Available online at www.refaad.com
Gen.Lett. Math., 12 (1) (2022), 31-39
Research Article

General Letters in Mathematics (GLM)

Journal Homepage: https://www.refaad.com/Journal/Index/1
ISSN: 2519-9277 (Online) 2519-9269 (Print)

A New Accurate Approximate Solution of Singular Two-Point Boundary Value Problems

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Abstract

In this research paper, A new accurate solutions are obtained for classes of singular two-point boundary value problems (BVPs) using a new solution procedure based on the construction of the auxiliary functions of the standard optimal homotopy asymptotic method (OHAM). This procedure give us accurate approximate solutions using only one order of approximation compared with the solutions obtained previously by the standard OHAM approximation of three order. The obtained numerical results which are displayed in tables and plotted graphically in figures leads to concluded that this procedure is efficient and of third order reliable for finding the solutions of singular two BVPs.

Keywords: OHAM, Singular Two Point Boundary Value Problems, Series Solution
2010 MSC: MSC 34K28, 35A24, 35F30

1. Introduction

In the last years, a lot of interest for solving singular two BVPs has been taken under consideration [1, 2, 3, 4, 5]. A large wide of phenomena in biological and physical science such as oxygen diffusion in cell and tumor growth, isothermal gas sphere, thermal explosion are modelled by singular two BVPs [6, 7, 8]. The main goal for this study is to propose an effective solution procedure for solving singular two-BVPs of the following form:

$$\frac{1}{p}u''(x) + \frac{1}{q(x)}u'(x) + \frac{1}{r(x)} = g(x), \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

with it is boundary conditions

$$u(0) = \alpha_1, \quad u(1) = \beta, \quad \text{or} \quad u'(0) = \alpha_2, \quad u(1) = \beta, \tag{1.2}$$

where p, q, r and g are continuous functions on $(0, 1]$ and the parameters $\alpha_1, \alpha_2, \beta$ are real constants by using anew form of the auxiliary function in the construction of the homotopy equation of the standard OHAM. Obtaining exact solution of the singular two point boundary value problems of form (1.1) and (1.2) is difficult in most cases, Therefore, many numerical procedures were employed to find their

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doi:10.31559/glm2022.12.1.4

Received 12 Feb 2022 : Revised : 16 Apr 2022 Accepted: 22 Apr 2022

solutions such as Legendre reproducing kernel, continuous genetic algorithm, Differential transformation method, hybrid methods, block methods, optimisation technique, Sumudu transform with modified homotopy perturbation method [9, 10, 11, 12, 13]

In 2008, Marinca et al. proposed and developed a numerical method called optimal homotopy asymptotic method (OHAM) based on the homotopy analysis method (HAM) [14, 15, 16, 17] to become in more general form of HAM and also of the homotopy perturbation method (HPM) [18, 19], they are applying it successfully for obtaining approximate solutions of many nonlinear problems in a series of papers [20, 21, 22]. An advantage of OHAM is that it was built in similar criteria of HAM and it does not require to identify the ħ-curve for the purpose of the convergence as in HAM. OHAM offers a convenient method for controlling and adjusting the convergence region. Additionally, it was built using the same convergence criteria as HAM but with additional flexibility. This method has been applied successfully for the solutions of different types of differential equations and problems by many researchers during the last years [23, 24, 25, 26, 27, 28, 29, 30].

In this research article, a new form of OHAM approximate solutions procedure is employed Based on the creation of the homotopy equation, a new type of OHAM approximation solutions approach is used, this form of the solutions enable us to obtain accurate results using only one term compared by the results obtained previously by same method but of three order of approximate and by other methods in literature.

This paper is formulated and structured in four Sections, The analysis of the presented procedure described in Section 2, numerical examples are illustrated in Section 3, results and dissections in Section 4, and the conclusions of this study in the last section.

2. OHAM Procedure

Examine the following equation to demonstrate the fundamental concept of OHAM [3].

$$L(u(x)) + g(x) + N(u(x)) = 0, \quad B\left(u, \frac{du}{dx}\right) = 0, \tag{2.1}$$

where L and N are the linear and nonlinear operators, $u(x)$ is the dependent variable, x denotes to the independent variable, $g(x)$ is the dependent variable and B is a boundary operator.

