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A Stochastic Maximum Principle for a Minimization Problem Under Partial Information

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Abstract

In this paper, we establish a stochastic maximum principle for a stochastic minimization problem under partial information. With the Backward stochastic differential equations (in short BSDE's), we establish a sufficient condition of optimality to characterize and determine an optimal control. This is done instead of using the Hamiltonian which is a deterministic function. The equations translating the dynamics of the state variables of the controlled system contain an BSPDE (Backward stochastic partial differential equation) which can be the unnormalized conditional density like the Zakai equation born from a problem of passage from partial to full information.

Keywords: Stochastic maximum principle, Backward stochastic differential equation, Stochastic partial differential equation, Hamiltonian, partial information, minimization problem, Zakai equation.
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1. Introduction

Control theory is now applied in various fields: Optimization of air traffic ([4, 18, 5]), railway, road, transport, optimization of radar coverage, response responsiveness, inventory management, troops, equipment, distances in the military field, and many others (See [17, 21]). Coupled with filtering theory, it is applied to fields such as: Radio astronomy ([17]), passive tracking ([17]), estimation of the position of a satellite ([18]), signal filtering, sound or image, putting and keeping a satellite in orbit ([18]), guiding spacecraft (rockets, planes, drones etc. [18, 21, 5, 22]), portfolio management in finance ([2, 9, 13, 14, 19]) and many more ([17]).
One of the fundamental problems in control theory ([14, 1, 9, 7, 8, 3, 20]) is the determination of the optimal control. This problem is generally solved by the minimization ([13, 12]) or the maximization ([9, 16, 2, 14, 6]) of an associated function called criterion, objective function or cost function. The goal of control theory is to drive a system from one state to another or to stabilize it. In stochastic control theory and under partial information ([2, 1, 9, 8, 14]), we face systems evolving under conditions of uncertainty and we must control under the basis of perceived information. For example driving and keeping an

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airplane ([4, 5, 21]), a rocket ([18]) on its trajectory, a satellite in orbit based on perceived information such as gas ejection rate, wind speed, atmospheric conditions which are random variables or in fiance ([14, 2, 15, 9, 12]) with portfolio management on the basis of certain market data such as the price of risky assets or interest rates and many more. Filtering theory ([1]) aims to estimate a state from observations. Nowadays, both control and filtering theory are applied in many fields ([22, 21, 17, 4]) such as: Industry, signal processing, aeronautics, finance, aerospace, classical mechanics and many more. In control theory, the functions to be optimized (integrals, expectations) are not generally sufficiently regular functions as in classical optimization. As a result, new instruments, techniques and methods are needed.
One of the basic methods established was Pontryagin's maximum principle applied first to deterministic control and later to stochastic control ([14, 15, 19]). The basic idea of this method is to bring a control problem (deterministic or stochastic) to a classical optimization problem.
When coupling a problem of control and filtering simultaneously, the principles of stochastic maximum generally established ([15]) do not make it possible to solve the problem. Indeed, usually ([14, 2, 19]) in the dynamics of the state variables of the controlled system, we obtain SPDEs (Stochastic partial differential equations) with a combined objective function of a mathematical expectation and one or more integrals ([14]). Mainly in the case of control theory under partial information ([14, 1, 9]), the unnormalized conditional density ([3, 8, 1]) of the unobservable variable that must be added to the dynamics of the variables of states of the controlled system is rather the solution of stochastic partial differential equations of the Zakai, Kuschner-Shatonovitch or Bellman type.
The authors in ([25, 24, 27, 26, 23]) proceeded to the homogenization of the stochastic PDEs and had for intention the application to the stochastic control.
In this paper, we propose to establish a stochastic maximum principle as a sufficient condition of optimality of a stochastic optimization problem with an objective function that depends on the variables whose dynamics of two of them are governed by SDE (Stochastic Differential Equations) and the dynamics of one governed by an SPDE (Stochastic Partial Differential Equation). The authors in [14, 15, 1] considered this problem but with only one observable variable, the SDE or a maximization problem. We use the SDE, the SPDE and the BSDE with the Hamiltonian and the adjoint equations which are BSDE(Backward stochastic differential equations) to establish a sufficient optimality condition for the associated minimization problem. The following section 2 presents more of the motivations, the model, the problem and the basic elements. Section 3 is devoted to the results.

2. Motivations, model, problem

When coupling a control and filtering problem, it is necessary to add to the state variables of the controlled system a conditional density whose dynamics are rather governed by a SPDE. Unfortunately, the majority of established stochastic maximum principles are made in the framework of deterministic control. In the framework of stochastic control under partial information ([14, 9, 15, 19, 8, 10]) where we couple a control and filtering problem with n observable variables $Y^1, Y^2 ... Y^n$ and p unobservable variables $Z^1, Z^2 ... Z^p$ with the dynamics:

$$dY_t = h(t, Z_t, Y_t)dt + \sigma(t, Y_t)dW_t, \quad Y_0 = y, \tag{2.1}$$

$$dZ_t = g(t, Z_t, Y_t)dt + \alpha(t, Z_t, Y_t)dW_t + \gamma(t, Z_t, Y_t)dW_t^{\perp}, \quad Z_0 = \varepsilon \tag{2.2}$$

where W and $W^{\perp}$ are two given independent Brownian movements respectively p and $q-$ dimensional. Assuming:

Assumption 2.1. .

i) $h(t, z, y) : [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is globally continuous and of linear growth (in z and y: $|h(t, z, y)| \leqslant k(1+ \mid z \mid)+ \mid y \mid)$.

ii) $g(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is uniformly continuous in z, y is bounded and twice continuously differentiable with bounded derivatives.

iii)$\sigma(t,y) : [0,T] \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ is uniformly continuous, bounded, three times continuously differentiable with bounded derivative, satisfies the following: $\sigma\sigma^t \geqslant \lambda I$ [1] for all y and t, for some constant $\lambda > 0$ (uniform ellipticity condition).

iv) $\alpha(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$, $\gamma(t,z,y) : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^q)$ are uniformly continuous and α is uniformly elliptic.

v) h, σ, g and γ are globally lipschitz of y and z.

