(Cl(V)) \subseteq Cl(W)$. Since f is weakly r-supercontinuous, then there exists an r-open set U of X containing x such that $f(U) \subseteq Cl(V)$. Hence $g(f(U)) \subseteq g(Cl(V)) \subseteq Cl(W)$. Therefore, $g \circ f$ is weakly r-supercontinuous.

(3) Let $x \in X$ and A be an open set of Z containing $g(f(x))$. Since g is continuous, then $g^{-1}(A)$ is an open set of Y containing $f(x)$. But, f is weakly r-supercontinuous, then there exists an r-open set B of X containing x such that $f(B) \subseteq Cl(g^{-1}(A))$. Also, since g is continuous, then we obtain $(g \circ f)(B) \subseteq g(Cl(g^{-1}(A))) \subseteq Cl(A)$. Therefore, $g \circ f$ is weakly r-supercontinuous.

(4) Let $x \in X$ and W be an open subset of Z containing $g(f(x))$. Since g is weakly r-supercontinuous, there exists an r-open set U of Y containing $f(x)$ such that $g(U) \subseteq Cl(W)$. It is clear that $g^{-1}(Cl(W))$ is an r-open set of Y containing $f(x)$. Since f is continuous and open, then $f^{-1}(g^{-1}\ (Cl(W))) = (g \circ f)^{-1}(Cl(W))$ is an r-open set in X containing x and clearly $(g \circ f)(g \circ f)^{-1}(Cl(W)) \subseteq Cl(W)$. Hence $(g \circ f)$ is weakly r-supercontinuous. □

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