Now, we construct homotopy equation of the following form $h(\nu(x,p),p) : R \times [0,1] \to R$ which satisfies

$$(1-p)[L(\nu(x,p)) - u_0(x)] = H(p)[L(\nu(x,p)) + g(x) + N(\nu(x,p))], \tag{2.2}$$
$$B\left(\nu(x,p), \frac{\partial \nu(x,p)}{\partial x}\right) = 0, \tag{2.3}$$

where $x \in R$ and $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$ and $\nu(x,p)$ is the dependent variable. Obviously, when $p = 0$,we have $\nu(x,0) = u_0(x)$ and $p = 1$ we get $\nu(x,1) = u(x)$.

Therefore, the initial guess that satisfies the linear operator and the boundary conditions is given by the solution $\nu(x,p)$, which approaches from $u_0(x)$ to $u(x)$ as p varies from 0 to 1

$$L(u_0(x)) = 0, \quad B\left(u_0, \frac{du_0}{dx}\right) = 0. \tag{2.4}$$

After that, we choose the auxiliary function $H(p)$ in below form

$$H(p) = \sum_{k=1}^{m} (c_m x^{m-1}) q, \tag{2.5}$$

where the convergent control parameters $C_1, C_2, C_3, \ldots$ are evaluated afterwards.

To find the first order OHAM approximate solution, we use Taylor's series to expand $v(x, p, C_i)$ about p in the following manner,

$$v(x,p,\ C_i) = \sum_{k=0}^{1} u_k(x, C_1, C_2, \ldots, C_k)p^k. \tag{2.6}$$

Substituting (2.6) into (2.1) and equating the coefficient of like powers of p, we obtain the following linear equations.

$$L(u_1(x)) + g(x) = C_1 N_0(u_0(x)), \quad B\left(u_1, \frac{du_1}{dx}\right) = 0, \tag{2.7}$$

The m'th-order OHAM approximation is

$$\tilde{u}(x, C_1, C_2, C_3, \ldots, C_m) = \sum_{i=1}^{m} u_i(x, C_1, C_2, \ldots, C_i). \tag{2.8}$$

The residual produced by replacing (2.8) into (2.1) is as follows.

$$\begin{aligned} R\left(x, C_1, C_2, C_3, \ldots, C_m\right) &= L(\tilde{u}\left(x, C_1, C_2, C_3, \ldots, C_m\right)) + g\left(x\right) \\ &\quad + N(\tilde{u}\left(x, C_1, C_2, C_3, \ldots, C_m\right)). \end{aligned} \tag{2.9}$$

$\tilde{u}$ will be the exact solution if $R = 0$. Since nonlinear problems often do not involve such a scenario, we minimize the functional in the following way.

$$J(C_1, C_2, C_3, \ldots, C_m) = \int_a^b R^2(x, C_1, C_2, C_3, \ldots, C_m)dx, \tag{2.10}$$

where a and b are the endpoints of the given problem. The unknown convergent control parameters C_i $(i = 1, 2, 3, \ldots, m)$ can identified from the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_m} = 0. \tag{2.11}$$

3. Numerical examples

In order to prove and demonstrate the efficiency and capability of the the presented procedure, three examples are considered.

3.1. *Example 1*

Consider the following BVP with two points in second order [3, 31].

$$\begin{aligned} & u''(x) + \frac{2}{x}u'(x) - 4u(x) = -2, \quad 0 < x \leqslant 1, \\ & u'(0) = 0, \quad u(1) = 5.5. \end{aligned} \tag{3.1}$$

We define the linear and nonlinear operators as follows in order to solve this problem using the OHAM approach outlined in the previous section.

$$L[v(x,p)] = \frac{d^2v(x,p)}{dx^2} \tag{3.2}$$

$$N[v(x,p)] = \frac{d^2v(x,p)}{dx^2} + \frac{2}{x}\frac{dv(x,p)}{dx} - 4v(x,p) + 2. \tag{3.3}$$

Then, the zeroth-order problem is given by:

$$u_0''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1, \tag{3.4}$$

and the first-order problem is:

$$\begin{aligned} u_1''(x) &= -20c_{10}x^9 - 20c_9x^8 - 20c_8x^7 - 20.c_7x^6 - 20c_6x^5 - 20c_5x^4 \\ &\quad -20c_4x^3 - 20c_3x^2 - 20c_2x - 20c_1, \quad u_1'(0) = 0,\ u_1(1) = 0, \end{aligned} \tag{3.5}$$

