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Existence and Ulam stability of solutions for Caputo-Hadamard fractional differential equations

Abduljawad K. Anwar[a], Shayma Adil Murad[a,*]

[a] *Department of Mathematics, College of Science, University of Duhok, Duhok 42001, IRAQ.*

Abstract

In this paper, we study the existence of solutions for fractional differential equations with the Caputo-Hadamard fractional derivative of order $\alpha \in (1, 2]$. The uniqueness result is proved via Banach's contraction mapping principle and the existence results are established by using the Schauder's fixed point theorem. Furthermore, the Ulam-Hyers and Ulam-Hyers-Rassias stability of the proposed equation is employed. Some examples are given to illustrate the results.

Keywords: Fractional differential equation, Caputo-Hadamard fractional derivative, Fixed point theorems, Hyers-Ulam stability.
2010 MSC: 26A33, 34A08, 34A12, 34B27, 34D20, 34K20.

1. Introduction

Fractional differential equations are considered as active field of research due to their applications in various areas including dynamics, computer science, and biological sciences. For instance see ([19, 24, 31, 32, 34, 37]). The existence theory is one of the most important topics in fractional calculus. Researchers have obtained many results about the existence and uniqueness solutions for the initial and boundary value problems of fractional differential equations in the sense of Riemann-Liouville and Caputo fractional derivatives, see ([1, 3, 4, 5, 6, 7, 14, 16, 17, 23, 26, 27, 28, 29, 38]).
Recently, several scientists have been interested in Hadamard-type fractional differential equations. The Hadamard fractional derivative is a specific type of fractional derivative assigned to Hadamard in 1892 [11]. This fractional derivative differs from the Riemann-Liouville and Caputo fractional derivatives in the sense that the kernel of the integral contains a logarithmic function of arbitrary exponent. The existence and uniqueness of mild solutions of boundary value problem for Caputo-Hadamard fractional differential equations with integral and anti-periodic conditions investigated by [20].
Rezapour et al. [35] investigated the existence results for solutions of a new class of the fractional boundary value problem in the Caputo–Hadamard settings. Abbas et al. [2] proved some existence results for a class of Caputo–Hadamard fractional differential equations, the results are based on Mönch's fixed point theorem associated with the technique of measure of non-compactness.

*Corresponding author
Email address: `shaymaadil@uod.ac` (Shayma Adil Murad)

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Liu et al. [22] dealt with the existence of solutions of the boundary value problems for nonlinear fractional differential equations.

$$\begin{cases} {}^{C}D^{\alpha}_{-1}\, D^{\beta}_{0+}y(t) + f(t, y(t)) = b, & 0 < t < 1, \\ y(0) = 0, \quad y'(1) = D^{\beta}_{0+}y(1) = 0, \end{cases}$$

where ${}^{C}D^{\alpha}_{-1}$ denote the Caputo fractional derivatives of orders α, with $0 < \alpha < 1$, D^{β}_{0+} is the Riemann-Liouville fractional derivative with $1 < \beta \leqslant 2$ and $\alpha + \beta > 2$, and $b > 0$ is a constant real number.
Wang et al. [39] employed the upper and lower solution method along with the fixed point theorem (FPT) of a cone to investigate the existence and uniqueness of a positive solution for

$$\begin{cases} D^{\alpha}_{0+}y(t) + f(t, y(t)) = 0, & 0 < t < 1, \\ y(0) = 0, \quad y'(1) = \int_0^1 y(s)ds, & \alpha \in (1, 2]. \end{cases}$$

Recently, more researchers are interested in applying the Ulam-Hyers stability see ([8, 9, 10, 15, 40]).
Murad et al. [30] studied the existence, Ulam-Hyers and Ulam-Hyers-Rassias theorems of solutions to a differential equation of mixed Caputo-Riemann fractional derivatives.
Muniyappan and Rajan [25] discussed Ulam-Hyers and Ulam-Hyers-Rassias stability for the fractional differential equation with boundary condition

$$\begin{aligned} &D^{\alpha}y(t) = f(t, y(t)), \quad 0 < \alpha \leqslant 1, \\ &ay(0) + by(T) = c, \end{aligned}$$

where D^{α} is Caputo fractional derivative of order α. Liu et al. [21] researched the stability of generalized Liouville–Caputo fractional differential equations in Ulam-Hyers sense.
Patil et al. [33] concerned the existence and uniqueness of positive solutions to the fractional differential equation.

$${}^{C}D^{\alpha}_{0+}y(t) + f(t, y(t)) = 0, \qquad 0 < x < 1,$$

with nonlocal integral boundary conditions

$$\begin{cases} y(0) = y'(0) + g(y), \\ y(1) = \int_0^1 y(s)ds, \end{cases}$$

where $1 < \alpha \leqslant 2$, ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative of order α, $f : [0, 1] \times R^{+} \longrightarrow R^{+}$, and $g : C[0, 1] \longrightarrow R^{+}$.

Motivated by the above work and the researches going on in this direction, in this paper, we study existence and uniqueness of solution of Caputo-Hadamard fractional differential equation.

$${}^{CH}D^{\alpha}_{1+}y(t) = f(t, y(t)), \quad t \in J = [1, e], \tag{1.1}$$

with the boundary condition

$$\begin{cases} y(1) = y'(1), \\ y(e) = \int_1^e y(t)\frac{dt}{t}, \end{cases} \tag{1.2}$$

where ${}^{CH}D^{\alpha}_{1+}$ Caputo-Hadamard fractional derivative, with $1 < \alpha \leqslant 2$ and $f : [1, e] \times R \to R$ is continuous function. We apply Schauder and Banach-fixed-point theorems to obtain the existence and the uniqueness of solution for the (1.1)-(1.2) under certain hypotheses. Furthermore, some stability theorems such as Ulam-Hyers and Ulam-Hyers-Rassias stability are proved. A few examples are presented as an application to illustrate the main results.

2. Preliminaries

Let us introduce some definitions and lemmas that are basic and needed in various places of this work.

Definition 2.1. ([19]) (Riemann-Liouville fractional derivative). Let $f : (0,\infty) \to R$ be a continuous function. Then the Riemann-Liouville fractional derivative of order $\alpha > 0, n = [\alpha]+1$, ($[\alpha]$ denotes the integer part of the real number α) defined as

$$^{RL}D^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_0^x (x-\tau)^{n-\alpha-1}f(\tau)d\tau,$$

Where $n-1 < \alpha < n$.

Definition 2.2. ([19]) (Caputo fractional derivative). Let $f : (0,\infty) \to R$ be a continuous function. Then Caputo fractional derivative of order $\alpha > 0, n = [\alpha]+1$, defined as

$$^{C}D^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,$$

Where $n-1 < \alpha < n$.

Definition 2.3. ([19]) The Hadamard fractional integral of order $\alpha \in R$ for a continuous function f is defined as

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_{a+}^x \left(\log(\frac{x}{\tau})\right)^{\alpha-1}\frac{f(\tau)}{\tau}d\tau,$$

Definition 2.4. ([19]) The Hadamard derivative of fractional order $\alpha \in R$ for a continuous function f is defined as

$$D^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(n-\alpha)}(x\frac{d}{dx})^{n}\int_{a+}^x \left(\log\frac{x}{\tau}\right)^{n-\alpha-1}\frac{f(\tau)}{\tau}d\tau,$$

where $n-1 < \alpha < n$,$n = [a]+1$ where $[\alpha]$ denotes the integer part of the real number α

Lemma 2.5. *[18] Let $\alpha > 0$, and $n = [\alpha]+1$. If $y \in AC^n_\delta[a,b]$, then the differential equation $^{CH}D^{\alpha}y(t) = 0$, has solutions*

$$h(t) = \sum_{k=0}^{n-1} c_k(\log\frac{t}{a})^k,$$

and the following formula holds: $I^{\alpha}_{a+}\ {}^{CH}D^{\alpha}_{a+}y(t) = y(t) - \sum_{k=0}^{n-1} c_k(\log\frac{t}{a})^k$,
where $c_k \in R$, $k = 1,2,...,n-1$.