Let $D_t = D(t, Y_t) = (\sigma\sigma^t)(t, Y_t)$ which we assume to be symmetrical and invertible. φ_t defined by :

$$d\varphi_t = -\varphi_t h^t(t, Z_t, Y_t) D_t^{-\frac{1}{2}}(t, Y_t) dW_t, \quad \varphi_0 = 1^{2}$$

φ is a supermartingale with $\mathbb{E}[\varphi_t] = 1 \quad \forall t \in [0,T]$. Therefore by Girsanov's theorem, we define a new probability measure $\tilde{\mathbb{P}}$ such that $\forall t \in [0,T]$

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathfrak{F}_t} := \varphi_t \quad (\; d\tilde{\mathbb{P}} = \varphi_t d\mathbb{P} \quad \text{on} \quad \mathfrak{F}_t, \forall t \in [0,T]).$$

and there exists a Brownian motion $\tilde{W}$ under $\tilde{\mathbb{P}}$ such that: $dY_t = \sigma(t, Y_t) d\tilde{W}_t$

$$dZ_t = \left[g(t, Z_t, Y_t) - \alpha^t(t, Z_t, Y_t) h^t(t, Z_t, Y_t) D_t^{-\frac{1}{2}} \right] dt + +\alpha^t(t, Z_t, Y_t) D_t^{-\frac{1}{2}} dY_t + \gamma(t, Z_t, Y_t) dW_t^{\perp}.$$

; Let $(\tilde{Y}_t, t \in [0,T])$ be the process defined by:

$$d\tilde{Y}_t = D_t^{-\frac{1}{2}} dY_t. \tag{2.3}$$

Then $\tilde{Y}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and $\tilde{Y}$ and $W^{\perp}$ are two independent Brownian motions(see[1]). In addition, using the fact that D_t is invertible, we have : $\mathfrak{F}_t^Y = \mathfrak{F}_t^{\tilde{Y}}$.
Let us set

$$\begin{aligned} K_t =: \frac{1}{\rho_t} &= \exp\left\{ \int_0^t h^t(s, Z_s, Y_s) D_s^{-\frac{1}{2}} dW_s + \frac{1}{2} \int_0^t h^t(s, Z_s, Y_s D_s^{-1} h(s, Z_s, Y_s) d_s \right\} \\ &= \exp\left\{ \int_0^t h^t(s, Z_s, Y_s) D_s^{-1} dY_s + \frac{1}{2} \int_0^t h^t(s, Z_s, Y_s) D_s^{-1} h(s, Z_s, Y_s) d_s \right\} \end{aligned}$$

Then K_t is a martingale.
Let $\phi = (\phi(t,z,w), (t,z,w) \in [0,T] \times \mathbb{R}^d \times \Omega)$, be a process such that for all $f \in C_0^\infty(\mathbb{R}^d)$ [3]. We have:

$$\tilde{E}\left[f(Z_t) K_t | \mathfrak{F}_t^Y \right] = \int_{\mathbb{R}^d} f(z) \phi(t,z) dz, \tag{2.4}$$

$\tilde{E}$ its expectation under $\tilde{\mathbb{P}}$. Then $\phi(t,z)$ is called the unormalized conditional density of Z_t given $\mathfrak{F}_t^Y$ Applying the theorem of Îto to $(K_t f(Z_t))$and using the integration by parts, ϕ satisfy a backward stochastic

[1] σ^t denote the transposed of σ
[2] h^t the transposed vector of h
[3] $C_0^\infty(\mathbb{R}^d)$ = space of functions C^∞ on $\mathbb{R}^d$ with compact support

partial differential equation precisely the following Zakai equation (see [16], [9], [14], [1]):

$$\begin{aligned} d\phi(t,z) =& \left[\frac{1}{2}\sum_{i,j}\sum \frac{\partial^2}{\partial z_i \partial z_j}\left\{\left[\gamma\gamma^t+\alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} - \sum_i \frac{\partial g_i \phi(t,z)}{\partial z_i}\right] dt + \\ & \left[h - \sum_i \frac{\partial}{\partial z_i}(\alpha^i \phi(t,z))\right] d\tilde{Y}_t \\ =& L_Z^* \phi(t,z)dt + M^*\phi(t,z)d\tilde{Y}_t, \end{aligned} \tag{2.5}$$

With initial condition $\phi(0,z) = \xi(z)$ where $\xi(z)$ is the density of Z_0

$$\begin{aligned} L_Z^*\phi(t,z) &= \frac{1}{2}\sum_{i,j}\sum \frac{\partial^2}{\partial z_i \partial z_j}\left\{\left[\gamma\gamma^t+\alpha\alpha^t\right]_{i,j}\phi(t,z)\right\} - \sum_i \frac{\partial\left(g_i\phi(t,z)\right)}{\partial z_i} \\ &= \mathfrak{L}\phi(t,z) + i\phi(t,z) \\ M^*\phi(t,z) &= h - \sum_i \frac{\partial}{\partial z_i}(\alpha^i\phi(t,z)) \end{aligned}$$

$\phi(t,z)$ is called the unnormalized conditional density of Z.

In this work, we establish a stochastic maximum principle where the dynamics of one of the state variables of the controlled system is governed by an SPDE(Zakai Equation for example). Unlike in [14, 11, 1, 15], we will use here two observable variables and a variable whose dynamics are given by a stochastic partial differential equation.