Substituting the solutions of Eqs. (3.4) and (3.5) into Eq. (2.8),when $m = 1$, yields to the following first-order OHAM approximate solution

$$\begin{aligned} \tilde{u}(x) &= -0.181818c_{10}x^{11} - 0.222222c_9x^{10} - 0.277778c_8x^9 - 0.357143c_7x^8 \\ &\quad -0.47619c_6x^7 - 0.666667c_5x^6 - 1.c_4x^5 - 1.66667c_3x^4 - 3.33333c_2x^3 - 10.c_1x^2 \\ &\quad +10.c_1 + 3.33333c_2 + 1.66667c_3 + 1.c_4 + 0.666667c_5 + 0.47619c_6 + 0.357143c_7 \\ &\quad +0.277778c_8 + 0.222222c_9 + 0.181818c_{10} + 5.5 \end{aligned} \tag{3.6}$$

Based on Eq. (2.9), the residual error is

$$\begin{aligned} R &= 0.727273c_{10}x^{11} + 0.888889c_9x^{10} + 1.11111c_8x^9 - 24.c_{10}x^9 + 1.42857c_7x^8 \\ &\quad -24.4444c_9x^8 + 1.90476c_6x^7 - 25.c_8x^7 + 2.66667c_5x^6 - 25.7143c_7x^6 + 4.c_4x^5 \\ &\quad -26.6667c_6x^5 + 6.66667c_3x^4 - 28.c_5x^4 + 13.3333c_2x^3 - 30.c_4x^3 + 40.c_1x^2 \\ &\quad -33.3333c_3x^2 - 40.c_2x - 100.c_1 - 13.3333c_2 - 6.66667c_3 - 4.c_4 - 2.66667c_5 \\ &\quad -1.90476c_6 - 1.42857c_7 - 1.11111c_8 - 0.888889c_9 - 0.727273c_{10} - 20. \end{aligned} \tag{3.7}$$

The values of the convergent control parameters $C_i's, i = 1, 2, 3, ...$, we will be determined by using the following formula

$$J = \int_0^1 R^2 dx \tag{3.8}$$

and then by minimizing it as

$$\frac{dJ}{dc_1} = \frac{dJ}{dc_2} = \cdots = \frac{dJ}{dc_{10}}.$$

we obtain, the following optimal values of C_i's

$$\begin{aligned} &c_1 = -0.183814, \quad c_2 = 3.566403891625021 \times 10^{-9}, \quad c_3 = -0.220577, \\ &c_4 = 9.8145 \times 10^{-9}c_5 - 0.0525148, \quad c_6 = -0.0000231278, \quad c_7 = -0.0053736, \\ &c_8 = -0.000125626, \quad c_9 = -0.000196294, \quad c_{10} = -0.0000615801. \end{aligned}$$

Using these values into Eq. (3.6), the new OHAM approximate solution of first order will be

$$\begin{aligned} \tilde{u}(x) &= 0.0000111964x^{11} + 0.0000436209x^{10} + 0.0000348961x^9 + 0.00191914x^8 \\ &\quad +0.0000110133x^7 + 0.0350099x^6 - 9.814498240293585 \times 10^{-9}x^5 \\ &\quad +0.367628x^4 - 1.1888012972083402 \times 10^{-8}x^3 + 1.83814x^2 + 3.25721. \end{aligned}$$

Which is very agree to the exact solution

$$u(x) = 0.5 + \frac{5Sinh2x}{xSinh(2)}. \tag{3.9}$$

Table 1: Comparison of exact solution and OHAM solution for Example 1.

x	Exact Solution	Prsented Method Absolute Error	Standered OHAM [3] Absolute Error	Cubic Spline [31] Absolute Error
0.001	3.25721	8.88×10^{-16}	1.84×10^{-6}	7.4×10^{-5}
0.200	3.33132	2.93×10^{-13}	7.65×10^{-5}	7.3×10^{-5}
0.400	3.56086	2.29×10^{-12}	8.36×10^{-5}	6.5×10^{-5}
0.600	3.96825	3.19×10^{-12}	1.71×10^{-5}	5.4×10^{-5}
0.800	4.59371	1.51×10^{-12}	4.61×10^{-5}	3.9×10^{-5}
1.000	5.50000	0.00	1.77636×10^{-15}	0