Definition 2.6. [36] The equation (1.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C^1(J,R)$ of the inequality

$$|^{CH}D^{\alpha}z(t) - f(t,z(t))| \leqslant \varepsilon, \quad t \in J, \tag{2.1}$$

there exists a solution $y \in C^1(J,R)$ of equation (1.1) with

$$|z(t) - y(t)| \leqslant c\varepsilon, \quad t \in J.$$

Definition 2.7. [36] The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C^1(J,R_+)$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C(J,R)$ of the inequality

$$|^{CH}D^{\alpha}z(t) - f(t,z(t))| \leqslant \varepsilon\varphi(t), \quad t \in J, \tag{2.2}$$

there exists a solution $y \in C^1(J,R)$ of equation (1.1) with

$$|z(t) - y(t)| \leqslant c\varepsilon\varphi(t), \quad t \in J. \tag{2.3}$$

Theorem 2.8. *[12] (Banach contraction mapping principle). Let H be a Banach space. If $Z : H \to H$ is a contraction, then Z has a unique fixed point in H.*

Theorem 2.9. *[12](Schauder Fixed Point theorem). Let H be closed, bounded and convex subset of Banach space X and the mapping $Z : H \to H$ is a continuous map such that the set $\{z_x : x \in H\}$, is relatively compact.Then Z has at least one fixed point.*

Lemma 2.10. *[13] (Gronwall Inequality). Let $u(t)$ and $f(t)$ be non-negative, continuous functions on $I = [0, \infty]$ for each inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds, \qquad t \in I, \tag{2.4}$$

holds, where u_0 is a non-negative constant. Then

$$u(t) \leqslant u_0\, e^{\int_0^t f(s)ds}, \quad t \in I. \tag{2.5}$$

Lemma 2.11. *For any $y(t) \in C(J, R)$, $1 < \alpha \leqslant 2$, then the boundary value problem (1.1)-(1.2) has a solution*

$$y(t) = 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s} - \alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, y(s))\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s}. \tag{2.6}$$

Proof. Applying Lemma 2.5, we can reduce the problem (1.1)-(1.2) to an equivalent integral equation

$$y(t) = c_0 + c_1 \log(t) + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s},$$

to find c_0 and c_1, from the first boundary condition $y(1) = y'(1)$, we get

$$y(t) = c_0(\log(t)+1) + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s},$$

by using the condition $y(e) = \int_1^e y(t)\frac{dt}{t}$, the result is

$$y(e) = 2c_0 + \int_1^e \frac{(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s},$$

and

$$\int_1^e y(t)\frac{dt}{t} = \frac{3}{2}c_0 + \int_1^e\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s}\frac{dt}{t},$$

by using Fubini's theorem, the following is obtained

$$\int_1^e y(t)\frac{dt}{t} = \frac{3}{2}c_0 + \int_1^e \frac{(\log\frac{e}{s})^{\alpha}}{\alpha\Gamma(\alpha)} f(s, y(s))\frac{ds}{s}.$$

Hence

$$c_0 = 2\int_1^e \frac{(\log\frac{e}{s} - \alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, y(s))\frac{ds}{s},$$

this implies that

$$y(t) = 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s} - \alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, y(s))\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s},$$

and this complete the poof. □

3. Existence and Uniqueness of Solutions

Let $C = C(J, R)$ denote the Banach space of all continuous functions from J to R, with the norm defined by

$$\|y\| = \sup\{|y(t)|, t \in J\}.$$

To prove the main results, we introduce the following assumption:
(**H1**) There exists constant $\mu > 0$, such that

$$\left|f(t, y(t))\right| \leqslant \mu\, |y(t)|.$$

(**H2**) There exists constant $k > 0$, such that

$$\left|f(t, x(t)) - f(t, y(t))\right| \leqslant k|x(t) - y(t)|.$$

First result is based on Banach contraction principle. For the sake of convenience, we set the notation:

$$\lambda = \frac{5\alpha + 9}{\Gamma(\alpha+2)}.$$

Theorem 3.1. *Assume that (**H2**) hold. If $\lambda k < 1$. Then the boundary value problem (1.1)-(1.2) has a unique solution on J.*

Proof. Define the operator $B : C(J, R) \to C(J, R)$ by

$$(By)(t) = 2(\log(t) + 1)\int_1^e \frac{(\log\frac{e}{s} - \alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, y(s))\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\frac{ds}{s}, \tag{3.1}$$

let's set $r \geqslant \dfrac{M\,\lambda}{(1-k\lambda)}$, and show that $BG_r \subset G_r$, when $G_r = \{y \in C(J, R) : \|y\| \leqslant r\}$, and $M = \sup|f(t, 0)|$, for $t \in G_r$, we have

$$|B(y)(t)| \leqslant 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s} - \alpha)|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s, y(s))|\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s, y(s))|\frac{ds}{s},$$

$$|B(y)(t)| \leqslant 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s} - \alpha)|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s, y(s)) - f(s,0)|\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s, 0)|\frac{ds}{s}$$
$$+ \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s, y(s)) - f(s,0)|\frac{ds}{s} + 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s} - \alpha)|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s, 0)|\frac{ds}{s},$$
$$\|B(y)(t)\| \leqslant \left[4\int_1^e \frac{|(\log\frac{e}{s} - \alpha)|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]\big(k\,r + M\big),$$
$$\|B(y)(t)\| = \lambda\big(k\,r + M\big) \leqslant r.$$

Therefore, $BG_r \subset G_r$. Now to that B is a contraction mapping, let $y_1, y_2 \in G_r$ and for each $t \in J$, we obtain

$$|B(y_1)(t) - B(y_2)(t)| \leqslant 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s}) - \alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s, y_1(s) - f(s, y_2(s))|\frac{ds}{s}$$
$$+ \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s, y_1(s)) - f(s, y_2(s))|\frac{ds}{s},$$
$$\leqslant \left[2(\log(e)+1)\int_1^e \frac{(\log\frac{e}{s} - \alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]k\|y_1(t) - y_2(t)\|,$$
$$\leqslant \left[4\left(\int_1^e \frac{|(\log\frac{e}{s})^{\alpha}|}{\Gamma(\alpha+1)}\frac{ds}{s} + \int_1^e \frac{\alpha(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}\right) + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]k\|y_1(t) - y_2(t)\|,$$
$$|B(y_1)(t) - B(y_2)(t)| \leqslant \lambda k\|y_1(t) - y_2(t)\|.$$

Therefore, it follows from condition $\lambda k \leqslant 1$, that B is a contraction operator. Thus we conclude by Banach contraction mapping principle that operator B has a unique fixed point, which is the unique solution to the problem (1.1) and (1.2). □

Next, the result is based on Schauder's fixed point theorem.

Theorem 3.2. *Assume that (**H1**)-(**H2**) hold. Then the boundary value problem (1.1)-(1.2) has at least one solution.*

Proof. : This proof will be presented in four steps.
Step 1: We will show that B is continuous.
Assume that $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, R)$, then for each $t \in J$ we have

$$\begin{aligned}
|B(y_n)(t) - B(y)(t)| &\leqslant 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s,y_n(s)-f(s,y(s))|\frac{ds}{s}\\
&+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s))-f(s,y(s))|\frac{ds}{s},\\
&\leqslant\left[2(\log(e)+1)\int_1^e \frac{|\log\frac{e}{s}-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]k\|y_n(t)-y(t)\|,\\
&\leqslant\left[4\left(\int_1^e \frac{|(\log\frac{e}{s})^{\alpha}|}{\Gamma(\alpha+1)}\frac{ds}{s}+\int_1^e \frac{\alpha(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}\right)+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right],\\
|B(y_n)(t) - B(y)(t)| &\leqslant \lambda k\ \|y_n(t)-y(t)\|.
\end{aligned}$$

Then according to Lebesgue dominated convergence theorem this implies that $\|B(y_n)(t) - B(y)(t)\|_\infty \to 0$ as $n \to \infty$.
Step2: B maps the bounded sets into the bounded sets in $C(J, R)$.
For any $d > 0$, such that $H_d = \{y \in C : \|y\|_\infty \leqslant d\}$. It is obvious that H_d is closed, convex subset of $C(J, R)$. Suppose that $y \in C$ then for each $t \in J$, we have

$$\begin{aligned}
|B(y)(t)| &\leqslant 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s,y(s))|\frac{ds}{s}+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|\frac{ds}{s},\\
|B(y)(t)| &\leqslant \left[2(\log(e)+1)\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]\mu\ \|y(t)\|,\\
|B(y)(t)| &\leqslant \left[4\int_1^e \frac{(\log\frac{e}{s})^{\alpha}}{\Gamma(\alpha+1)}\frac{ds}{s}+4\int_1^e \frac{\alpha(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}+\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{ds}{s}\right]\mu\ r,\\
\|B(y)(t)\|_\infty &\leqslant\ \lambda\ \mu\ r = L.
\end{aligned}$$

Thus, $\|B(y)(t)\|_\infty \leqslant L$, for some constant L.
Step 3: B maps $C(J, R)$ into an equicontinuous set of $C(J, R)$.
Let $y \in C(J, R)$ and $t_1, t_2 \in J$ with $t_1 < t_2$, then

$$\begin{aligned}
|B(y)(t_2) - B(y)(t_1)| &\leqslant \int_{t_1}^{t_2} \frac{(\log\frac{t_2}{s})^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))|\frac{ds}{s}+2\left|\log(\frac{t_2}{t_1})\right|\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}|f(s,y(s))|\frac{ds}{s}\\
&+\int_1^{t_1} \frac{|(\log\frac{t_2}{s})^{\alpha-1}-(\log\frac{t_1}{s})^{\alpha-1}|}{\Gamma(\alpha)}|f(s,y(s))|\frac{ds}{s},\\
&\leqslant\left[\frac{(\log\frac{t_2}{t_1})^{\alpha}}{\alpha\Gamma(\alpha)}+2\left|\log(\frac{t_2}{t_1})\right|\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)}\frac{ds}{s}+\int_1^{t_1} \frac{|(\log\frac{t_2}{s})^{\alpha-1}-(\log\frac{t_1}{s})^{\alpha-1}|}{\Gamma(\alpha)}\frac{ds}{s}\right]\mu,\\
|B(y)(t_2) - B(y)(t_1)| &\leqslant \left(\frac{(\log\frac{t_2}{t_1})^{\alpha}}{\alpha\Gamma(\alpha)}+2\left|\log(\frac{t_2}{t_1})\right|\frac{\alpha+2}{\Gamma(\alpha+2)}+\frac{|(\log\frac{t_2}{t_1})^{\alpha}-(\log(t_2))^{\alpha}+(\log(t_1))^{\alpha}|}{\Gamma(\alpha+1)}\right)\mu,
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.
Step 4: Now, we should show that B is prior bound.
Let $u = \{y \in C(J,R) : y = \varphi By, \text{ for some } 0 < \varphi < 1\}$, we need to show that the set u is bounded.
Let $y \in u$ and for each $t \in J$, we have

$$\begin{aligned}|B(y)(t)| \leqslant & \varphi|2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s,y(s))\frac{ds}{s}| + \varphi|\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\frac{ds}{s}|,\\ & \leqslant 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} |f(s,y(s))|\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |f(s,y(s))|\frac{ds}{s},\\ ||B(y)(t)|| \leqslant & \lambda\, \mu,\end{aligned}$$

which implies that u is a bounded set. By Schauder's fixed point theorem, B must have at least one fixed point which is a solution of (1.1) and (1.2). □

4. Stability Theorems

In the following theorems, we will prove the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for equation(1.1) on the interval $J = [1, e]$.

Theorem 4.1. *If the assumptions (**H2**) hold. Then the boundary value problem (1.1)-(1.2) is Ulam-Hyers stable.*

Proof. For $\varepsilon > 0$, and each solution $w \in C(J,R)$ of the inequality

$$|^{CH}D^{\alpha}w(t) - f(t,w(t))| \leqslant \varepsilon, \quad t \in J,$$

Let $y \in C(J,R)$ be the unique solution of boundary value problem (1.1)-(1.2). Then $y(t)$ is given by

$$y(t) = 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s,y(s))\frac{ds}{s} + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))\frac{ds}{s}.$$

Then, we have

$$\begin{aligned}\Big|w(t)-y(t)\Big| \leqslant & \Big|w(t) - 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s,w(s))\frac{ds}{s} - \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s,w(s))\frac{ds}{s}\Big|\\ & + 2(\log(t)+1)\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} |f(s,w(s)) - f(s,y(s))|\frac{ds}{s}\\ & + \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |f(s,w(s)) - f(s,y(s))|\frac{ds}{s},\end{aligned}$$

and by using (**H2**), the result is

$$\leqslant \frac{\varepsilon(\log(t))^{\alpha}}{\Gamma(\alpha+1)} + 4k\int_1^e \frac{|(\log\frac{e}{s})-\alpha|(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} |w(s)-y(s)|\frac{ds}{s} + k\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |w(s)-y(s)|\frac{ds}{s},$$

where

$$|w(t) - 2(\log(t)+1)\int_1^e \frac{(\log\frac{e}{s}-\alpha)(\log\frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s,w(s))\frac{ds}{s} - \int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s,w(s))\frac{ds}{s}| \leqslant \frac{\varepsilon(\log(t))^{\alpha}}{\Gamma(\alpha+1)}.$$

Then, we get

$$\begin{aligned}|w(t)-y(t)| &\leqslant \frac{\varepsilon(\log(t))^{\alpha}}{\Gamma(\alpha+1)} + \frac{4k(\alpha+2)}{\Gamma(\alpha+2)}|w(t)-y(t)| + k\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |w(s)-y(s)|\frac{ds}{s},\\ |w(t)-y(t)| &\leqslant \frac{\varepsilon(\log(t))^{\alpha}}{(1-\mu_1)\,\Gamma(\alpha+1)} + D\int_1^t \frac{(\log\frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |w(s)-y(s)|\frac{ds}{s},\end{aligned}$$

where $D = \frac{k}{(1-\mu_1)}$ and $\mu_1 = \frac{4k(\alpha+2)}{\Gamma(\alpha+2)}$, by using Lemma (2.10), it becomes

$$|w(t) - y(t)| \leqslant \frac{\varepsilon(\log(t))^{\alpha}}{(1-\mu_1)\ \Gamma(\alpha+1)} e^{\frac{D}{\Gamma(\alpha+1)}},$$

set $c = \frac{e^{\frac{D}{\Gamma(\alpha+1)}}}{(1-\mu_1)\Gamma(\alpha+1)}$.
The inequality

$$|w(t) - y(t)| \leqslant c\varepsilon.$$

Hold, then the boundary value problem (1.1)-(1.2) is Ulam-Hyers stable. □

Theorem 4.2. *Assume (**H2**) and*
(**H4**) *The function $\phi \in C(J, R_+)$ is increasing and there exist $\Lambda_\phi > 0$ such that, for each $t \in J = [1, e]$ we have*

$$\int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds \leqslant \Lambda_\phi \phi(t),$$

hold, then the fractional differential equation (1.1) with the boundary condition (1.2) is Ulam-Hyers-Rassias stable with respect to ϕ on $[1, e]$.

Proof. Let $w \in C(J, R)$ be a solution of the inequality,

$$|{}^{CH}D^{\alpha} w(t) - f(t, w(t))| \leqslant \varepsilon\phi(t), \quad t \in J. \tag{4.1}$$

Denote $y \in C(J, R)$ be the unique solution of the boundary value problem (1.1)-(1.2), that is

$$y(t) = 2(\log(t) + 1) \int_1^e \frac{(\log \frac{e}{s} - \alpha)(\log \frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, y(s)) \frac{ds}{s} + \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) \frac{ds}{s}.$$