Let T be a fixed exercise date, $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{0\in[0,T]}, \mathbb{P})$ a filtered probability space on which we have two Brownian motions W^1 and W^2. We consider the controlled diffusion below which describes the dynamics of the various state processes:

$$\begin{cases} dY_t^1 = b_1(t,Y_t^1,u_t)dt + \sigma_{11}(t,Y_t^1,u_t)dW_t^1 + \sigma_{12}(t,Y_t^1,u_t)dW_t^2, & Y_0^1 = y_0^1 \\ dY_t^2 = b_2(t,Y_t^2,u_t)dt + \sigma_{21}(t,Y_t^2,u_t)dW_t^1 + \sigma_{22}(t,Y_t^2,u_t)dW_t^2, & Y_0^2 = y_0^2 \\ dX_t = b_3(t,X_t,Y_t^1,Y_t^2,u_t)dt + \sigma_{31}(t,X_t,u_t)dW_t^1 + \sigma_{32}(t,X_t,u_t)dW_t^2, & X_0 = x \\ d\phi(t,z) = [L\phi(t,z) + b_4(t,z,\phi(t,z),\frac{\partial\phi}{\partial z}(t,z),Y_t^1,Y_t^2,u_t)]dt & \\ \quad + \sigma_{41}(t,z,\phi(t,z),\frac{\partial\phi}{\partial z}(t,z),Y_t^1,Y_t^2,u_t)dW_t^1 & \phi(0,z)=\xi(z),\ z\in\mathbb{R}. \\ \quad + \sigma_{42}(t,z,\phi(t,z),\frac{\partial\phi}{\partial z}(t,z),Y_t^1,Y_t^2,u_t)dW_t^2, & \\ \quad \lim_{\|z\|\to+\infty} \phi(t,z) = 0 \ \ \forall t\in[0,T] & \end{cases} \tag{2.6}$$

where L is a linear differential operator. $b_1, b_2, b_3, b_4, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \sigma_{31}, \sigma_{32}, \sigma_{41}, \sigma_{42}$ the given functions satisfying the conditions of existence and uniqueness of strong solutions of the above system, and L^* the formal adjoint of L.

Let f and g be given functions C^1 in their arguments. We consider the objective function:

$$\begin{aligned} J(u) =& \mathbb{E}\left[\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t)dzd\mathbb{P}_{\bar{B}}dt \right. \\ & \left. + \int_{\mathbb{R}}\int_{\mathbb{R}} g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u_T)dzd\mathbb{P}_{\mathbb{B}}\right] \end{aligned} \tag{2.7}$$

We note U_{ad} the set of admissible controls contained in the set of controls $\mathfrak{F}_t-$ predictable such that the above system has a single strong solution and

$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} \left|f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t)\right| dzd\mathbb{P}_{\bar{B}}dt + \int_{\mathbb{R}}\int_{\mathbb{R}}\left|g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u_T)\right| dzd\mathbb{P}_{\mathbb{B}}\right] < \infty.$

Problem 2.1. Determine the value function

$$J(\hat{u}) = \inf_{u \in U_{ad}} J(u) \tag{2.8}$$

under conditions (2.6).
That is to say to seek the optimal control $\hat{u} \in U_{ad}$ which minimizes the objective function J.

3. Stochastic maximum principle for minimization problem

For the control problem (2.6), (2.7), (2.8) we define the Hamiltonian:

$$H : [0,T] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$$

$H(t,z,y_1,y_2,x,\phi,\phi',u,p_1,p_2,p_3,p_4,q_1,q_2,q_3,q_4,q_1',q_2',q_3',q_4') = H(t,z,y,x,\phi,\phi',u,p,q,q')$
. with $y = (y_1,y_2), p = (p_1,p_2,p_3,p_4), q = (q_1,q_2,q_3,q_4), q' = (q_1',q_2',q_3',q_4')$.

$$\begin{aligned} H(t,z,y_1,y_2,x,\phi,\phi',u,p,q,q') = &\int_{\mathbb{R}} f(t,z,y_1,y_2,x,\phi,\bar{b},u)d\mathbb{P}_{\bar{B}} + b_1(t,y_1,u)p_1 + b_2(t,y_2,u)p_2 \\ &+ b_3(t,y_1,y_2,x,u)p_3 + b_4(t,z,\phi,\phi',y_1,y_2,u)p_4 + \sigma_{11}(t,y_1,u)q_1 \\ &+ \sigma_{21}(t,y_2,u)q_2 + \sigma_{31}(t,x,u)q_3 + \sigma_{41}(t,z,\phi,\phi',y_1,y_2,u)q_4 + \sigma_{12}(t,y_1,u)q_1' \\ &+ \sigma_{22}(t,y_2,u)q_2' + \sigma_{32}(t,x,u)q_3' + \sigma_{42}(t,z,\phi,\phi',y_1,y_2,u)q_4' \end{aligned} \tag{3.1}$$

Where $\phi' = \frac{\partial \phi}{\partial z}, \phi'' = \frac{\partial^2 \phi}{\partial^2 z}$.We suppose that H is differentiable in the variables y_1, y_2, x, ϕ and ϕ'.
For $u \in U_{ad}$, we consider the adjoint processes satisfying the backward stochastic differential equations in the unknown $p_1(t,z), q_1(t,z), q_1'(t,z), p_2(t,z), q_2(t,z), q_2'(t,z), p_3(t,z), q_3(t,z), q_3'(t,z)$, $p_4(t,z), q_4(t,z), q_4'(t,z) \in \mathbb{R}$ with the system of adjoint equations:

$$\begin{cases} -dp_1 = \frac{\partial H}{\partial y_1}(t,z)dt - q_1 dW_t^1 - q_1' dW_t^2, & p_1(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_2 = \frac{\partial H}{\partial y_2}(t,z)dt - q_2 dW_t^1 - q_2' dW_t^2, & p_2(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial y_2}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_3 = \frac{\partial H}{\partial x}(t,z)dt - q_3 dW_t^1 - q_3' dW_t^2, & p_3(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial x}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ -dp_4 = \left[\frac{\partial H}{\partial \phi}(t,z) + L^* p_4(t,z) - \frac{\partial}{\partial z}\left(\frac{\partial H(t,z)}{\partial \phi'}\right)\right] dt - q_4 dW_t^1 - q_4' dW_t^2, & p_4(T,z) = \int_{\mathbb{R}} \frac{\partial g}{\partial \phi}(z,\bar{b})d\mathbb{P}_{\bar{B}} \\ \quad , \lim_{\|x\| \to +\infty} p_4(t,z) = 0 \end{cases} \tag{3.2}$$

With short notations:$g(z,\bar{b}) = g(z, Y_T^1, Y_T^2, X_T, \phi(T,z), \bar{b}, u_T)$ and
$H(t,z) = H(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\phi'(t,z),u_t,p(t,z),q(t,z),q'(t,z))$

Remark 3.1. .
If we suppose for example that the coefficients of the controlled diffusion, and our functions are fairly regular then there is existence and uniqueness of classical strong solutions of our backward stochastic differential equations as well as of the backward stochastic partial differential equation constituting the process of associated adjunct equations.

