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Extensions of Nil-Reversible Rings with an Endomorphism α

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Abstract

The concept of an $\alpha-$nil reversible ring is a generalization of $\alpha-$reversible ring as well as an extension of nil reversible rings. We first consider basic properties of $\alpha-$nil reversible rings. Then we investigate extensions of $\alpha-$nil reversible, including trivial extension, Dorroh extension and Jordan extension.

Keywords: nil reversible ring, polynomial ring, matrix ring, endomorphism, trivial extension.
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1. Introduction

Throughout this paper, every rings are associative with identity. R represents a ring, and the set of all nilpotent elements in R is denoted by $N(R)$. Z(resp., Z_n) denotes the ring of integers (resp., modulo n). We denote the n by n full matrix (with $n \geqslant 2$) (resp., upper triangular matrix) ring over R by the symbol $Mat_n(R)$ (resp., $U_n(R)$), and let the symbol $D_n(R)$ be the ring of all matrices in $U_n(R)$ having diagonal entries equal. And let $R[x]$ (resp., $R[[x]]$) be the ring of polynomials (resp., power series ring) with an indeterminate x over the ring R. A ring is called reduced if it has no non-zero nilpotent elements. It is well-known that if R is reduced, then the next condition holds:

$$ab = 0 \text{ implies } ba = 0 \text{ for } a, b \in R.$$

However, Cohn [1] called a ring reversible if the above condition holds. It is clearly verified that reversible rings are abelian (i.e., all idempotent is central). Recently, the concept of reversibility is extended to $\alpha-$reversibility and $\alpha-$skew CNZ, where α is an endomorphism. Thus, if $a, b \in R$, such that $ab = 0$ implies $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), then α is termed as right (resp., left) reversible, and the ring R is called right (resp., left) $\alpha-$reversible if there exists a right (resp., left) reversible endomorphism α of R. A ring is $\alpha-$reversible [2] if it is both left and right $\alpha-$reversible. And if $a, b \in N(R)$, such that $ab = 0$ implies $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), then α is termed as right (resp., left) skew CNZ, and the ring R is called right (resp., left) $\alpha-$skew CNZ if there exists a right (resp., left) skew CNZ endomorphism α of R. A ring R is $\alpha-$skew CNZ [3] if it is both left and right $\alpha-$skew CNZ. Rings in which $N(R)$ is an ideal are said to be NI. The insertion-of-factors-property (simply, IFP) is a well-known property between NI and commutative, according to Bell [4], a right (or left) ideal I of R is said to have the IFP if for $a, b \in R$,

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such that $ab \in I$ implies $aRb \subseteq I$, therefore a ring R is called IFP if the zero ideal of R has the IFP. A ring R is nil-IFP [5] if for $a, b \in R$, such that $ab \in N(R)$implies $aRb \subseteq N(R)$. Due to Krempa [6], an endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ for $a \in R$ implies $a = 0$, and also the ring R is called $\alpha-$rigid [7] if there exists a rigid endomorphism α of R. A ring R is called $\alpha-$compatible [8] if $ab = 0 \Leftrightarrow a\alpha(b) = 0$, for each $a, b \in R$. If R is $\alpha-$compatible, then it is clear the endomorphism α is a monomorphism. We call a ring R nil reversible [9] if $l(a) = r(a)$ for any $a \in N(R)$. In other way, a ring R is nil reversible if $ab = 0$ for $a \in N(R)$, $b \in R$ then $ba = 0$. The concept of an $\alpha-$nil reversible ring is a generalization of $\alpha-$reversible ring.

In this paper, we introduce a class of rings called $\alpha-$nil reversiblerings which is a strict generalization of $\alpha-$reversiblerings as well as an extension of nil reversible rings, and then study the structure of right $\alpha-$nil reversible rings and their related properties. And also the relationship between $\alpha-$nil reversible rings and generalized Armendariz rings is investigated. As a result, several known findings are obtained as corollaries of our findings.

Throughout this paper, α denotes a non-zero endomorphism of a given ring, unless specified otherwise.

2. Basic properties of right $\alpha-$nil reversible rings

In this section, we study the basic structure of right $\alpha-$nil reversiblerings, investigating several related ring examples and properties.