On the other hand, for each $t \in J$, we have

$$\begin{aligned}
|w(t) - y(t)| &\leqslant \Big|w(t) - 2(\log(t) + 1) \int_1^e \frac{(\log \frac{e}{s} - \alpha)(\log \frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, w(s)) \frac{ds}{s} - \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, w(s)) \frac{ds}{s}\Big| \\
&\quad + 2(\log(t) + 1) \int_1^e \frac{|(\log \frac{e}{s}) - \alpha|(\log \frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} |f(s, w(s)) - f(s, y(s))| \frac{ds}{s} \\
&\quad + \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |f(s, w(s)) - f(s, y(s))| \frac{ds}{s}, \\
&\leqslant \varepsilon\Lambda_\phi \phi(t) + 4k \int_1^e \frac{(\log \frac{e}{s} - \alpha)(\log \frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} |w(s) - y(s)| \frac{ds}{s} + k \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |w(s) - y(s)| \frac{ds}{s},
\end{aligned}$$

where

$$\left| w(t) - 2(\log(t) + 1) \int_1^e \frac{(\log \frac{e}{s} - \alpha)(\log \frac{e}{s})^{\alpha-1}}{\Gamma(\alpha+1)} f(s, w(s)) \frac{ds}{s} - \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} f(s, w(s)) \frac{ds}{s} \right| \leqslant \varepsilon\ \Lambda_\phi \phi(t),$$

which yields that

$$|w(t) - y(t)| \leqslant \frac{\varepsilon\Lambda_\phi \phi(t)}{(1-\mu_1)} + D \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} |w(s) - y(s)| \frac{ds}{s},$$

where $D = \frac{k}{(1-\mu_1)}$ and $\mu_1 = \frac{4k(\alpha+2)}{\Gamma(\alpha+2)}$, by using Gronwall Lemma(2.10), the result is

$$|w(t) - y(t)| \leqslant \frac{\varepsilon\Lambda_\phi \phi(t)}{(1-\mu_1)}\ e^{D \int_1^t \frac{(\log \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s}}.$$

Set $c = \frac{\Lambda_\phi \, e^{\frac{D}{\Gamma(\alpha+1)}}}{\left(1-\mu_1\right)}$.
The inequality

$$|w(t) - y(t)| \leqslant \varepsilon\phi(t).$$

Hold, then the boundary value problem (1.1)-(1.2) is Ulam-Hyers-Rassias stable. □

5. An Examples

This paper involves an examples to illustrate our main result.

Example 5.1. Consider the boundary value problem

$$\begin{aligned} D^{\frac{5}{3}}y(t) =& e^{-t} + \frac{y}{20+\cos(t)}. \\ y(1) =& y'(1), \qquad y(e) = \int_1^e y(s)\frac{ds}{s}, \end{aligned} \tag{5.1}$$

where $\alpha = \frac{5}{3}$, and $f(t, y(t)) = e^{-t} + \frac{y}{20+\cos(t)}$, by using (H_2) the result is obtained

$$|f(t, y_1(t)) - f(t, y_2))| \leqslant \frac{1}{20+\cos(t)}|y_1 - y_2|.$$

Since $k = 0.205$, from Theorem (3.1), we have $\lambda k = 0.60353384 < 1$, then the problem (5.1) has a unique solution on $[1, e]$.

Example 5.2. Consider the following problem of Caputo–Hadamard fractional differential equation

$$D^{\frac{7}{4}}y(t) = \frac{y+\log(t)}{17+e^{3t}}, \tag{5.2}$$

with the boundary conditions

$$\begin{cases} y(1) = y'(1), \\ y(e) = \int_1^e y(s)\frac{ds}{s}, \end{cases} \tag{5.3}$$

where $\alpha = \frac{7}{4}$, and $f(t, y) = \frac{y+\log(t)}{17+e^{3t}}$ for all $t \in [1, e]$, by using Lipshitz condition, we have

$$|f(t, y_1(t)) - f(t, y_2(t))| \leqslant \frac{1}{18}|y_1 - y_2|,$$

$k = \frac{1}{18}$, then $\lambda k = 0.297584754 < 1$. from Theorem (3.1), the problem (5.2)-(5.3) has a unique solution on $[1, e]$.

6. Conclusion

In this paper, we studied existence and uniqueness of solutions of Caputo-Hadamard fractional differential equations with boundary conditions. Our results are based on some classical fixed point theorems such as Banach contraction mapping principle and Schauder fixed point theorems. At last, we have presented two examples for the illustration of main results.

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