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Epsilon-Diskcyclic Operators

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Abstract

In this paper, we will introduce a new concept of cyclic phenomena that is called ε-diskcyclic operator and construct examples to show the relationship between other types. In particular, for each ε in $(0, 1)$, we will construct an ε-diskcyclic operator that is not diskcyclic. These examples illustrate the main differences between the new definition and other types.

Keywords: ε-hypercyclic operator, diskcyclic operators, weighted shift operator.
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1. Introduction

Let $\mathcal{H}$ be an infinite dimentional separable complex Hilbert space, X be an infinite separable complex Banach space, and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. We will refer to the closed unit disk in the complex plane by $\mathbb{D}\setminus\{0\}$, i.e., $\mathbb{D}\setminus\{0\} := \{\gamma \in \mathbb{C}; |\gamma| \leqslant 1\}$, the open unit disk by $\mathbb{U} := \{\gamma \in \mathbb{C}; |\gamma| < 1\}$ and the unit circle by $\mathbb{T} := \{\gamma \in \mathbb{C}; |\gamma| = 1\}$. An operator $T \in \mathcal{B}(X)$ is called hypercyclic if there exists a vector $x \in X$, such that $Orb(T, x) = \{T^n x : n \geqslant 0\}$ is dense in X, where such a vector is called a hypercyclic vector for T. The definition of hypercyclicity was already studied by G.D. Birkhoff in 1922 [7]. Furthermore he proved the first characterization of hypercyclic operators, which is a direct application of the Baire category theorem. Now it is often referred to as Birkhoff's transitivity Theorem. In 1929 G.D. Birkhoff [6] gave a historical example of a hypercyclic operators. Later G.R. MacLane [5] found the same phenomenon for the differentiation operator. In the beginning of eighteenth, linear operator theory is rapidly evolving in a branch of functional analysis with the Toronto Ph.D. thesis of C. Kitai [2]. She discovered the sufficient condition for hypercyclic (the Hypercyclic Criterion). In the early seventies, Hilden and Wallen [9] introduced the definition of supercyclicity. An operator $T \in \mathcal{B}(X)$ is called supercyclic if there exists a vector $x \in X$, such that $\mathbb{C}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, n \geqslant 0\}$ is dense in X, where such a vector is called a supercyclic vector for T. Zeana in her Ph.D. thesis [14] partition the cone orbit $\mathbb{C}Orb(T, x)$ into three parts as follows: diskcyclic operators if $|\alpha| \leqslant 1$, circle cyclic operators if $|\alpha| = 1$ and codiskcyclic operators if $|\alpha| \geqslant 1$. In other words, a vector $x \in X$ is called diskcyclic vector if the set $\{\alpha T^n x : \alpha \in \mathbb{D}\setminus\{0\}, n \geqslant 0\}$ is dense in X. An operator $T \in \mathcal{B}(X)$ is called a diskcyclic if it has a diskcyclic vector $x \in X$.

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Zeana [14] introduced a brief description of a cyclic phenomena on Hilbert spaces. She proposed a question "Is every circle cyclic a hypercyclic operator?". León and Müller in [3] proved that T is a hypercyclic operator if and only if T is a circle cyclic operator in the following Theorem.

Theorem 1.1 (León-Müller Theorem)**.**
Let $T \in \mathcal{B}(X)$. Then $x \in X$ is hypercyclic for T if and only if x is a $\mathbb{T}$-supercyclic for T.

In 2016, Yu-Xia Liang and Ze-Hua Zhou introduced diskcyclic and codiskcyclic tuples of the adjoint weighted composition operators on Hilbert spaces [13].

One of the most important results from [14] was the Diskcyclicity Criterion. After that, Bamerni, Kılıcman, and Noorani in [10] introduced another characterization that is equivalent to the one introduced in [14].

Theorem 1.2 (Diskcyclicity Criterion)**.** *[10] Let $T \in \mathcal{B}(X)$. We assume that there exist two dense subsets $X_0, Y_0 \subset X$, an increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, and maps $S_{n_k} : Y_0 \to X$ such that for any $x \in X_0$ and any $y \in Y_0$ the following holds:*

1. $T^{n_k}x \to 0$ *as* $k \to \infty$;
2. $S_{n_k}y \to 0$ *as* $k \to \infty$;
3. $T^{n_k}S_{n_k}y \to y$ *as* $k \to \infty$.

Then T is diskcyclic.

Later, in 2016, Bamerni [11] presented a simpler equivalent version of the Diskcyclicity Criterion.

Theorem 1.3 (Second Diskcyclicity Criterion)**.** *[11] Let $T \in \mathcal{B}(X)$. If there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ and two dense sets $D_1, D_2 \subset X$ such that*

1. *for each $y \in D_2$, there is a sequence $\{x_k\}$ in X such that $x_k \to 0$ and $T^{n_k}x_k \to y$,*
2. $\|T^{n_k}x\|\|x_k\| \to 0$ *for all* $x \in D_1$.

Then T is diskcyclic.

The following definition was also introduced in [10]

Definition 1.1. *[10] A bounded linear operator $T \in X$ is called disk transitive if for any pair U, V of non-empty open subsets of X, there exist $\alpha \in \mathbb{C}; 0 < |\alpha| \leqslant 1$, and $n \geqslant 0$ such that $T^n(\alpha U) \cap V \neq \phi$ or equivalently, there exist $\alpha \in \mathbb{C}; |\alpha| \geqslant 1$, and $n \geqslant 0$ such that $T^{-n}(\alpha U) \cap V \neq \phi$.*

The following lemma is a very useful tool in proving our results according to this article.

Lemma 1.1. *[10] Let $T \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent.*

1. *T is a diskcyclic operator on $\mathcal{H}$.*
2. *T is disk transitive.*
3. *For each $x, y \in \mathcal{H}$, there exist sequences $\{x_k\}$ in $\mathcal{H}$, $\{n_k\}$ in $\mathbb{N}$, and $\{\alpha_k\}$ in $\mathbb{C}; 0 < |\alpha_k| \leqslant 1$ such that $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$.*
4. *For each $x, y \in \mathcal{H}$ and each neighbourhood W of zero in H, there exist $z \in \mathcal{H}, n \in \mathbb{N}$, and $\alpha \in \mathbb{C}; 0 < |\alpha| \leqslant 1$ such that $x - z \in W$ and $T^n\alpha z - y \in W$.*

In 2002, Feldman [12] investigated the density of the orbit in another point of view. He characterized the hypercyclicity by researching a small set that has an orbit under T to still ensure hypercyclicity. He proved that the operator is hypercyclic if and only if there exist $d > 0$ and a vector $x \in \mathcal{H}$ having a d-dense orbit.

In [1], C. Badea, S. Grivaux, and V. Müller have investigated a weaker version of Feldman's result which is called ε-hypercyclic operator. An ε-hypercyclic operator has a vector x whose orbit intersects every cone of the aperture $\varepsilon \in (0, 1)$.

Definition 1.2. *[1] Let $\varepsilon \in (0,1)$ and let $T \in \mathcal{B}(X)$. A vector $x \in X$, is called an ε-hypercyclic vector for T if for every nonzero vector $y \in X$ there exists a nonnegative integer n such that*

$$\|T^n x - y\| \leqslant \varepsilon \|y\|.$$

An operator T is called ε-hypercyclic if it has an ε-hypercyclic vector.

In addition, they proved that for each $\varepsilon \in (0,1)$, there is an ε-hypercyclic operator that is not a hypercyclic operator on $\ell^1(\mathbb{N})$ space. After that, Bayart [4] extended this result to Hilbert spaces.

Definition 1.3. *Let $\varepsilon \in (0,1)$ and let $T \in \mathcal{B}(X)$. A vector $x \in X$, is called an ε-diskcyclic vector for T if for every nonzero vector $y \in X$ there exist $\gamma \in \mathbb{D} \setminus \{0\}$ and a nonnegative integer n such that*

$$\|\gamma T^n x - y\| \leqslant \varepsilon \|y\|.$$

The operator T is called an ε-diskcyclic if it has an ε-diskcyclic vector.

***Remark* 1.** *It suffices to put $\gamma = 1$ in the ε-diskcyclic operator's definition to observe that every ε-hypercyclic operator is an ε-diskcyclic operator.*

Looking thoroughly at this definition, we observe that the disk orbit of x must intersect every cone of a fixed $\varepsilon \in (0,1)$. It is obviously weaker than the definition of diskcyclic operator. So, it is natural to propose the following questions.

Question 1. *Suppose that $T \in \mathcal{B}(X)$ is an ε-diskcyclic operator for some $\varepsilon \in (0,1)$. Is T is diskcyclic?*

The main results of this paper investigate the operators on Hilbert spaces and give a negative answer to Question 1.

Question 2. *Suppose that $T \in \mathcal{B}(X)$ is a diskcyclic operator for some $\varepsilon \in (0,1)$. Is T is ε-hypercyclic?*

Additionally, the main results give a negative answer for Question 2.

2. Main Results

Theorem 2.1. *Let $T \in \mathcal{B}(X)$ and $x \in X$, for any nonzero complex number λ, if x is an ε-diskcyclic vector for T, then λx is an ε-diskcyclic vector for T.*

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be fixed. Suppose that x is an ε-diskcyclic vector for an operator T and $y \in X$ is any nonzero vector in X. Then there is $n \in \mathbb{N}$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that

$$\|\alpha T^n x - \frac{1}{\lambda} y\| \leqslant \varepsilon \|\frac{1}{\lambda} y\|.$$

This implies

$$\frac{1}{|\lambda|}\|\lambda \alpha T^n x - y\| \leqslant \varepsilon \frac{1}{|\lambda|}\|y\|.$$

Then

$$\|\alpha T^n \lambda x - y\| \leqslant \varepsilon \|y\|.$$

Hence, λx is an ε-diskcyclic vector for T. □

By the linearity of the operator and the previous Theorem, one can observe the following corollary.

Corollary 2.1. *Let $T \in \mathcal{B}(X)$ be an ε-diskcyclic operator. Then λT is an ε-diskcyclic operator for any nonzero complex number λ.*

Next, we show that every diskcyclic operator is an ε-diskcyclic operator.

Proposition 2.1. *Let $T \in \mathcal{B}(X)$ be a diskcyclic operator, Then T is an ε-diskcyclic operator.*

Proof. Let $\delta > 0$ be given, $\varepsilon \in (0,1)$ and a nonzero vector $y \in X$. We can find a positive integer N such that $\varepsilon^N \delta < \|y\|$. Now, since T is a diskcyclic operator with a diskcyclic vector x. Then there is a $\gamma \in \mathbb{D} \setminus \{0\}$ and $n \in \mathbb{N}$ such that

$$\|\gamma T^n x - \frac{1}{\varepsilon^{N+1}} y\| \leqslant \delta.$$

That is

$$\|\varepsilon^{N+1}\gamma T^n x - y\| \leqslant \varepsilon^{N+1}\delta = \varepsilon\varepsilon^N\delta < \varepsilon\|y\|.$$

Clearly, $\varepsilon^{N+1}\gamma \in \mathbb{D} \setminus \{0\}$.
Hence, T is an ε-diskcyclic operator with an ε-diskcyclic vector x. □

Theorem 2.2. *Let $T \in \mathcal{B}(X)$, if T is an ε-diskcyclic for every $\varepsilon \in (0,1)$, then T is a diskcyclic operator.*

Proof. Suppose that T is ε-diskcyclic operator on X for every ε. We are going to prove that T is disk transitive. Let U and V be two non-empty open sets in X. Let $u \in U$ and $v \in V$ be two non-zero vectors in U and V respectively. Let $m = \min\{\|u\|, \|v\|\}$. Since U and V are open sets then we can find $0 < \delta < m$ such that $B(u,\delta) \subset U$ and $B(v,\delta) \subset V$. Put $\varepsilon_0 := \frac{\delta}{6\max\{\|u\|,\|v\|\}}$. Clearly $\varepsilon_0 \in (0,1)$. Since T is ε-diskcyclic for every $\varepsilon \in (0,1)$, then we can find an ε_0-diskcyclic vector $x \in X$, $n_0 \in \mathbb{N}$ and $\gamma_0 \in \mathbb{D} \setminus \{0\}$ such that $\|\gamma_0 T^{n_0}x - u\| \leqslant \varepsilon_0\|u\| \leqslant \frac{\delta}{6} < \delta$. Hence, the vector $\gamma_0 T^{n_0}x \in U$. Similarly, there are $n_1 \in \mathbb{N}$ and $\gamma_1 \in \mathbb{D} \setminus \{0\}$ such that $\|\gamma_1 T^{n_1}x - \frac{1}{\gamma_0}v\| \leqslant \frac{\varepsilon_0}{|\gamma_0|}\|v\| \leqslant \frac{\delta}{6|\gamma_0|} < \frac{\delta}{|\gamma_0|}$. Hence, $\|\gamma_0\gamma_1 T^{n_1}x - v\| < \delta$. Hence, the vector $\gamma_0\gamma_1 T^{n_1}x \in V$. If $n_1 \geqslant n_0$, then $T^{n_1-n_0}(\gamma_1\gamma_0 T^{n_0}x) = (\gamma_0\gamma_1 T^{n_1}x) \in V$. Now, we have $\gamma_1\gamma_0 T^{n_0}x \in \gamma_1 U$. Hence, $T^{n_1-n_0}(\gamma_1 U) \cap V \neq \phi$. On the other hand if $n_1 < n_0$ we have two cases. The first is: there is $n_k > n_0$ and $\gamma_k \in \mathbb{D} \setminus \{0\}$ such that $\|\gamma_k T^{n_k}x - \frac{1}{\gamma_0}v\| \leqslant \frac{\varepsilon_0}{|\gamma_0|}\|v\| < \frac{\delta}{|\gamma_0|}$. So, $\|\gamma_0\gamma_k T^{n_k}x - v\| < \delta$ which implies that $\gamma_0\gamma_k T^{n_k}x$ belong to V. As above we have $T^{n_k-n_0}(\gamma_k U) \cap V \neq \phi$. The second case is there is no $n_k > n_0$ such that $\gamma_0\gamma_k T^{n_k}x$ belong to V. In this case we observe that the number of n_k's that satisfies $\gamma_0\gamma_k T^{n_k}x$ belong to V are finite numbers call them $n_1, n_2, \ldots, n_k$. Now, we are going to get a contradiction. For every $v' \in V$ such that $\|v - v'\| < \frac{\delta}{2}$ there is $n_{v'} \in \mathbb{N}$ and $\gamma_{v'} \in \mathbb{D} \setminus \{0\}$ such that $\|\gamma_{v'} T^{n_{v'}}x - \frac{1}{\gamma_0}v'\| \leqslant \frac{\varepsilon_0\|v'\|}{|\gamma_0|} < \frac{3\varepsilon_0\|v\|}{|\gamma_0|} < \frac{3\delta\|v\|}{|\gamma_0| 6\max\{\|u\|\|v\|\}} < \frac{\delta}{2|\gamma_0|}$ and then $\|\gamma_0\gamma_{v'} T^{n_{v'}}x - v'\| < \frac{\delta}{2}$. Since, $\|\gamma_{n_{v'}}\gamma_0 T^{n_{v'}}x - v\| \leqslant \|\gamma_{n_{v'}}\gamma_0 T^{n_{v'}}x - v'\| + \|v' - v\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Hence, $\gamma_{n_{v'}}\gamma_0 T^{n_{v'}}x$ belong to V, then $n_{v'} \in \{n_1, \ldots, n_k\}$. We can conclude that the ball $B(v, \frac{\delta}{2})$ can be covered by finite balls $B(\gamma_0\gamma_1 T^{n_1}x, \frac{\delta}{2}), \ldots, B(\gamma_0\gamma_k T^{n_k}x, \frac{\delta}{2})$ which is impossible in infinite dimensional spaces. Hence, there are infinitely many n_k's with $\|\gamma_0\gamma_k T^{n_k}x - v\| < \delta$. So we can find $n_k > n_0$ with γ_k such that $\gamma_0\gamma_k T^{n_k}x$ belongs to V. So, $T^{n_k-n_0}(\gamma_k\gamma_0 T^{n_0}x) = (\gamma_k\gamma_0 T^{n_k}x) \in V$. Hence, $T^{n_k-n_0}(\gamma_k U) \cap V \neq \phi$. Hence, T is a disk transitive operator. Thanks to Lemma (1.1) to observe that T is diskcyclic operator on X. □

Here we ask, for every $\varepsilon \in (0,1)$, is there an ε-diskcyclic operator on the separable Banach space which is not a diskcyclic?
Next, we investigate this question on Hilbert spaces and remain the question on every separable Banach space is open.

Theorem 2.3. *For every $\varepsilon \in (0,1)$, there exists an ε-diskcyclic operator on the separable Hilbert space which is not a diskcyclic.*

In order to prove theorem 2.3, for every $\varepsilon \in (0,1)$, we will introduce an example of an ε-diskcyclic operator on the separable Hilbert space that is not a diskcyclic operator. Firstly, we will construct a diskcyclic operator on a separable Hilbert space. Recall that every infinite dimintional Hilbert space is isomorphic to $\ell^2(\mathbb{N})$. In what follows, let $\mathcal{H}$ be Hilbert space which is the direct sum of countably many copies of $\ell^2(\mathbb{N})$, (i.e., $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \ldots$). We shall construct a diskcyclic operator on the Hilbert space $\mathcal{H}$. We call $(e_n)_{n\geqslant 0}$ the canonical basis of $\ell^2(\mathbb{N})$.

Example 2.1. *Let $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \dots$. Define the backward weighted shift operator $B : \mathcal{H} \to \mathcal{H}$ as,*

$$B(x_0, x_1, x_2, \dots) = (T_1 x_1, T_2 x_2, \dots),$$

where $T_i x_j = 2x_j$ for every $x_j \in \ell^2(\mathbb{N})$. Then B is a diskcyclic operator on $\mathcal{H}$.

Proof. Let $F : \mathcal{H} \to \mathcal{H}$ be the forward weighted shift operator defined as:

$$F(x_0, x_1, \dots) = (0, T_1^{-1}x_0, T_2^{-1}x_1, \dots) = (0, 2^{-1}x_0, 2^{-1}x_1, \dots).$$

Since $\mathcal{H}$ is separable, then we can find a countable dense sequence $(y^{(k)})_{k\in\mathbb{N}}$ in $\mathcal{H}$ of the form $y^{(k)} = (y_0^{(k)}, y_1^{(k)}, \dots, y_{k-1}^{(k)}, 0, \dots)$, $k \geqslant 1$, where each $y_i^{(k)} \in \ell^2(\mathbb{N})$, $i = 0, 1, 2, \dots$. We shall construct an increasing sequence of integers $(m_k)_{k\geqslant 1}$ such that $m_{k+1} > m_k + k$ and satisfies (1). Also take such a sequence $(\gamma_k)_{k\geqslant 1} \subset \mathbb{D} \setminus \{0\}$ such that $|\gamma_1| < |\gamma_2| < |\gamma_3| < \dots$. Now, the two sequences $(\gamma_k)_k$ and $(m_k)_k$ must satisfy the following statements:

1. $\|\frac{1}{\gamma_k} F^{m_k} y^{(k)}\| < \frac{1}{2^k}$,
2. $\frac{|\gamma_i|}{|\gamma_k|}\|B^{m_i} F^{m_k} y^{(k)}\| < \frac{1}{2^k}$, when $i < k$,

In order to check that, we can firstly find a suitable m_1, to satisfy (1). After that, we will follow the following procedure:

1. Consider m_1 was chosen above.
2. Take $m_k > m_{k-1} + k - 1$, $k \geqslant 2$.
3. Check if $\|\frac{1}{\gamma_k} F^{m_k} y^{(k)}\| < \frac{1}{2^k}$ and if $\frac{|\gamma_{k-1}|}{|\gamma_k|}\|B^{m_{k-1}} F^{m_k} y^{(k)}\| < \frac{1}{2^k}$.
    a. if (3) is not true, increase m_k, and return to check (3).
    b. if (3) is true, save m_k, and return to (2) to find m_{k+1}.

It is worth mentioning that $\|B^{m_i} F^{m_k} y^{(k)}\| = 0$, when $i > k$. Now, we can say that for every $y^{(k)}$ we have a corresponding $\gamma_k \in \mathbb{D} \setminus \{0\}$ and m_k.

Take the vector

$$x = \sum_{k=1}^{\infty} \frac{1}{\gamma_k} F^{m_k} y^{(k)}. \tag{2.1}$$

$\|x\| = \|\sum_{k=1}^{\infty} \frac{1}{\gamma_k} F^{m_k} y^{(k)}\| \leqslant \sum_{k=1}^{\infty} \|\frac{1}{\gamma_k} F^{m_k} y^{(k)}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$. So, $x \in \mathcal{H}$. We claim that x is a diskcyclic vector for B. Let $k \in \mathbb{N}$, take the vector $y^{(k)}$, we have the corresponding $\gamma_k \in \mathbb{D} \setminus \{0\}$ and an integer m_k such that,

$$\begin{aligned}
\|\gamma_k B^{m_k} x - y^{(k)}\| &= \|\gamma_k B^{m_k} (\sum_{j=0}^{\infty} \frac{1}{\gamma_j} F^{m_j} y^{(j)}) - y^{(k)}\| \\
&= \| \sum_{j=0}^{\infty} \frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)} - y^{(k)}\| \\
&= \| \sum_{j<k} \frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)} + \frac{\gamma_k}{\gamma_k} B^{m_k} F^{m_k} y^{(k)} - y^{(k)} + \sum_{j>k} \frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)}\| \\
&\leqslant \| \sum_{j<k} \frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)}\| + \|\frac{\gamma_k}{\gamma_k} B^{m_k} F^{m_k} y^{(k)} - y^{(k)}\| + \| \sum_{j>k} \frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)}\| \\
&\leqslant \sum_{j<k} \|\frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)}\| + \|y^{(k)} - y^{(k)}\| + \sum_{j>k} \|\frac{\gamma_k}{\gamma_j} B^{m_k} F^{m_j} y^{(j)}\| \\
&< 0 + 0 + \sum_{j>k} \frac{1}{2^j} = \frac{1}{2^k} \xrightarrow{as \quad k\to\infty} 0.
\end{aligned}$$

Now, for any $y \in \mathcal{H} \setminus \{0\}$, there is $y^{(k)}$ such that, $\|y^{(k)} - y\| \xrightarrow{as\ \ k\to\infty} 0$. So we have,

$$\|\gamma_k B^{m_k} x - y\| \leqslant \|\gamma_k B^{m_k} x - y^{(k)}\| + \|y^{(k)} - y\| \quad \xrightarrow{as\ \ k\to\infty} \quad 0+0=0,$$

where $\gamma_k \in \mathbb{D} \setminus \{0\}$. Therefore, x is a diskcyclic vector for B. □

Bayart in [4] constructed a wonderful example of an ε-hypercyclic operator on a Hilbert space. So by Remark (1), we will consider Bayart's example in [4] is an ε-diskcyclic operator.

Example 2.2. *[4] Let $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \dots$ and let $\varepsilon \in (0,1)$ be fixed and a be a positive integer such that $2^{-a} < \varepsilon$. Define the operator B on $\mathcal{H}$ by*

$$B(x_0, x_1, \dots) = (S_1^{-1}x_1, S_2^{-1}x_2, \dots).$$

where the sequence $(S_j)_{j\geqslant 1}$ of bounded operators on $\ell^2(\mathbb{N})$. Let a sequence of vectors $(z^{(k)})$ in $\mathcal{H}$ such that $z^{(k)} = (z_0^{(k)}, \dots, z_{k-1}^{(k)}, 0, \dots)$ satisfies the following properties:

(a) For any $k \geqslant 1$, $\|z^{(k)} - y^{(k)}\| \leqslant 2^{-a}\|y^{(k)}\|$.
(b) Each operator S_j is bounded, invertible, upper triangular with $\|S_j^{-1}\| \leqslant 2$.
(c) $\|S_j S_{j-1} \dots S_1\| \leqslant 2^a$ for every $j \in \mathbb{N}$.
(d) $S_j e_0 = e_0$ for every $j \geqslant 1$.
(e) $\|S_{n_k+j-p} \dots S_{j+1} z_j^{(k)}\| \leqslant 2^{-k}$ for every $k \geqslant 1$, every $j = 0, \dots, k-1$ and every $p \leqslant n_{k-1}$.
(f) $S_{n'_k} \dots S_2 S_1 = I$ for every $k \in \mathbb{N}$.
(g) Let $k \geqslant 1$, $p > k^2$ and $i \in \{n'_{k-1}, \dots, n'_k - 1\}$. Then $S_1^{-1} \dots S_i^{-1} e_p = 2^{i-n'_{k-1}} e_p$.

where $(n_k)_k$ and $(n'_k)_k$ are two increasing sequences of integers. For more details about the construction of those sequences, see Section 4 in [4]. After all previous constructions were done, we set $x^{(k)} = (\underbrace{0, \dots, 0}_{n_k}, S_{n_k} \dots S_1 z_0^{(k)}, S_{n_k+1} \dots S_2 z_1^{(k)}, \dots, S_{n_k+k-1} \dots S_k z_{k-1}^{(k)}, 0, \dots)$. Then the vector $x = \sum_{k=1}^{\infty} x^{(k)}$ is an ε-hypercyclic operator.

Proof. By Remark (1), we conclude that the operator B is an ε-diskcyclic operator. It remains to prove that it is not diskcyclic operator. We will do that by show that there is no diskcyclic vector for B in $\mathcal{H}$. Suppose on the contrary, that there is a vector $y = (y_0, y_1, \dots) \in \mathcal{H}$ which is a diskcyclic for B. Then there is a sequence $(\gamma_j)_{j\in\mathbb{N}}$ in $\mathbb{D} \setminus \{0\}$ and an increasing sequence $(m_j)_{j\geqslant 0}$ of integers such that $\|\gamma_j B^{m_j} y - (e_0, 0, \dots)\| \to 0$ as $j \to \infty$. Then,

$$\begin{aligned}
&\|\gamma_j B^{m_j}(y_0, y_1, \dots) - (e_0, 0, \dots)\| \\
&\quad = \|\gamma_j (S_1^{-1} S_2^{-1} \dots S_{m_j}^{-1} y_{m_j}, S_2^{-1} S_3^{-1} \dots S_{m_j+1}^{-1} y_{m_j+1}, \dots) - (e_0, 0, \dots)\| \\
&\quad = \|(\gamma_j S_1^{-1} S_2^{-1} \dots S_{m_j}^{-1} y_{m_j}, \gamma_j S_2^{-1} S_3^{-1} \dots S_{m_j+1}^{-1} y_{m_j+1}, \dots) - (e_0, 0, \dots)\| \\
&\quad = \|(\gamma_j S_1^{-1} S_2^{-1} \dots S_{m_j}^{-1} y_{m_j} - e_0, \gamma_j S_2^{-1} S_3^{-1} \dots S_{m_j+1}^{-1} y_{m_j+1}, \dots), \dots)\| \to 0
\end{aligned}$$

After we take the first coordinate, we have,

$$\|\gamma_j S_1^{-1} S_2^{-1} \dots S_{m_j}^{-1} y_{m_j} - e_0\| \to 0.$$

From (c) and (d) of Example (2.2), $\|S_j S_{j-1} \dots S_1\| \leqslant 2^a$ and $S_j e_0 = e_0$ for every $j \geqslant 1$, So, we have

$$\begin{aligned}
\|\gamma_j y_{m_j} - e_0\| &= \|\gamma_j y_{m_j} - S_{m_j} \dots S_1 e_0\| \\
&= \|(S_{m_j} \dots S_1)(S_1^{-1} \dots S_{m_j}^{-1} \gamma_j y_{m_j} - e_0)\| \\
&\leqslant \|S_{m_j} \dots S_1\| \|S_1^{-1} \dots S_{m_j}^{-1} \gamma_j y_{m_j} - e_0\| \\
&\leqslant 2^a \|S_1^{-1} \dots S_{m_j}^{-1} \gamma_j y_{m_j} - e_0\| \\
&= 2^a \|\gamma_j S_1^{-1} \dots S_{m_j}^{-1} y_{m_j} - e_0\| \to 0.
\end{aligned}$$

Hence $\|\gamma_j y_{m_j}\| \to \|e_0\| = 1$, it implies that $\|y_{m_j}\| \geqslant 1$ and then $\|y\| \to \infty$. In other words, $y \notin \mathcal{H}$ which is a contradiction with $y \in \mathcal{H}$. Therefore, B is not a diskcyclic operator. □

After we complete the proof of this example, we get the proof of Theorem 2.3 as a desire and give a negative answer to Question 1.

One can ask what about the relation between the diskcyclic operators and ε-hypercyclic? In order to answer this question, we will construct an example of diskcyclic operator that is not ε-hypercyclic operator. This gives a negative answer to Question 2.

Example 2.3. *Let $F_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be the forward weighted shift operator with weight sequence*

$$\omega_n = \begin{cases} 2 \quad , \quad n \geqslant 0 \\ 4 \quad , \quad \text{otherwise.} \end{cases}$$

Then F_ω is diskcyclic but is not ε-hypercyclic operator.