We have the following theorem:

Theorem 3.2. .
Let $\hat{u} \in U_{ad}$ with the corresponding solutions $\hat{Y}_t^1, \hat{Y}_t^2, \hat{X}_t, \hat{\phi}(t,z)$ of(2.6); $\hat{p} = (\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4), \hat{q} = (\hat{q}_1,\hat{q}_2,\hat{q}_3,\hat{q}_4), \hat{q}' = (\hat{q}_1',\hat{q}_2',\hat{q}_3',\hat{q}_4')$ of (2.8) and (3.2). Let the following conditions be satisfied:

1. *The function $g : (y_1,y_2,x,\phi,u) \longmapsto g(z,y_1,y_2,x,\phi,\bar{b},u)$ is convex in y_1, y_2, x and ϕ for all $z \in \mathbb{R}, \bar{b} \in \mathbb{R}, u \in U_{ad}$*
2. $H(t,z,y_1,y_2,x,\hat{u},\phi,\phi',\hat{p},\hat{q},\hat{q}') = \min_{u \in U_{ad}} H(t,z,y_1,y_2,x,phi,\phi',u,\hat{p},\hat{q},\hat{q}') \quad \forall y_1,y_2,x,\phi,\phi'$

3. *The function* $h : (y_1, y_2, x, \phi, \phi') \longmapsto H(t, z, y_1, y_2, x, \phi, \phi', \hat{u}, \hat{p}, \hat{q}, \hat{q}')$ *is convex in* y_1, y_2, x, ϕ *and* ϕ' *for all* $(t,z) \in [0,T] \times \mathbb{R}$
4. *With the integrability conditions below satisfied:*
$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left\{(\sigma_{11} - \hat{\sigma}_{11})^2 \hat{p}_1^2(t,z) + (\sigma_{12} - \hat{\sigma}_{12})^2 \hat{p}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q}_1^2(t,z) + \left(Y_t^1 - \hat{Y}_t^1\right)^2 \hat{q'}_1^2(t,z) + (\sigma_{21} - \hat{\sigma}_{21})^2 \hat{p}_2^2(t,z) + (\sigma_{22} - \hat{\sigma}_{22})^2 \hat{p}_2^2(t,z) + \left(Y_t^2 - \hat{Y}_t^2\right)^2 \hat{q}_2^2(t,z) + \left(Y_t^2 - \hat{Y}_t^2\right)^2 \hat{q'}_2^2(t,z) + (\sigma_{31} - \hat{\sigma}_{31})^2 \hat{p}_3^2(t,z) + (\sigma_{32} - \hat{\sigma}_{32})^2 \hat{p}_3^2(t,z) + \left(X_t - \hat{X}_t\right)^2 \hat{q}_3^2(t,z) + \left(X_t - \hat{X}_t\right)^2 \hat{q'}_3^2(t,z)\right\} dtdz\right] < \infty.$ *and*
$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left\{(\sigma_{41} - \hat{\sigma}_{41})^2 \hat{p}_4^2(t,z) + (\sigma_{42} - \hat{\sigma}_{42})^2 \hat{p}_4^2(t,z) + \left(\phi(t,z) - \hat{\phi}_1(t,z)\right)^2 \hat{q}_4^2(t,z) + \left(\phi(t,z) - \hat{\phi}_1(t,z)\right)^2 \hat{q'}_4^2(t,z)\right\} dtdz\right] < \infty.$

Then the control $\hat{u}$ *is optimal for the problem* (2.8).

Proof. .
. Let us show that $J(\hat{u}) \leqslant J(u) \quad \forall u \in \mathcal{U}_{ad}$.
Recall that

$$\begin{aligned} H(t,z,y_1,y_2,x,\phi,\phi',u,p,q,q') = &\int_{\mathbb{R}} f(t,z,y_1,y_2,x,\phi,\bar{b},u) d\mathbb{P}_{\bar{B}} + b_1(t,y_1,u)p_1 + b_2(t,y_2,u)p_2 \\ &+ b_3(t,y_1,y_2,x,u)p_3 + b_4(t,z,\phi,\phi',Y_1,Y_2,u)p_4 + \sigma_{11}(t,y_1,u)q_1 \\ &+ \sigma_{21}(t,y_2,u)q_2 + \sigma_{31}(t,x,u)q_3 + \sigma_{41}(t,z,\phi,\phi',Y_1,Y_2,u)q_4 + \sigma_{12}(t,y_1,u)q_1' \\ &+ \sigma_{22}(t,y_2,u)q_2' + \sigma_{32}(t,x,u)q_3' + \sigma_{42}(t,z,\phi,\phi',Y_1,Y_2,u)q_4' \end{aligned}$$