Definition 2.1. An endomorphism α of a ring R is called a right (resp., left) nil reversible if whenever $ab = 0$ for $a \in N(R)$ and $b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), and the ring R is called right (resp., left) $\alpha-$nil reversible if there exist a right (resp., left) nil reversible endomorphism α of R. A ring R is called $\alpha-$nil reversible if it is both right and left $\alpha-$nil reversible.

Remark 2.2. (1) A ring R is nil reversible if R is I_R-nil reversible, where I_R is the identity endomorphism of R. (2) All subring S with $\alpha(S) \subseteq S$ of an $\alpha-$nil reversible ring is also an $\alpha-$nil reversible ring.

The next example demonstrates that $\alpha-$nil reversibility is not left-right symmetric.

Example 2.3. *Consider a ring* $R = U_2(Z_4)$, *then* $N(R) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \in \{0, 2\},\ b \in Z_4 \right\}$. *Let* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$. *We have* $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, *but* $BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. *So* R*is not nil reversible.*

(1) *Let* $\alpha : R \to R$ *be an endomorphism defined by:*

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that $AB = 0$ *for* $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$. *Then* $aa' = 0$ *and it implies that* $B\alpha(A) = 0$ *since* Z_4 *is commutative. Therefore* R *is right* $\alpha-$*nil reversible. However, for* $A = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ *with* $AB = 0$, *we have* $\alpha(B)A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$, *and thus* R *is not left* $\alpha-$*nil reversible.*

(2) *Let* $\beta : R \to R$ *be an endomorphism defined by:*

$$\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Then R *is not right* $\beta-$*nil reversible but left* $\beta-$*nil reversible via similar calculation to (1) .*

However, we have the following:

Theorem 2.4. *For a ring* R *with an endomorphism* α*, we have the following.*

(1) *Let* R *be a reversible ring. Then* R *is right* $\alpha -$ *nil reversible if and only if* $ab = 0$ *for* $a \in N(R)$ *and* $b \in R$*, implies* $aR\alpha^n(b) = 0$ *and* $\alpha^n(a)Rb = 0$*, for any non-negative integer* n *if and only if* R *is left* $\alpha -$ *nil reversible.*

(2) *Let* R *be a nil reversible ring. Then* R *is right* $\alpha -$ *nil reversible if and only if* R *is* $\alpha -$ *nil reversible.*

Proof. (1) Assume that R be right $\alpha -$ nil reversible and $ab = 0$ for $a \in N(R)$, $b \in R$. It is enough to show that $aR\alpha(b) = 0$ and $\alpha(a)Rb = 0$. Since R is reversible and right $\alpha -$ nil reversible, $ab = 0$ implies $b\alpha(a) = 0$ and $ba = 0$. If $b\alpha(a) = 0$, then $b\alpha(a)c = 0$ for any $c \in R$ and so $\alpha(a)cb = 0$, and thus $\alpha(a)Rb = 0$. From $ba = 0$, we have $bac = 0$ for any $c \in R$, and so $aR\alpha(b) = 0$. For the case R being left $\alpha -$ nil reversible the proof is similar.

It is simple to obtain the converse by using the fact that R is reversible.

(2) Assume that R is nil reversible and right $\alpha -$ nil reversible and $ab = 0$ for $a \in N(R)$, $b \in R$. Then $ba = 0$ and so $a\alpha(b) = 0$. Therefore $\alpha(b)a = 0$ since $\alpha(b) \in N(R)$. So R is left $\alpha -$ nil reversible, implying that R is $\alpha -$ nil reversible. The converse is obvious. □

Note that for a commutative ring R there exists an endomorphism α which is not $\alpha -$ nil reversible by the following example.

Example 2.5. *Let* Z_4 *be the ring of integers modulo* 4 *and consider a ring* $R = Z_4 \oplus Z_4$, $N(R) = \{(a,b) : \ a, b \in \{0,2\}\}$ *with the usual addition and multiplication. Since* R *is commutative,* R *is nil reversible. Now suppose that* $\alpha : R \to R$ *be defined by* $\alpha((a,b)) = (b,a)$*. Then* α *is an automorphism of* R*. For* $a = (2,0) \in N(R)$, $b = (0,1) \in R$, $ab = (0,0)$ *but* $b\alpha(a) = (0,1)\ (0,2) = (0,2) \neq 0$*, and thus* R *is not* $\alpha -$ *nil reversible.*

It is evident that an $\alpha -$ reversible is $\alpha -$ nil reversible. However, there exists an $\alpha -$ nil reversible ring which is not $\alpha -$ reversible as given in the next example.

Example 2.6. *Consider a ring* $R = U_2(Z_3)$*. Then* $N(U_2(Z_3)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in Z_3 \right\}$.

Let $\alpha : R \to R$ *be an endomorphism defined by:*

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

It is obvious that R *is* $\alpha -$ *nil reversible. However, for* $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \in R$*, with* $AB = 0$*. We have* $B\alpha(A) \neq 0$*, and thus* R *is not* $\alpha -$ *reversible.*

Every $\alpha -$ nil reversible is $\alpha -$ skew CNZ, but the converse is not true as shown in the next example:

Example 2.7. *Assume that* $R = Mat_2(Z_2)$ *and an endomorphism* $\alpha : R \to R$ *defined by:*

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

R *is (*$\alpha -$ *skew* CNZ*) by [3, Example 2.5] but not* $\alpha -$ *nil reversible for* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, $AB = 0$ *but* $B\alpha(A) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq 0$*. Therefore* R *is not right* $\alpha -$ *nil reversible, thus* R *is not* $\alpha -$ *nil reversible.*

Now consider a right $\alpha-$nil reversible ring R in its reverse condition:

$$(*) \qquad a\alpha(b) = 0 \text{ for } a \in N(R),\ b \in R \text{ implies } ba = 0.$$

In Example 2.3(1), the ring $R = U_2(Z_4)$ with the endomorphism α is right $\alpha-$nil reversible but it does not satisfy the condition $(*)$. For $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in R$, we have $A\alpha(B) = 0$ but $BA = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$.

Proposition 2.8. (1) *For a ring R with a monomorphism α. R is satisfies the condition $(*)$ above if and only if R is right $\alpha-$nil reversible.*

(2) *For an $\alpha-$compatible ring R. R is satisfies the condition $(*)$ above if and only if R is right $\alpha-$nil reversible.*

Proof. (1) Let R satisfies the condition$(*)$. Suppose that $ab = 0$ for $a \in N(R)$, $b \in R$. Then $\alpha(a)\alpha(b) = \alpha(ab) = 0$ and so $b\alpha(a) = 0$ by assumption. So R is right $\alpha-$nil reversible.

Conversely, suppose that R is right $\alpha-$nil reversible. If $a\alpha(b) = 0$ for $a \in N(R), b \in R$, then $\alpha(b)\alpha(a) = 0$ since $\alpha(b) \in R$. Hence $ba = 0$ because α is a monomorphism. Consequently, R satisfies the condition$(*)$.

(2) Is an instantaneous result of (1) because α is a monomorphism when R is $\alpha-$compatible. □

For a nonempty subset S of a ring R, $r_R(S) = \{c \in R : Sc = 0\}$ is said to be the right annihilator of S in R. And the left annihilator is likewise defined and written by $l_R(S)$, then we type $r_R(a)$(resp., $l_R(a)$) instead of $r_R(\{a\})$ (resp., $l_R(\{a\})$).

Proposition 2.9. *For a ring R with an endomorphism α of R, the following are equivalent:*

(1) *R is right $\alpha-$nil reversible.*

(2) $r_R(S) \subseteq l_R(\alpha(S))$ *for any nonempty subset S of N(R).*

(3) *For each $a \in N(R)$, $r_R(a) \subseteq l_R(\alpha(a))$.*

(4) $AB = 0$ *implies* $B\alpha(A) = 0$ *for any nonempty subsets A of N(R) and B of R.*

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow from the definition of right $\alpha-$nil reversible, and (4) $\Rightarrow$ (1) is straightforward.

Now for (1) $\Rightarrow$ (4): Suppose that (1) holds. Assume that $AB = 0$, for two nonempty subsets A of N(R) and B of R. Then $ab = 0$ for all $a \in A$ and $b \in B$, therefore $b\alpha(a) = 0$ by (1). Hence, $B\alpha(A) = \sum\limits_{a\in A, b\in B} b\alpha(a) = 0$. □

$\alpha-$nil reversibility is closed under subdirect products and direct products, by the next proposition.

Proposition 2.10. (1) *Let α_λ be an endomorphism of a ring R for each $\lambda \in \Delta$, where Δ is some index set. The subdirect product of an arbitrary family of $\alpha-$nil reversible rings is $\bar{\alpha}-$nil reversible.*

(2) *The direct product of $\alpha-$nil reversible rings is $\bar{\alpha}-$nil reversible ring.*

Proof. (1) Suppose that $\{I_\lambda : \lambda \in \Delta\}$ be a family of ideals of R such that $\bigcap\limits_{\lambda\in\Delta} I_\lambda = 0$ and R/I_λ's are right $\alpha-$nil reversible rings, take $a \in N(R)$ and $b \in R$ satisfying $ab = 0$. There is a positive integer m with the property $a^m = 0$. Thus $a + I_\lambda \in N(R/I_\lambda)$. Since $N(R/I_\lambda)$ is nil reversible and $(a + I_\lambda)\,(b + I_\lambda) = 0$, $(b + I_\lambda)\ \bar{\alpha}\ (a + I_\lambda) = (b + I_\lambda)\,(\alpha(a) + I_\lambda) = 0$ this implies $b\alpha(a) + I_\lambda = 0$. So $b\alpha(a) \in I_\lambda$ for all $\lambda \in \Delta$. That is, $b\alpha(a) \in \bigcap\limits_{\lambda\in\Delta} I_\lambda = 0$. Therefore $\bigcap\limits_{\lambda\in\Delta} I_\lambda$ is right $\bar{\alpha}-$nil reversible. Following similar argument, we have $\bigcap\limits_{\lambda\in\Delta} I_\lambda = 0$ is left $\bar{\alpha}-$nil reversible. Thus $\bigcap\limits_{\lambda\in\Delta} I_\lambda$ is $\bar{\alpha}-$nil reversible.

(2) It comes from the fact that $N(\prod\limits_{\lambda\in\Delta} R_\lambda) = \prod\limits_{\lambda\in\Delta} N(R_\lambda)$ [10, Proposition 2.13]. Suppose that $(a_1, a_2, \ldots) \in N(\prod\limits_{\lambda\in\Delta} R_\lambda)$ and $(b_1, b_2, \ldots) \in \prod\limits_{\lambda\in\Delta} R_\lambda$ such that $(a_1, a_2, \ldots)\,(b_1, b_2, \ldots) = 0$. Thus for each $\lambda = 1, 2, \ldots$, $a_\lambda b_\lambda = 0$, since R_λ is right $\alpha-$nil reversible then $b_\lambda \bar{\alpha}(a_\lambda) = 0$ For each $\lambda = 1, 2, \ldots$, so we get $(b_1, b_2, \ldots)\ \bar{\alpha}\ (a_1, a_2, \ldots) = 0$. The method can be similarly proved for left $\bar{\alpha}-$nil reversible rings. □

Proposition 2.11. *Let* R *be an abelian ring with* $\alpha(e) = e$ *for any* $e^2 = e \in R$*. Then* eR *and* $(1-e)R$ *are right* $\alpha-$*nil reversible if and only if* R *is right* $\alpha-$*nil reversible.*

Proof. Since any subring S with $\alpha(S) \subseteq S$ of a right $\alpha-$nil reversible ring is also right $\alpha-$nil reversible so eR and $(1-e)R$ are right $\alpha-$nil reversible rings if R is right $\alpha-$nil reversible.

Conversely, suppose that $ab = 0$ for $a \in N(R)$, $b \in R$ then $eaeb = 0$ and $(1-e)a(1-e)b = 0$. Since eR and $(1-e)R$ are right $\alpha-$nil reversible, then $eb\alpha(ea) = 0$ and $(1-e)b\alpha((1-e)a) = 0$. Hence
$eb\alpha(ea) + (1-e)b\alpha((1-e)a)$
$= eb\alpha(a) + (1-e)b\alpha(a)$
$= eb\alpha(a) + b\alpha(a) - eb\alpha(a)$
$= b\alpha(a) = 0$.
Thus R is right $\alpha-$nil reversible. □

Proposition 2.12. *If* R *is an* $\alpha-$*compatible ring. Then* R *is an* $\alpha-$*nil reversible ring.*

Proof. