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On Nano S_β-Open Sets In Nano Topological Spaces

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Abstract

The aim of this work is to introduce a new class of nano semi-open sets in nano topological spaces which called nano S_β-open sets in nano topological spaces. Also, we study the relationship between some types of nano near open sets with this new class. The class of nano S_β-open sets exactly lie between nano S_p-open sets and nano semi-open sets.

Keywords: Nano Topology, Nano semi-open sets, Nano β-open sets, Nano S_β-open sets.
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1. Introduction

The notion of nano topological space (briefly NTS) introduced by Thivagar and Carmel [1] with respect to a subset X of a universe; which is defined in terms of lower and upper approximations. Levine [2] introduced the notions of semi-open. Abd El-Monsef [3] introduced the notion of β-open sets in topology. Later, nano semi-open sets introduced by Thivagar Carmel [1], also nano β-open sets introduced by Revathy and lango [4]. During this work, we introduce the concept nano S_β-open sets as a strong form of nano semi-open sets, since every nano S_β-open sets is nano semi-open sets and the relationship with some class of nano near open sets. Various forms of family of nano S_β-open sets under various cases of approximations idea also derived.

2. Preliminaries

Definition 2.1. [5] A non-empty finite set U of objects called the universe and R is an equivalence relation on the set U named the indiscernibility relation. Elements which in the same equivalence class are called indiscernible with one another. The couple (U, R) is said to be the approximation space. Let $X \subseteq U$:

1. The lower approximation of X with respect to R denoted by $L_R(X)$ and defined by $L_R(X) = \bigcup_{x \in U} \{R(x) ; R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R denoted by $U_R(X)$ and defined by $U_R(X) = \bigcup_{x \in U} \{R(x) ; R(x) \bigcap X \neq \phi\}$.

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3. The boundary region of X with respect to R denoted by $B_R(X)$ and defined by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [1] Suppose that U denote the universe and R is an equivalence relation on U and $\tau_R(X) = \{\phi, U, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then the following axioms hold for $\tau_R(X)$:

1. U and $\phi \in \tau_R(X)$
2. The union of members of $\tau_R(X)$ is in $\tau_R(X)$.
3. The finite intersection of members of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a Topology on U and named the Nano Topology on the set U with respect to X. The pair $(U, \tau_R(X))$ called the NTS. The members of $\tau_R(X)$ are called nano open sets and $[\tau_R(X)]^c$ is called the nano dual topology of $\tau_R(X)$.

Definition 2.3. [4]
$nint(A) = \cup\{G; G \in \tau_R(X) \text{ and } G \subseteq A\}$
$ncl(A) = \cap\{F; F \in [\tau_R(X)]^c \text{ and } A \subseteq F\}$

Definition 2.4. Let $(U, \tau_R(X))$ be a NTS and $M \subseteq U$. The set M is said to be Nano:

1. regular-open [1], if $M = nint(ncl(M))$.
2. α-open [1], if $M \subseteq nint(ncl(nint((M))))$.
3. semi-open [1], if $M \subseteq ncl(nint(M))$.
4. pre-open [1], if $M \subseteq nint(ncl(M))$.
5. γ-open [1], if $M \subseteq nint(ncl(M)) \cup ncl(nint(M))$
6. β- open (nano semi pre-open) [4], if $M \subseteq ncl(nint(ncl(M)))$.
7. S_P-open [6], if M is nano semi-open and $M = \cup\{F_\alpha;\ F_\alpha \text{ is nano pre-closed set}\}$.
8. $\delta\beta$-open [7], if $M \subseteq ncl\left(nint\left(ncl^\delta(M)\right)\right)$.
9. θ-open [8], if for each $x \in M$, there exists a nano open set G such that $x \in G \subseteq ncl(M) \subseteq M$.

The set of all Nano regular-open (resp. Nano α-open, Nano semi-open, Nano pre-open, Nano γ-open, Nano β-open, Nano θ-open, Nano S_P-open and Nano $\delta\beta$-open) sets denoted by $nRO(U, X)$ (resp. $n\alpha O(U, X)$, $nSO(U, X)$, $nPO(U, X)$, $n\gamma O(U, X)$, $n\beta O(U, X)$, $n\theta O(U, X)$, $nS_PO(U, X)$ and $n\delta\beta O(U, X)$).

Definition 2.5. [6] A NTS $(U, \tau_R(X))$ is said to be:

1. Nano locally indiscrete space, if every $nano$-open set is nano closed.
2. A topological space is called extremally disconnected if the closure of any open subset is still an open subset.
3. Nano clopen if $nint(A) = ncl(A)$.

Theorem 2.6. *[1] Let $(U, \tau_R(X))$ be a NTS. if $U_R(X) = U$ and $L_R(X) \neq \phi$, then ϕ, U, $L_R(X)$ and $B_R(X)$ are the only nS-open sets in a NTS U.*

Theorem 2.7. *[1] Let $(U, \tau_R(X))$ be a NTS. if $U_R(X) = U$ and $L_R(X) \neq \phi$, then ϕ, U, $L_R(X)$ and $B_R(X)$ are the only $n\alpha$-open sets in a NTS U.*

Theorem 2.8. *[1] Let $(U, \tau_R(X))$ be a NTS. If $U_R(X) \neq U$, $L_R(X) = \phi$, then ϕ and those sets A for which $U_R(X) \subseteq A$ are the only $n\alpha$-open sets in a NTS U.*

Theorem 2.9. *[1] Let $(U, \tau_R(X))$ be a NTS. If $U_R(X) = L_R(X) \neq U$, then ϕ and those sets A for which $U_R(X) \subseteq A$ are the only $n\alpha$-open sets in a NTS U.*

Theorem 2.10. *[1] Let $(U, \tau_R(X))$ be a NTS. If $U_R(X) \neq U$ and $L_R(X) = \phi$, then ϕ and those sets A for which $U_R(X) \subseteq A$ are the only nS-open sets in a NTS U.*

Theorem 2.11. *[1] Let* $(U, \tau_R(X))$ *be a* NTS. *If* $U_R(X) = L_R(X) \neq U$, *then* ϕ *and those sets* A *for which* $L_R(X) \subseteq A$ *are the only* nS*-open sets in a* NTS U.

Theorem 2.12. *[1] Let* $(U, \tau_R(X))$ *be a* NTS. *If* $U_R(X) \neq L_R(X)$ *where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$, *then* ϕ, $L_R(X), B_R(X)$ *and any set containing* $U_R(X)$, $L_R(X) \cup W$ and $B_R(X) \cup W$ *where* $W \subseteq [U_R(X)]^c$ *are the only* nS*-open sets in a* NTS U.

Theorem 2.13. *[1] The union of any two* nS*-open sets in* U *is* nS*-open sets in a* NTS U.

Theorem 2.14. *[4] In a* NTS $(U, \tau_R(X))$, *the following statements are true:*

1. *Every* nano*-open set in* $(U, \tau_R(X))$ *is* $n\beta$*-open set in a* NTS U.
2. *Every* nS*-open set in* $(U, \tau_R(X))$ *is* $n\beta$*-open set in a* NTS U.
3. *Every* nP*-open set in* $(U, \tau_R(X))$ *is* $n\beta$*-open set in a* NTS U.
4. *Every* $n\alpha$*-open set in* $(U, \tau_R(X))$ *is* $n\beta$*-open set in a* NTS U.
5. *Every* nR*-open set in* $(U, \tau_R(X))$ *is* $n\beta$*-open set in a* NTS U.

Theorem 2.15. *[4] The union of any two* $n\beta$*-open sets is* $n\beta$*-open set in a* NTS U.

Theorem 2.16. *[4] If* $U_R(X) \neq U$, *then* ϕ, U *and any set which intersect* $U_R(X)$ *are* $n\beta$*-open set in a* NTS U.

Theorem 2.17. *[4] If* $U_R(X) = U$ *in a* NTS, *then* $n\beta O(X)$ *is* $P(U)$.

Theorem 2.18. *Let* $(U, \tau_R(X))$ *be a* NTS *when* $U_R(X) = L_R(X) \neq U$ *and* $U_R(X) = \{x\}$, $x \in U$, *then* ϕ, $U \neq A \in n\beta C(U, X)$ *if and only if* $U_R(X) \cap A = \phi$.

Proof. Let ϕ, $U \neq A \in n\beta C(U, X)$ if and only if A^c is $n\beta$-open set if and only if $A^c \cap U_R(X) \neq \phi$, that is A^c containing x, since $U_R(X) = \{x\}$, so that $A \cap U_R(X) = \phi$. □

Theorem 2.19. *[6] Let* $(U, \tau_R(X))$ *be a nano topological space, then the following statements hold:*

1. *If* $M \in n\theta O(U, X) \Rightarrow M \in n\delta O(U, X)$.
2. *if* $M \in nRC(U, X) \Rightarrow M \in nSO(U, X) \cap nPO(U, X)$.
3. *If* $M \in n\theta O(U, X) \Rightarrow M \in nS_pO(U, X)$.
4. *If* $M \in n\theta O(U, X) \cap nSO(U, X) \Rightarrow M \in nS_pO(U, X)$.
5. *If* $M \in nRC(U, X) \Rightarrow M \in nS_pO(U, X)$.
6. *If* $M \in nS_pO(U, X) \Rightarrow M \in nSO(U, X)$.

Theorem 2.20. *[7] Every* $n\beta$*-open is* $n\delta\beta$*-open.*

3. Nano S_β-Open Sets

Definition 3.1. A nano *semi*-open set A of a NTS $(U, \tau_R(X))$ is said to be nano S_β-open set, if for each $x \in A$, there exist a nano β-closed set F such that $x \in F \subseteq A$. The set of all nano S_β-open sets denoted by $nS_\beta O(U, X)$.

Definition 3.2. The complement of nS_β-open sets are called nano S_β-closed sets. The set of all nano S_β-closed sets denoted by $nS_\beta C(U, X)$.

Proposition 3.3. *If* $A \in nS_\beta O(U, X)$, *then* $A \in nSO(U, X)$.

Proof. Follows from Definition 3.1 □

Remark 3.4. The Proposition 3.3 Shows that every nS_β-open set is nS-open set, but the converse may no be true general, as it shown in the next example.

Example 3.5. *Let* $U=\{a, b, c\}$ with $U/R=\{\{a\}, \{b, c\}\}$ *and* $X=\{a\}$*, then* $\tau_R(X)=\{\phi, U, \{a\}\}$ *and* $nS_\beta O(U,X)=\{\phi, U\}$*. Consider the set* $\{a\}\in nSO(U,X)$ *but not in* $nS_\beta O(U,X)$*.*

Proposition 3.6. *A subset* A *of a* NTS $(U, \tau_R(X))$ *is* nS_β*-open set if and only if* A *is* nS*-open and the union of* $n\beta$*-closed sets in a* NTS*.*

Proof. Follows from Definition 3.1. □

Proposition 3.7. *If* $\{A_i; i\in\Delta\}$ *is a family of* nS_β*-open sets in a* NTS $(U, \tau_R(X))$*, then* $\cup\{A_i; i\in\Delta\}$ *is also* nS_β*-open set.*

Proof. Let $\{A_i; i\in\Delta\}$ be a family of nS_β-open sets, by Proposition 3.3 and Theorem 2.13, the for each $x\in\cup A_i\subseteq nSO(U, X)$, there exists $n\beta$-closed set such that $x\in F\subseteq A_i\subseteq \cup A_i$. This implies that $x\in F\subseteq \cup A_i$. Therefore, $\cup\{A_i; i\in\Delta\}$ is also nS_β-open set. □

Remark 3.8. The intersection of two nS_β-open sets in a NTS U, may not be nS_β-open set in general, as it shown in the next example.

Example 3.9. *Let* $U=\{a, b, c, d\}$ *with* $U/R=\{\{a, b\}, \{c\}, \{d\}\}$ *and* $X=\{a, c\}$*.*
Then $\tau_R(X)=\{\phi, U, \{c\}, \{a, b, c\}, \{a, b\}\}$ *and* $nS_\beta O(U,X)=\{\phi, U, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$*, then* $\{c, d\}$ *and* $\{a, b, d\}$ *are* nS_β*-open sets, but* $\{c, d\}\cap\{a, b, d\}=\{d\}$ *which is not* nS_β*-open set in* U*.*

Remark 3.10. Class of $nS_\beta O(U, X)$ is supra topology on U.

Theorem 3.11. *If* $A_1, A_2\in nS_\beta O(U,X)$ *and* $nSO(U,X)$ *forms a nano topology on* U*, then* $A_1\cap A_2\in nS_\beta O(U,X)$ *and* $nS_\beta O(U,X)$ *forms a nano topology on* U*.*

Proof. Suppose that $A_1, A_2\in nS_\beta O(U,X)$ and $nSO(U,X)$ forms a nano topology on U. By Proposition 3.3, A_1, A_2 are in $nSO(U,X)$ and $nSO(U,X)$ forms a nano topology, then $A_1\cap A_2\in nSO(U,X)$. Let $x\in A_1\cap A_2$, then $x\in A_1$ and $x\in A_2$. So there exist $n\beta$-closed sets F_1 and F_2 such that $x\in F_1\subseteq A_1$ and $x\in F_1\subseteq A_2$. Hence $x\in F_1\cap F_2$, and by Theorem 2.15, the intersection of two $n\beta$-closed sets is $n\beta$-closed set, it follows that $A_1\cap A_2$ is nS_β-open. Therefore, the collection of $n\beta$-open sets form a nano topology on U. □

Remark 3.12. The concept of nano-open sets and nS_β-open sets are independent in general. From Example 3.5, $\{a\}\in\tau_R(X)$, but $\{a\}\notin nS_\beta O(U,X)$. Also, form Example 3.9, $\{c, d\}\in nS_\beta O(U,X)$, but $\{c, d\}\notin\tau_R(X)$.

Theorem 3.13. *Let* $(U, \tau_R(X))$ *be a* NTS*, then the following statements are true:*

1. *If* $A\in nS_pO(U, X)\Rightarrow A\in nS_\beta O(U, X)$.
2. *If* $A\in n\theta O(U, X)\Rightarrow A\in nS_\beta O(U, X)$.
3. *If* $A\in nRC(U, X)\Rightarrow A\in nS_\beta O(U, X)$.
4. *If* $A\in nRO(U, X)\Rightarrow A\in nS_\beta C(U, X)$.
5. *If* $A\in n\theta O(U, X)\cap nSO(U, X)\Rightarrow A\in nS_\beta O(U, X)$.

Proof.

1. Suppose that A is nS_P-open set, then there exists a nP-closed set F such that $F\subseteq A$ for each x in A. By Theorem 2.14 (3), A is $n\beta$-closed set such that $F\subseteq A$ for each x in A. Hence A is nS_β-open set.
2. Suppose that A is $n\theta$-open set, then by Theorem 2.19 (3), F is also nS_p-open set, by Part (1), A is nS_β-open set.
3. Suppose that A is nR-closed set, that is $A = ncl(nint(A))$, but $A\subseteq ncl(nint(A))$, it is implies that A is nS-open set. The set A is nR-closed, then A^C is nR-open set, by Theorem 2.14 (4), we get A^C is $n\beta$-open set, that is A is $n\beta$-closed set. Now, A is nS-open set and also $n\beta$-closed. Thus, A is nS_β-open set.
4. It follows form part (3).

5. Suppose that $A \in n\theta O(U, X) \cap nSO(U, X)$, then by Theorem 2.19 (5), A is nS_p-open set, then by Part (1), A is nS_β-open set.

□

Remark 3.14. The converse of each part of above Theorem 3.13 may not be true in general, as it shown in the next example.

Example 3.15. *Let* $U=\{a, b, c, d\}$ *with* $U/R=\{\{a\}, \{b, c\}, \{d\}\}$ *and* $X=\{a, b\}$.
Then $\tau_R(X)=\{\phi, U, \{a\}, \{b, c\}\ \{a, b, c\}\}$, $n\theta O(U, X)=\{\phi, U\}$, $nRO(U, X)=\{\phi, U, \{a\}, \{b, c\}\}$, $nS_P O(X)=\{\phi, U, \{a, d\}, \{b, c\}, \{b, c, d\}\}$ *and* $nS_\beta O(X)=nSO(U, X)$. *Take* $A=\{a\}$, *then* A *is* nS_β*-open but not* nS_P*-open set. Take* $B=\{a, d\}$, *then* B *is* nS_β*-open set but* $\{a, d\} \notin n\theta O(U, X)$. *Take* $C=\{b, c\}$, *then* C *is* nS_β*-open but not* nR*-closed in* U. *Take* $D=\{a, d\}$, *then* D *is* nS_β*-open set but* $\{a, d\} \notin n\theta O(U, X) \cap nSO(U, X)$.