Let's pose
$H(t,z) = H(t,z,Y_t^1,Y_t^2,X_t,u_t,\phi(t,z),\phi'(t,z),\hat{p},\hat{q},\hat{q}')$,
$\hat{H}(t,z) = H(t,z,\hat{Y}_t^1,\hat{Y}_t^2,\hat{X}^t,\hat{u}_t,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{p},\hat{q},\hat{q}')$,
$f(t,z,\bar{b}) = f(t,z,Y_t^1,Y_t^2,X_t,\phi(t,z),\bar{b},u_t), \hat{f}(t,z,\bar{b}) = f(t,z,\hat{Y}_t^1,\hat{Y}_t^2,\hat{X}_t,\hat{\phi}(t,z),\bar{b},\hat{u})$,
$g(z,\bar{b}) = g(z,Y_T^1,Y_T^2,X_T,\phi(T,z),\bar{b},u), \hat{g}(z,\bar{b}) = g(z,\hat{X}_T,\hat{Y}_T^1,\hat{Y}_T^2,\hat{\phi}(T,z),\bar{b},u)$

$$b_1(t) = b_1(t,Y_t^1,u_t), \quad \hat{b}_1(t) = \hat{b}_1(t,\hat{Y}_t^1,\hat{u}_t) \quad b_2(t) = b_2(t,Y_t^1,u_t), \quad \hat{b}_2(t) = \hat{b}_2(t,\hat{Y}_t^1,\hat{u}_t)$$
$$b_3(t) = b_1(t,X_t,Y_t^1,Y_t^2,u_t) \quad \hat{b}_3(t) = \hat{b}_3(t,\hat{X}_t,\hat{Y}_t^1,\hat{Y}_t^2,\hat{u}_t)$$
$$b_4(t,z) = b_4(t,z,\phi(t,z),\phi'(t,z)),Y_t^1,Y_t^2,u_t) \quad \hat{b}_4(t,z) = \hat{b}_4(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),Y_t^1,Y_t^2,\hat{u}_t)$$
$$\sigma_{11}(t) = \sigma_{11}(t,Y_t^1,u_t), \quad \hat{\sigma}_{11}(t) = \hat{\sigma}_{11}(t,\hat{Y}_t^1,\hat{u}_t)\sigma_{12}(t) = \sigma_{12}(t,Y_t^1,u_t), \quad \hat{\sigma}_{12}(t) = \hat{\sigma}_{12}(t,\hat{Y}_t^1,\hat{u}_t)$$

$$\sigma_{21}(t) = \sigma_{21}(t,Y_t^2,u_t), \quad \hat{\sigma}_{21}(t) = \hat{\sigma}_{21}(t,\hat{Y}_t^2,\hat{u}_t),\sigma_{22}(t) = \sigma_{22}(t,Y_t^2,u_t), \quad \hat{\sigma}_{22}(t) = \hat{\sigma}_{22}(t,\hat{Y}_t^2,\hat{u}_t)$$
$$\sigma_{31}(t) = \sigma_{31}(t,X_t,u_t), \quad \hat{\sigma}_{31}(t) = \hat{\sigma}_{31}(t,\hat{X}_t,\hat{u}_t),\sigma_{32}(t) = \sigma_{32}(t,X_t,u_t), \quad \hat{\sigma}_{32}(t) = \hat{\sigma}_{32}(t,\hat{X}_t,\hat{u}_t)$$
$$\sigma_{41}(t,z) = \sigma_{41}(t,z,\phi(t,z),\phi'(t,z),Y_t^1,Y_t^2,u_t) \quad \hat{\sigma}_{41}(t,z) = \hat{\sigma}_{41}(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{Y}_t^1,\hat{Y}_t^2,\hat{u}_t)$$
$$\sigma_{42}(t,z) = \sigma_{42}(t,z,\phi(t,z),\phi'(t,z),Y_t^1,Y_t^2,u_t) \quad \hat{\sigma}_{42}(t) = \hat{\sigma}_{42}(t,z,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{Y}_t^1,\hat{Y}_t^2,\hat{u}_t)$$

$\hat{p}_1 = \hat{p}_1(t,z), \hat{p}_2 = \hat{p}_2(t,z), \hat{p}_3 = \hat{p}_3(t,z), \hat{p}_4 = \hat{p}_4(t,z), \hat{q}_1 = \hat{q}_1(t,z), \hat{q}_2 = \hat{q}_2(t,z), \hat{q}_3 = \hat{q}_3(t,z), \hat{q}_4 = \hat{q}_4(t,z), \hat{q'}_1 = \hat{q'}_1(t,z), \hat{q'}_2 = \hat{q'}_2(t,z), \hat{q'}_3 = \hat{q'}_3(t,z), \hat{q'}_4 = \hat{q'}_4(t,z)$, we have :

$$\begin{aligned} \int_{\mathbb{R}} f(t,z,\bar{b}) d\mathbb{P}_{\bar{B}} = &H(t,z) - b_1(t)\hat{p}_1 - b_2(t)\hat{p}_2 - b_3(t)\hat{p}_3 - b_4(t)\hat{p}_4 - \sigma_{11}\hat{q}_1 \\ &- \sigma_{21}\hat{q}_2 - \sigma_{31}\hat{q}_3 - \sigma_{41}\hat{q}_4 - \sigma_{12}\hat{q'}_1 - \sigma_{22}\hat{q'}_2 - \sigma_{32}\hat{q}_3' + \sigma_{42}\hat{q}_4' \end{aligned} \tag{3.3}$$

$$\begin{aligned} \int_{\mathbb{R}} \hat{f}(t,z,\bar{b}) d\mathbb{P}_{\bar{B}} = &\hat{H}(t,z) - \hat{b}_1(t)\hat{p}_1 - \hat{b}_2(t)\hat{p}_2 - \hat{b}_3(t)\hat{p}_3 - \hat{b}_4(t)\hat{p}_4 - \hat{\sigma}_{11}\hat{q}_1 \\ &- \hat{\sigma}_{21}\hat{q}_2 - \hat{\sigma}_{31}\hat{q}_3 - \hat{\sigma}_{41}\hat{q}_4 - \hat{\sigma}_{12}\hat{q'}_1 - \hat{\sigma}_{22}\hat{q'}_2 - \hat{\sigma}_{32}\hat{q}_3' + \hat{\sigma}_{42}\hat{q}_4' \end{aligned} \tag{3.4}$$

$$J(\hat{u})-J(u)=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{f}(t,z,\bar{b})-f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}}+\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{g}(z,\bar{b})-g(z,\bar{b})\big)dzd\mathbb{P}_{\mathbb{B}}\right] \quad (3.5)$$
$$=I_1+I_2$$

From (3.3), (3.4) and (3.9) we have:

$$\begin{aligned}
I_1 =&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\hat{f}(t,z,\bar{b})-f(t,z,\bar{b})\big)dtdzd\mathbb{P}_{\mathbb{B}}\right]\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{b}_1(t)-b_1(t)\big)\hat{p}_1(t,z)-\big(\hat{b}_2(t)-b_2(t)\big)\hat{p}_2(t,z)\right.\\
&-\big(\hat{b}_3(t)-b_3(t)\big)\hat{p}_3(t,z)-\big(\hat{b}_4(t,z)-b_4(t,z)\big)\hat{p}_4(t,z)-\big(\hat{\sigma}_{11}(t)-\sigma_{11}(t)\big)\hat{q}_1(t,z)-\big(\hat{\sigma}_{21}(t)-\sigma_{21}(t)\big)\hat{q}_2(t,z)\\
&-\big(\hat{\sigma}_{31}(t)-\sigma_{31}(t)\big)\hat{q}_3(t,z)-\big(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z)\big)\hat{q}_4(t,z)-\big(\hat{\sigma}_{12}(t)-\sigma_{12}(t)\big)\hat{q'}_1(t,z)\\
&\left.-\big(\hat{\sigma}_{22}(t)-\sigma_{22}(t)\big)\hat{q'}_2(t,z)-\big(\hat{\sigma}_{32}(t)-\sigma_{32}(t)\big)\hat{q'}_3(t,z)-\big(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\big)\hat{q'}_4(t,z)\Big\}dtdz\right]
\end{aligned} \quad (3.6)$$

$$\begin{aligned}
I_2 =&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\big\{\hat{g}(z,\bar{b})-g(z,\bar{b})\big\}dzd\mathbb{P}_{\bar{\mathbb{B}}}\right]\\
\leqslant&\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\frac{\partial\hat{g}}{\partial x}(z,\bar{b})\big(\hat{X}_T-X_T\big)+\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})\big(\hat{Y}^1_T-Y^1_T\big)+\frac{\partial\hat{g}}{\partial y_2}(z,\bar{b})\big(\hat{Y}^2_T-Y^2_T\big)\right.\\
&\left.+\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})\big(\hat{\phi}(T,z)-\phi(T,z)\big)\Big\}dzd\mathbb{P}_{\mathbb{B}}\right] \text{ according to the convexity of } g \text{ in } \hat{X},\hat{Y}^1,\hat{Y}^2 \text{ and } \hat{\phi}\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\Big\{\big(\hat{X}_T-X_T\big)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial x}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}+\big(\hat{Y}^1_T-Y^1_T\big)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}\right.\\
&\left.+\big(\hat{Y}^2_T-Y^2_T\big)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_2}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}+\big(\hat{\phi}(T,z)-\phi(T,z)\big)\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})d\mathbb{P}_{\mathbb{B}}\Big\}dz\right]\\
&\qquad\text{since } X,Y^1,Y^2 \text{ and } \phi \text{ thus depend on } \bar{b}\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\Big\{\big(\hat{Y}^1_T-Y^1_T\big)\hat{p}_1(T,z)+\big(\hat{Y}^2_T-Y^2_T\big)\hat{p}_2(T,z)+\big(\hat{X}_T-X_T\big)\hat{p}_3(T,z)\right.\\
&\left.+\big(\hat{\phi}(T,z)-\phi(T,z)\big)\hat{p}_4(T,z)\Big\}dz\right] \qquad \text{hence from(3.2) } \hat{p}_1(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_1}(z,\bar{b})d\mathbb{P}_{\bar{B}},\\
&\hat{p}_2(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial y_2}(z,\bar{B})d\mathbb{P}_{\bar{B}},\quad \hat{p}_3(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial x}(z,\bar{b})d\mathbb{P}_{\bar{B}} \text{ and } \quad \hat{p}_4(T,z)=\int_{\mathbb{R}}\frac{\partial\hat{g}}{\partial\phi}(z,\bar{b})d\mathbb{P}_{\bar{B}},\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\Big\{d\big(\hat{Y}^1_t-Y^1_t\big)\hat{p}_1(t,z)+\big(\hat{Y}^1_t-Y^1_t\big)d\hat{p}_1(t,z)+d<\hat{Y}^1_t-Y^1_t,\hat{p}_1(t,z)>\right.\\
&+d\big(\hat{Y}^2_t-Y^2_t\big)\hat{p}_2(t,z)+\big(\hat{Y}^2_t-Y^2_t\big)d\hat{p}_2(t,z)+d<\hat{Y}^2_t-Y^2_t,\hat{p}_2(t,z)>\\
&+d\big(\hat{X}_t-X_t\big)\hat{p}_3(t,z)+\big(\hat{X}_t-X_t\big)d\hat{p}_3(t,z)+d<\hat{X}_t-X_t;\hat{p}_3(t,z)>\\
&\left.+d\big(\hat{\phi}(t,z)-\phi(t,z)\big)\hat{p}_4(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)d\hat{p}_4(t,z)+d<\hat{\phi}(t,z)-\phi(t,z),\hat{p}_4(t,z)>\Big\}dzdt\right]\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\Big\{\big((\hat{b}_1(t)-b_1(t))dt+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))dW_1(t)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))dW^2_t\big)\hat{p}_1(t,z)+\big((\hat{b}_2(t)-b_2(t))dt\right.\\
&+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))dW^1_t+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))dW^2_t\big)\hat{p}_2(t,z)\\
&+\big((\hat{b}_3(t)-b_3(t))dt+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))dW^1_t+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))dW^2_t\big)\hat{p}_3(t,z)\\
&\big((L(\hat{\phi}(t,z)-\phi(t,z))+(\hat{b}_4(t,z)-b_4(t,z))dt+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))dW^1_t
\end{aligned}$$

$$
\begin{aligned}
&+\left(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\right)dW_t^2\big)\,\hat{p}_4(t,z)+\left(\hat{Y}_t^1-Y_t^1\right)\big(-\frac{\partial\hat{H}}{\partial y_1}(t,z)dt+\hat{q}_1(t,z)dW_t^1+\hat{q}_1^{'}(t,z)dW_t^2\big)\\
&+\left(\hat{Y}_t^2-Y_t^2\right)\big(-\frac{\partial\hat{H}}{\partial y_2}(t,z)dt+\hat{q}_2(t,z)dW_t^1+\hat{q}_2^{'}(t,z)dW_t^2\big)\\
&+\left(\hat{X}_t-X_t\right)\big(-\frac{\partial\hat{H}}{\partial x}(t,z)dt+\hat{q}_3(t,z)dW_t^1+\hat{q}_3^{'}(t,z)dW_t^2\big)\\
&+\left(\hat{\phi}(t,z)-\phi(t,z)\right)\Big(\big(-\frac{\partial\hat{H}}{\partial\phi}(t,z)-L^*\hat{p}_4(t,z)+\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)dt+\hat{q}_4(t,z)dW_t^1+\hat{q}_4^{'}(t,z)dW_t^2\Big)\\
&+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))\hat{q}_1(t,z)dt+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))\hat{q}_1^{\prime}(t,z)dt+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))\hat{q}_2(t,z)dt\\
&+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))\hat{q}_2^{\prime}(t,z)dt+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))\hat{q}_3(t,z)dt+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))\hat{q}_3^{'}(t,z)dt\\
&+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))\hat{q}_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))\hat{q}_4^{'}(t,z)\Big\}dz\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\Big\{\left(\hat{b}_1(t)-b_1(t)\right)\hat{p}_1(t,z)+\left(\hat{b}_2(t)-b_2(t)\right)\hat{p}_2(t,z)+\left(\hat{b}_3(t)-b_3(t)\right)\hat{p}_3(t,z)dt\right.\\
&+\left(\hat{b}_4(t)-b_4(t)\right)\hat{p}_4(t,z)+L\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)+\left(\hat{Y}_t^1-Y_t^1\right)\big(-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big)\\
&+\left(\hat{Y}_t^2-Y_t^2\right)\big(-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big)+\left(\hat{X}_t-X_t\right)\big(-\frac{\partial\hat{H}}{\partial x}(t,z)\big)+\\
&,+\left(\hat{\phi}(t,z)-\phi(t,z)\right)\big(-\frac{\partial\hat{H}}{\partial\phi}(t,z)-L^*\hat{p}_4(t,z)+\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\big)\\
&+\left(\hat{\sigma}_{11}(t)-\sigma_{11}(t)\right)\hat{q}_1(t,z)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))\hat{q}_1^{\prime}(t,z)+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))\hat{q}_2(t,z)\\
&+(\hat{\sigma}_{22}(t)-\sigma_{22}(t))\hat{q}_2^{\prime}(t,z)+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))\hat{q}_3(t,z)+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))\hat{q}_3^{'}(t,z)\\
&\left.+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))\hat{q}_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))\hat{q}_4^{'}(t,z)\Big\}dtdz\right]
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7) we have :

$$
\begin{aligned}
J(\hat{u})-J(u)\leqslant\ & I_1+I_2\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\left(\hat{b}_1(t)-b_1(t)\right)\hat{p}_1(t,z)-\left(\hat{b}_2(t)-b_2(t)\right)\hat{p}_2(t,z)\right.\\
&-\left(\hat{b}_3(t)-b_3(t)\right)\hat{p}_3(t,z)-\left(\hat{b}_4(t,z)-b_4(t,z)\right)\hat{p}_4(t,z)-\left(\hat{\sigma}_{11}(t)-\sigma_{11}(t)\right)\hat{q}_1(t,z)\\
&-\left(\hat{\sigma}_{21}(t)-\sigma_{21}(t)\right)\hat{q}_2(t,z)-\left(\hat{\sigma}_{31}(t)-\sigma_{31}(t)\right)\hat{q}_3(t,z)-\left(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z)\right)\hat{q}_4(t,z)\\
&-\left(\hat{\sigma}_{12}(t)-\sigma_{12}(t)\right)\hat{q'}_1(t,z)-\left(\hat{\sigma}_{22}(t)-\sigma_{22}(t)\right)\hat{q'}_2(t,z)-\left(\hat{\sigma}_{32}(t)-\sigma_{32}(t)\right)\hat{q'}_3(t,z)\\
&-\left(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z)\right)\hat{q'}_4(t,z)+\left(\hat{b}_1(t)-b_1(t)\right)\hat{p}_1(t,z)+\left(\hat{b}_2(t)-b_2(t)\right)\hat{p}_2(t,z)\\
&+\left(\hat{b}_3(t)-b_3(t)\right)\hat{p}_3(t,z)+\left(\hat{b}_4(t)-b_4(t)\right)\hat{p}_4(t,z)+L\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)\\
&-\left(\hat{Y}_t^1-Y_t^1\right)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\left(\hat{Y}_t^2-Y_t^2\right)\frac{\partial\hat{H}}{\partial y_2}(t,z)-\left(\hat{X}_t-X_t\right)\frac{\partial\hat{H}}{\partial x}(t,z)\\
&-\left(\hat{\phi}(t,z)-\phi(t,z)\right)\frac{\partial\hat{H}}{\partial\phi}(t,z)-\left(\hat{\phi}(t,z)-\phi(t,z)\right)L^*\hat{p}_4(t,z)\\
&+\left(\hat{\phi}(t,z)-\phi(t,z)\right)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)+(\hat{\sigma}_{11}(t)-\sigma_{11}(t))q_1(t,z)+(\hat{\sigma}_{12}(t)-\sigma_{12}(t))q_1^{\prime}(t,z)\\
&+(\hat{\sigma}_{21}(t)-\sigma_{21}(t))q_2(t,z)+((\hat{\sigma}_{22}(t)-\sigma_{22}(t))q_2^{\prime}(t,z)+(\hat{\sigma}_{31}(t)-\sigma_{31}(t))q_3(t,z)\\
&\left.+(\hat{\sigma}_{32}(t)-\sigma_{32}(t))q_3^{'}(t,z)+(\hat{\sigma}_{41}(t,z)-\sigma_{41}(t,z))q_4(t,z)+(\hat{\sigma}_{42}(t,z)-\sigma_{42}(t,z))q_4^{'}(t,z)\Big\}dtdz\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{Y}_1(t)-Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}_2(t)-Y_2(t)\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&-\big(\hat{X}(t)-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\\
&\left.+L\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]\\
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\big(\hat{Y}_1(t)-Y_1(t)\big)\frac{\partial\hat{H}}{\partial y_1}(t,z)-\big(\hat{Y}_2(t)-Y_t^2\big)\frac{\partial\hat{H}}{\partial y_2}(t,z)\right.\\
&-\big(\hat{X}(t)-X_t\big)\frac{\partial\hat{H}}{\partial x}(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial\hat{H}}{\partial\phi}(t,z)+\big(\hat{\phi}(t,z)-\phi(t,z)\big)\frac{\partial}{\partial z}\left(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\right)\\
&\left.+L^*\left(\hat{\phi}(t,z)-\phi(t,z)\right)\hat{p}_4(t,z)-\big(\hat{\phi}(t,z)-\phi(t,z)\big)L^*\hat{p}_4(t,z)\Big\}dtdz\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}_t^1-Y_t^1\big)-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}_t^2-Y_t^2\big)\right.\\
&\left.-\frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t-X_t\big)-\frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z)-\phi(t,z)\big)-\frac{\partial\hat{H}}{\partial\phi'}(t,z)\big(\hat{\phi}'(t,z)-\phi'(t,z)\big)\Big\}dtdz\right]
\end{aligned}
\tag{3.8}
$$

. Recall that:$H(t,z)=H(t,z,Y_t^1,Y_t^2,X_t,u_t,\phi(t,z),\phi'(t,z),\hat{p},\hat{q},\hat{q'})$,
$\hat{H}(t,z)=H(t,z,\hat{Y}_t^1,\hat{Y}_t^2,\hat{X}_t,\hat{u}_t,\hat{\phi}(t,z),\hat{\phi}'(t,z),\hat{p},\hat{q},\hat{q'})$,
$H(t,z,X,Y^1,Y^2,\phi,\phi',\hat{u},\hat{p},\hat{q})=\min_{u\in U_{ad}}H(t,z,X,Y^1,Y^2,\phi,\phi',u,\hat{p},\hat{q}) \quad \forall X,Y^1,Y^2,\phi,\phi'$
$\hat{H}(t,z)=\min_{u\in U_{ad}}H(t,z,\hat{X},\hat{Y}^1,\hat{Y}^2,\hat{\phi},\hat{\phi}',u,\hat{p},\hat{q})$; $\hat{H}(t,z)=H(t,z,\hat{X},\hat{Y}^1,\hat{Y}^2,\hat{\phi},\hat{\phi}',\hat{u},\hat{p},\hat{q})$,
$H(t,z)$ depends on u and therefore is different from the function $\hat{H}(t,z)$ which depends on $\hat{u}$. We therefore cannot directly apply the convexity of $\hat{H}$. But we have:

$$
\begin{aligned}
H(t,z,Y^1,Y^2,X,\phi,\phi',u,\hat{p},\hat{q}) &\geqslant H(t,z,Y^1,Y^2,X,\phi,\phi',\hat{u},\hat{p},\hat{q})\\
\hat{H}(t,z)-H(t,z,Y^1,Y^2,X,\phi,\phi',u,\hat{p},\hat{q}) &\leqslant \hat{H}(t,z)-H(t,z,Y^1,Y^2,X,\phi,\phi',\hat{u},\hat{p},\hat{q})
\end{aligned}
$$

Let: $\hat{H}(t,z)-H(t,z)\leqslant\hat{H}(t,z)-H(t,z,X,Y^1,Y^2,\phi,\phi',\hat{u},\hat{p},\hat{q})$. We can now apply the concavity of h to the points $M_0=(\hat{Y}^1,\hat{Y}^2,\hat{X},\hat{\phi},\hat{\phi})'$ and $M=(Y^1,Y^2,X,\phi,\phi')$ in effect $h(\hat{X},\hat{Y}^1,\hat{Y}^2,\hat{\phi},\hat{\phi}')=\hat{H}$ According to (3.8) and the convexity of h we have :

$$
\begin{aligned}
J(\hat{u})-J(u)\leqslant&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z,X,Y^1,Y^2,\phi,\phi',\hat{u},\hat{p},\hat{q})-\frac{\partial\hat{H}}{\partial y_1}(t,z)\big(\hat{Y}_t^1-Y_t^1\big)\right.\\
&-\frac{\partial\hat{H}}{\partial y_2}(t,z)\big(\hat{Y}_t^2-Y_t^2\big)-\frac{\partial\hat{H}}{\partial x}(t,z)\big(\hat{X}_t-X_t\big)-\frac{\partial\hat{H}}{\partial\phi}(t,z)\big(\hat{\phi}(t,z)-\phi(t,z)\big)\\
&\left.-\frac{\partial\hat{H}}{\partial\phi'}(t,z)\big(\hat{\phi}'(t,z)-\phi(t,z)\big)\Big\}dtdz\right]\\
\leqslant&0
\end{aligned}
\tag{3.9}
$$

□

4. Conclusion

We have established a stochastic optimization result allowing to characterize and determine an optimal control. This in a stochastic control problem under partial information where the value function of the problem resorts to a minimization of the objective function. The result was established using backward stochastic differential equations and in the case where the dynamics of one of the state variables of the control system is governed by a stochastic partial differential equation. As a perspective, we intend to make a generalization of the results in higher dimension in the case where the control is not only a scalar function but a vector function with several state variables governed by SDE, BSDE, SPDE and make applications.

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