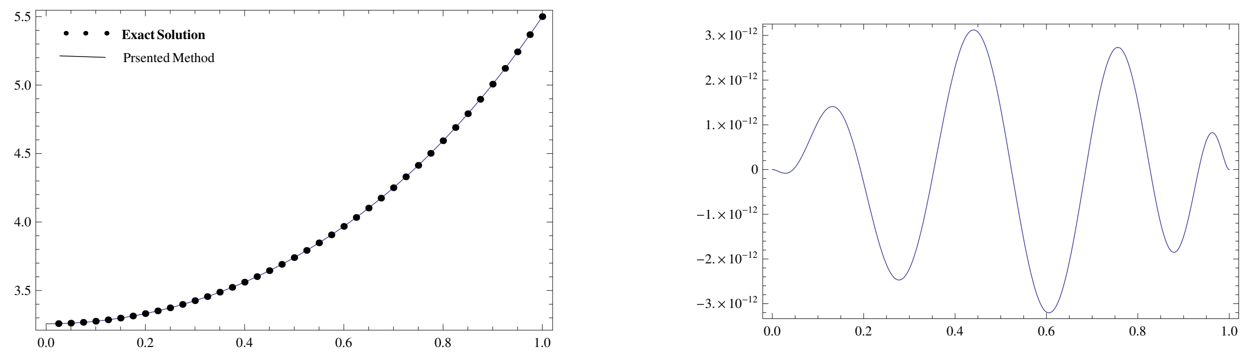


Figure 1: (a) Plot of the Exact and approximate Solution (b) Absolute Errors Related to Example 1

3.2. *Example 2*

Let us consider the singular two-point BVP [3],

$$u''(x) + \frac{1}{x}u'(x) + u(x) = \frac{5}{4} + \frac{x^2}{16}, \quad 0 \leqslant x \leqslant 1, \tag{3.10}$$

$$u'(0) = 0,\ u(1) = \frac{17}{16}. \tag{3.11}$$

According to the OHAM formulation described in above section, we start with

$$L[v(x,p)] = x\frac{d^2v(x,p)}{dx^2} \tag{3.12}$$

$$N[v(x,p)] = x\frac{d^2v(x,p)}{dx^2} + \frac{dv(x,p)}{dx} + xv(x,p) - xg(x). \tag{3.13}$$

To obtain approximate solution of first order using OHAM prcedure, we have the following zeroth and first order problem:

$$u_0''(x) = 0, \quad u_0'(0) = 0, \quad u_0(1) = \frac{17}{16}. \tag{3.14}$$

$$\begin{aligned} u_1''(x) = & -\frac{1}{16}c_7x^8 - \frac{c_6x^7}{16} - \frac{c_5x^6}{16} - \frac{3c_7x^6}{16} - \frac{c_4x^5}{16} - \frac{3c_6x^5}{16} - \frac{c_3x^4}{16} \\ & -\frac{3c_5x^4}{16} - \frac{c_2x^3}{16} - \frac{3c_4x^3}{16} - \frac{c_1x^2}{16} - \frac{3c_3x^2}{16} - \frac{3c_2x}{16} - \frac{3c_1}{16}, \\ & u_0'(0) = 0, \quad u_0(1) = 0. \end{aligned} \tag{3.15}$$

By substituting the solutions of the above order problems into Eq. (2.8), when $m = 1$, The first order OHAM approximation is obtained

$$\begin{aligned}\tilde{u}(x) &= -\frac{c_7x^{10}}{1440}-\frac{c_6x^9}{1152}-\frac{c_5x^8}{896}-\frac{3c_7x^8}{896}-\frac{c_6x^7}{224}-\frac{c_4x^7}{672}-\frac{c_5x^6}{160}\\&-\frac{c_3x^6}{480}-\frac{c_2x^5}{320}-\frac{3c_4x^5}{320}-\frac{c_3x^4}{64}-\frac{c_1x^4}{192}-\frac{c_2x^3}{32}-\frac{3c_1x^2}{32}\\&+\frac{19c_1}{192}+\frac{11c_2}{320}+\frac{17c_3}{960}+\frac{73c_4}{6720}+\frac{33c_5}{4480}+\frac{43c_6}{8064}+\frac{163c_7}{40320}+\frac{17}{16}\end{aligned} \tag{3.16}$$