Proposition 3.16. *If a* NTS $(U, \tau_R(X))$ *is locally indiscrete, then every* nS*-open set is* nS_β*-open set.*

Proof. Let A be a nS-open set in U, then $A \subseteq nint(ncl(A))$. Since U is locally indiscrete, then $nint(A)$ is nano closed set and hence $nint(A) = ncl(A)$ which it means that A is nR-closed in U. Therefore, by Theorem 3.13 (1), A is nS_β-open set. □

Remark 3.17. Let $A \subseteq U$. If $A \in nS_\beta O(U, X)$ and A is the union of nP-closed sets, then $A \in nS_p O(U, X)$.

Theorem 3.18. *Let* $(U, \tau_R(X))$ *be a* NTS, *then the following statements are true:*

1. *Every* nS_β*-open set is* $n\gamma$*-open set.*
2. *Every* nS_β*-open set is* $n\beta$*-open set.*
3. *Every* nS_β*-open set is* $n\delta\beta$*-open set.*

Proof.

1. Suppose that A is nS_β-open set, then by Proposition 3.3, A is nS-open set, hence $A \subseteq nint(cl(A)) \cup ncl(nint(A))$.
2. Suppose that A is nS_β-open set, then by Proposition 3.3, A is nS-open set, then by Theorem 2.14 (2), A is $n\beta$-open set.
3. Suppose that A is nS_β-open set, then by part (2), A is $n\beta$-open set, by Theorem 2.20, A is $n\delta\beta$-open set.

□

Remark 3.19. The converse of each part of above Theorem 3.18 is not true in general, as it shown in the following examples.

Example 3.20. *Let* $U=\{a, b, c\}$ *with* $U/R=\{\{a\}, \{b, c\}\}$ *and* $X=\{a\}$, *then* $\tau_R(X)=\{\phi, U, \{a\}\}$. *The nano* γ*-open sets are* $\{\phi, U, \{a\}, \{a, b\}, \{a, c\}\}$ *and* $nS_\beta O(X)=\{\phi, U\}$. *Consider at least the set* $\{a\}$ *is* γ*-open set but not* nS_β*-open set.*

Example 3.21. *Let* $U=\{a, b, c, d\}$ *with* $U/R=\{\{a, b\}, \{c\}, \{d\}\}$ *and* $X=\{a, c\}$.
Then, $\tau_R(X)=\{\phi, U, \{c\}, \{a, b, c\}, \{a, b\}\}$,$nSO(X)=\{\phi, U, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$, $n\beta O(X) = P(U) - \{d\}$, *then* $nS_\beta O(X) = \{\phi, U, \{c\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}\}$. *Consider* $\{b, c\} \in n\beta O(X)$, *but* $\{b, c\} \notin nSO(X)$. *Also,* $\{b, c\} \in n\beta O(X)$, *but* $\{b, c\} \notin nS_\beta O(X)$.

Example 3.22. *Let* $U=\{a, b, c, d\}$ *with* $U/R=\{\{a, b\}, \{c\}, \{d\}\}$ *and* $X=\{c, d\}$, *then* $\tau_R(X)=\{\phi, U, \{c, d\}\}$. $A=\{b\}$ *is* $n\delta\beta$*-open set but not* nS_β*-open set.*

Proposition 3.23. *In a* NTS $(U, \tau_R(X))$ *if* $\tau_R(X)$ $(resp.\ nSO(U, X)) = \{\phi, U\}$,
then $nS_\beta O(U, X) = \{\phi, U\}$.

Proof. Clear. □

Remark 3.24. The converse of above Proposition 3.23 may not to be true in general. That is, if $nS_\beta O(U, X) = \{\phi, U\}$ is does not mean $nSO(U, X) = \{\phi, U\}$, as it is clear in Example 3.5.

4. Family of Nano nS_β Open Sets in Term of $U_R(X), L_R(X)$ and $B_R(X)$

In this section, we consider nS_β-open sets by study of $U_R(X)$, $L_R(X)$ and $B_R(X)$ approximations with respect to X. In this view, we can easily find nS_β-open sets in NTSs.

Theorem 4.1. *Suppose that $U_R(X) = U$ and $L_R(X) \neq \phi$ in a NTS $(U, \tau_R(X))$, then $\tau_R(X) = \tau^S_R(X) = \tau^{S_\beta}_R(X) = \tau^\alpha_R(X)$.*

Proof. Suppose that $U_R(X) = U$ and $L_R(X) \neq \phi$, then $\tau_R(X) = \{\phi, U, L_R(X), B_R(X)\}$. Then by Theorem 2.17, $n\beta O(X) = P(U)$. Then $n\beta C(U,X) = P(U)$ but $nSO(U,X) \subseteq P(U) = n\beta C(U,X)$, so $nSO(U,X) = nS_\beta O(U,X)$, then by Theorem 2.6 and Theorem 2.10, we get $\tau_R(X) = \tau^S_R(X) = \tau^{S_\beta}_R(X) = \tau^\alpha_R(X)$. □

Theorem 4.2. *Suppose that $U_R(X) = U$ and $L_R(X) = \phi$ in a NTS $(U, \tau_R(X))$, then $\tau_R(X) = nS_\beta O(U,X) = \{U, \phi\}$.*

Proof. If $U_R(X) = U$ and $L_R(X) = \phi$, then $\tau_R(X) = \{\phi, U\}$. Then by Proposition 3.23, $nS_\beta O(U,X) = \{U, \phi\}$. □

Theorem 4.3. *Suppose that $(U, \tau_R(X))$ is a NTS. Let M and N be two nS_β-open set, then $M \cap N$ is nS_β-open in a NTS if $L_R(X) \neq \phi$ and $U_R(X) = U$.*

Proof. Follows from Preposition 4.1. □

Theorem 4.4. *Suppose that $(U, \tau_R(X))$ is a NTS. If $U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$, then $nS_\beta O(U,X) = \{\phi, U\}$.*

Proof. Suppose that $U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$, then $\tau_R(X) = \{\phi, U, \{x\}\}$. By Theorem 2.11, ϕ and those sets A for which $L_R(X) \subseteq A$ are the only nS-open sets in U, so all nS-open sets contains x, but by Theorem 2.16, ϕ, U and any set which intersect $U_R(X)$ are $n\beta$-open set in U, then by Theorem 2.18, any proper subset which contains $U_R(X) = \{x\}$ is not $n\beta$-closed set in U, but since all nS-open set contains x and there is no $n\beta$-closed set containing x, hence $nS_\beta O(U,X) = \{\phi, U\}$. □

Theorem 4.5. *Suppose that $U_R(X) \neq U$ in a NTS $(U, \tau_R(X))$. If M is nS-open set and $M^C \cap U_R(X) \neq \phi$, then M is nS_β-open set in a NTS U.*

Proof. Let $U_R(X) \neq U$ and M be a nS-open set such that $M^C \cap U_R(X) \neq \phi$. Then by Theorem 2.16, M^c is $n\beta$-open set in U, then M is $n\beta$-closed set. Hence M is nS_β-open set in a NTS U. □

Theorem 4.6. *$(U, \tau_R(X))$ be a NTS. If $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U, then ϕ and those sets M for which $U_R(X) \subseteq M$ are the only nS_β-open sets in NTS U.*

Proof. Suppose that $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U. By Theorem 2.11, ϕ and those sets M which $U_R(X) \subseteq M$ are the only nS-open sets in U. Let M is nS-open set, then $U_R(X) \subseteq M$. Let $x \in M$, then:

Case1. If $x \in U_R(X) \subseteq M$, by Theorem 2.16, $\{x\}$ is $n\beta$-open, since $\{x\} \cap U_R(X) \neq \phi$, then $\{x\}^c \cap U_R(X) \neq \phi$, since $U_R(X)$ contains more than one element of U. Hence $\{x\}^c$ is $n\beta$-open, so its complement $\{x\}$ is $n\beta$-closed. Thus, $x \in \{x\} \subseteq M$.

Case2. If $x \notin U_R(X) \subseteq M$, then $\{x\} \cap U_R(X) = \phi$, then $\{x\}^c \cap U_R(X) \neq \phi$, so $\{x\}^c$ is $n\beta$-open set, then its complement $\{x\}$ is $n\beta$-closed set. Thus, $x \in \{x\} \subseteq M$.

Therefore, ϕ and those sets M which $U_R(X) \subseteq M$ are the only nS_β-open sets in a NTS U. □

Theorem 4.7. *Suppose that $(U, \tau_R(X))$ is a NTS. If $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U, then ϕ and those sets M for which $U_R(X) \subseteq M$ are the only nS_β-open sets in a NTS U.*

Proof. $\tau_R(X) = \{\phi, U, U_R(X)\}$, then the proof similar with Theorem 4.6. □

Theorem 4.8. *Suppose that* $(U, \tau_R(X))$ *be a* NTS. *Let* M *and* N *are two* nS_β*-open sets in* U, *then* $M \cap N$ *is* nS_β*-open if* $U_R(X) = L_R(X) \neq U$ *and* $U_R(X)$ *contains more than one element of* NTS U.

Proof. Suppose that $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of a NTS U, then by Theorem 4.6, ϕ and any set containing $U_R(X)$ is nS_β-open set. If M or $N = \phi$, then the result is clear. Let M and $N \neq \phi$ be nS_β-open sets in NTS U, then $U_R(X) \subseteq M$ and $U_R(X) \subseteq N$ and hence $U_R(X) \subseteq M \cap N$. Thus, $M \cap N$ is nS_β-open set in NTS U. □

Theorem 4.9. *Suppose that* $(U, \tau_R(X))$ *is a* NTS. *If* M *and* N *are two* nS_β*-open set, then* $M \cap N$ *is* nS_β*-open if* $U_R(X) \neq U$, $L_R(X) = \phi$ *and* $U_R(X)$ *contains more than one element of a* NTS U.

Proof. The proof similar to Theorem 4.8. □

Theorem 4.10. *Let* $(U, \tau_R(X))$ *be a* NTS. *If* $L_R(X) \neq U_R(X)$ *where* $U_R(X) \neq U$ *and* $L_R(X) \neq \phi$, *then* ϕ, $L_R(X)$, $B_R(X)$ *and any set containing* $U_R(X)$, $L_R(X) \cup W$ and $B_R(X) \cup W$ *where* $W \subseteq [U_R(X)]^c$ *are the only* nS_β*-open sets in a* NTS U.

Proof. $\tau_R(X) = \{\phi, U, U_R(X), L_R(X), B_R(X)\}$. It is clear $U_R(X)$ contains more than one point of U. By Theorem 2.12, ϕ, $L_R(X)$, $B_R(X)$ and any set containing $U_R(X)$, $L_R(X) \cup W$ and $B_R(X) \cup W$ where $W \subseteq [U_R(X)]^c$ are the only nS-open sets in NTS U. Since $U_R(X)$ intersect any proper subset with less than one point of U, say G, so by Theorem 2.16, G is $n\beta$-open set and its complement is singleton $n\beta$-closed, that is, for any $x \in U$, $\{x\} \in n\beta C(U, X)$, then by Proposition 3.6, any nS-open set is nS_β-open set. Therefore, ϕ, $L_R(X)$, $B_R(X)$ and any set containing $U_R(X)$, $L_R(X) \cup W$ and $B_R(X) \cup W$ where $W \subseteq [U_R(X)]^c$ are the only nS_β-open sets in a NTS U. □

Theorem 4.11. *Let* A *be a subset of a* NTS $(U, \tau_R(X))$. *If* A *is* nano *clopen, then* A *is* nS_β *clopen in a* NTS U.

Proof. Suppose that G is a subset of NTS U which is nano clopen in U. Then $nint(G) = ncl(G)$ and $G \subseteq ncl(nint(ncl(G)))$. Hence, G is $n\beta$-open. Let $K = G^c$, then K is $n\beta$-closed. Since G is nano clopen, K is also nano clopen. So $K \subseteq ncl(nint(ncl(K)))$ and thus K is $n\beta$-open. Hence $G = K^c$ is $n\beta$-closed. Hence, G and K are $n\beta$-clopen in U. Again, G and K are nS-open as G and K both are nano open. Therefore, for nS-open set G, and for each $x \in G$ there exists a $n\beta$-closed set G such that $x \in G \subseteq G$. Thus, G is nS_β-open. By similar argument $K = G^c$ is nS_β-open. □

Theorem 4.12. *Let* $(U, \tau_R(X))$ *be a* NTS. *If* $U_R(X) \neq U$, $L_R(X) = \phi$ *and* $U_R(X)$ *contains more than one point of* U, *then* $nS_\beta O(U, X) = n\alpha O(U, X)$.

Proof. Follows from Theorem 2.11 and Theorem 4.7. □

Theorem 4.13. *Let* $(U, \tau_R(X))$ *be a* NTS. *If* $U_R(X) = L_R(X) \neq U$ *and* $U_R(X)$ *contains more than one point of* U, *then* $nS_\beta O(U, X) = n\alpha O(U, X)$.

Proof. Follows form Theorem 2.8 and Theorem 4.6. □

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