Proof. By the criterion of diskcyclic operator in [14], we have F_ω is a diskcyclic operator. Let $\varepsilon \in (0, 1)$ be fixed. To show that F_ω is not ε-hypercyclic operator, we can suppose that F_ω is ε-hypercyclic operator on $\ell^2(\mathbb{Z})$. Then we have at least one ε-hypercyclic vector $x \in \ell^2(\mathbb{Z})$. From the behaviour of the operator T we have $\|F_\omega^n x\| > \|x\|$ for every $n > 0$. In particular, take the vector $y = \frac{1}{5}x$. So we have $\|y\| = \frac{1}{5}\|x\| < \|F_\omega^n x\|$ for $n > 0$ and then $\varepsilon\|y\| = \frac{\varepsilon}{5}\|x\|$. For every $n \in \mathbb{N}$, $\|F_\omega^n x - y\| = \|F_\omega^n x - \frac{1}{5}x\| \geqslant \|F_\omega^n x\| - \frac{1}{5}\|x\| > \|x\| - \frac{1}{5}\|x\| = \frac{4}{5}\|x\| > \frac{\varepsilon}{5}\|x\| = \varepsilon\|y\|$ which is a contradiction with ε-hypercyclicity of the vector x. Recall that x was an arbitrary ε-hypercyclic vector. Hence, F_ω hasn't any ε-hypercyclic vector. Therefore, F_ω is not ε-hypercyclic operator. □

Hilden and Wallen in [9] introduced the concept of supercyclic operators as the operator which has a vector whose scaled orbit is dense.

Definition 2.1. *[9]*
An operator $T \in \mathcal{B}(X)$ is called supercyclic if there exists a vector $x \in X$ such that the scaled orbit of x, $\mathbb{C}\mathrm{Orb}(T, x) = \{\gamma T^n x | n \geqslant 0; \gamma \in \mathbb{C}\}$ is dense in X. In this case, x is called a supercyclic vector for T.

Next, we show the relation between the supercyclic and ε-diskcyclic operators.

Example 2.4. *Let $F_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be the forward weighted shift operator with weight sequence*

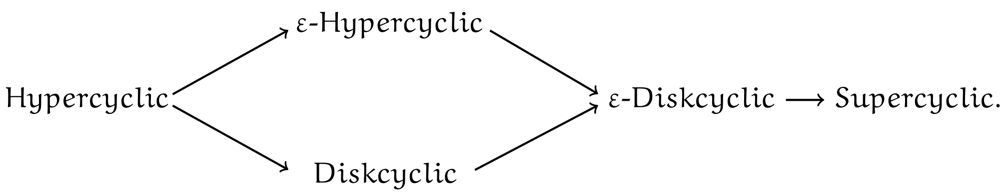
$$\omega_n = \begin{cases} \frac{1}{9} \quad n \geqslant 0 \\ \frac{1}{3} \quad \text{otherwise.} \end{cases}$$

Then F_ω is supercyclic but is not ε-diskcyclic operator.

Proof. By the criterion of supercyclic operator in [8], we have F_ω is supercyclic operator. Let $\varepsilon \in (0, 1)$ be fixed. To show that F_ω is not ε-diskcyclic operator, we can suppose that F_ω is ε-diskcyclic operator on $\ell^2(\mathbb{Z})$. Then we have at least one ε-diskcyclic vector $x \in \ell^2(\mathbb{Z})$. From the behaviour of the operator F_ω we have $\|F_\omega^n x\| < \|x\|$ for every $n > 0$. Thus, $|\gamma|\|F_\omega^n x\| \leqslant \|F_\omega^n x\| < \|x\|$, for every $\gamma \in \mathbb{D} \setminus \{0\}$. In particular, take the vector $y = \frac{1}{\varepsilon}x$. Thus, we have $\|y\| = \frac{1}{\varepsilon}\|x\| > \|x\| > \|\gamma F_\omega^n x\|$ for every $\gamma \in \mathbb{D} \setminus \{0\}$ and $n > 0$. For any $n \in \mathbb{N}$ and $\gamma \in \mathbb{D} \setminus \{0\}$, we have, $\|\gamma F_\omega^n x - y\| = \|\gamma F_\omega^n x - \frac{1}{\varepsilon}x\| \geqslant \frac{1}{\varepsilon}\|x\| - \|\gamma F_\omega^n x\| > \frac{1}{\varepsilon}\|x\| - \|x\| = (\frac{1}{\varepsilon} - 1)\|x\| > \|x\| = \varepsilon\|y\|$ which is a contradiction with ε-diskcyclicity of the vector x. Recall that x was an arbitrary ε-diskcyclic vector. Hence, T hasn't any ε-diskcyclic vector. Therefore, F_ω is not ε-diskcyclic operator. □

3. Conclusion

In conclusion: The ε-diskcyclicity phenomena lies in the midpoint between diskcyclicity and supercyclicity.



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