Based on Eq. (2.9), the residual error is

$$\begin{aligned}R &= -\frac{c_6x^9}{1152}-\frac{c_5x^8}{896}-\frac{c_4x^7}{672}-\frac{67c_6x^7}{896}-\frac{c_3x^6}{480}-\frac{87c_5x^6}{1120}-\frac{7c_6x^5}{32}-\frac{c_2x^5}{320}-\frac{79c_4x^5}{960}\\&-\frac{9c_5x^4}{40}-\frac{c_1x^4}{192}-\frac{29c_3x^4}{320}-\frac{7c_2x^3}{64}-\frac{15c_4x^3}{64}-\frac{c_3x^2}{4}-\frac{17c_1x^2}{96}-\frac{9c_2x}{32}+\frac{11c_2}{320}\\&+\frac{17c_3}{960}+\frac{73c_4}{6720}+\frac{33c_5}{4480}+\frac{43c_6}{8064}-\frac{53c_1}{192}-\frac{x^2}{16}-\frac{3}{16}.\end{aligned} \tag{3.17}$$

The values of the convergent control parameters $C_i's, i = 1, 2, 3, ...$, we will be determined by using the following formula

$$J = \int_0^1 R^2 dx \tag{3.18}$$

and then by minimizing it as

$$\frac{dJ}{dc_1} = \frac{dJ}{dc_2} = \cdots = \frac{dJ}{dc_{10}}.$$

we obtain, the following optimal values of C_i's

$$c_1 = -0.666665, \quad c_2 = -0.0000762973, \quad c_3 = 0.222567, \quad c_4 = 0.00176739,$$
$$c_5 = -0.0879007, \quad c_6 = 0.0303148$$

By taking into account these parameters our approximate solution of order one becomes,

$$\begin{aligned}\tilde{u}(x, C_1, C_2, C_3) &= -0.0000263149x^9 + 0.0000981035x^8 - 0.000137964x^7 + 0.000085699x^6\\&-0.0000163308x^5 - 5.392707410616688 \times 10^{-6}x^4\\&+2.3842894727431695 \times 10^{-6}x^3 + 0.0624998x^2 + 1.\end{aligned}$$

which is more accurate than the solutions obtained by other methods in literature compared with exact solution $u(x) = 1 + \frac{x^2}{16}$ and also with eight order of the HPM approximate solution obtained by us since it a special case of OHAM as stated previously.

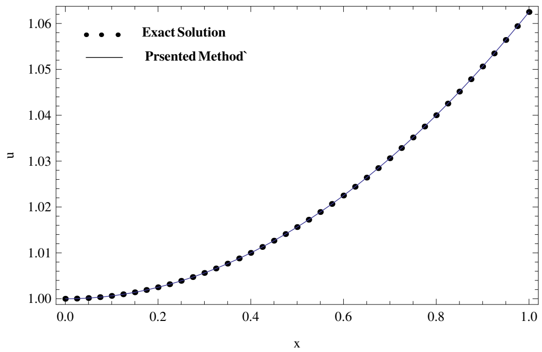
3.3. *Example 3*

Examine the singular two-point BVP [3, 33]

$$\begin{aligned}\left(1-\frac{x}{2}\right)u''(x)+\frac{3}{2}\left(\frac{1}{x}-1\right)u'(x)+\left(\frac{x}{2}-1\right)u(x) &= 5-\frac{29x}{2}+\frac{13x^2}{2}+\frac{3x^3}{2}-\frac{x^4}{2},\\ 0 \leqslant x \leqslant 1, \quad u'(0) = 0, \quad u(1) = 0.\end{aligned} \tag{3.19}$$

Table 2: comparison of exact solution and OHAM solution for Example 2.

x	Prsented Method Absolute Error	HPM Absolute Error	OHAM Absolute[3] Error	VIM solution [32] Absolute Error
0	6.08×10^{-10}	1.69×10^{-4}	5.45×10^{-7}	1.35×10^{-1}
0.2	1.22×10^{-9}	1.63×10^{-4}	4.67×10^{-7}	1.30×10^{-1}
0.4	3.09×10^{-10}	1.44×10^{-4}	2.49×10^{-7}	1.14×10^{-1}
0.6	1.68×10^{-9}	1.12×10^{-4}	1.01×10^{-7}	8.79×10^{-2}
0.8	1.87×10^{-9}	1.45×10^{-5}	6.14×10^{-8}	4.10×10^{-2}
1.0	2.22×10^{-16}	0	3.62×10^{-8}	0



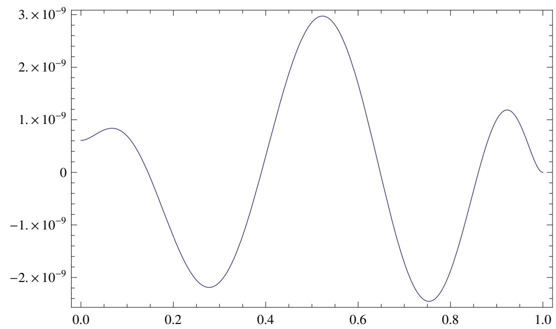


Figure 2: (a) Plot of the Exact and approximate Solution (b) Absolute Errors Related to Example 2

The following first order OHAM estimated solution is obtained by applying the same process used on the examples above

$$\begin{aligned}\tilde{u}(x) &= 5.27934 \times 10^{-11} + 0.999979x^2 - 0.997823x^3 - 0.0675558x^4 + 1.00927x^5 \\ &-8.74243x^6 + 48.3335x^7 - 180.225x^8 + 467.697x^9 - 856.702x^{10} + 1105.24x^{11} \\ &-980.651x^{12} + 559.784x^{13} - 165.509x^{14} - 7.17876x^{15} + 22.9015x^{16} \\ &-6.24672x^{17} + 0.27363x^{18} + 0.0852075x^{19} - 0.00686428x^{20}.\end{aligned}$$

which is accurate and agree very will with the exact solution $u(x) = x^2 - x^3$.

Table 3: comparison of exact solution and OHAM solution for Example 3.

x	Prsented Method Absolute Error	OHAM Absolute Error [3]	Sinc error [33] Absolute Error	He's HPM [33] Error
0	5.28×10^{-11}	9.841×10^{-4}	2.71×10^{-3}	8.936×10^{-4}
0.2	2.15×10^{-8}	2.801×10^{-3}	2.19×10^{-4}	8.631×10^{-4}
0.4	5.51×10^{-8}	1.831×10^{-3}	8.83×10^{-4}	7.408×10^{-4}
0.6	1.02×10^{-7}	2.707×10^{-4}	4.90×10^{-4}	5.435×10^{-4}
0.8	2.73×10^{-8}	9.266×10^{-5}	4.01×10^{-4}	2.878×10^{-4}
0	2.27×10^{-13}	1.917×10^{-4}	1.57×10^{-4}	1.467×10^{-9}

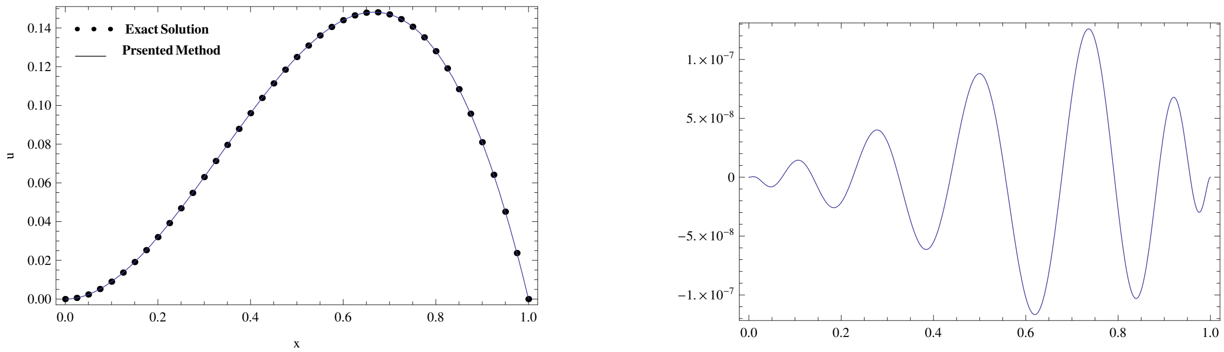


Figure 3: (a) Plot Exact and approximate Solution (b) Absolute Errors Related to Example 3

4. Results and Dissctions

Numerical results are obtained and presented in tables and graphically plotted also in figures, from that, it is clearly obvious that the new form of the OHAM approximate solution of first order is more accurate than that ones obtained by the other methods in literature which needs several terms to get better approximate solution with low remarkable error and also more accurate than the pervious standard OHAM solution of three order [3].

5. Conclusions

In this research article, a reliable procedure based on the standard OHAM has been applied successfully for obtaining more accurate approximate solutions of different classes of singular two point boundary value problems. The new results which are obtained leads to conclude that the presented procedure has the ability to control the convergent of the series solutions and acquire high accurate results with less computational work and a low remarkable error compared with previous OHAM solutions and other method in literature which is obviously clare in the numerical results displayed in tables and plotted graphically in Figures.

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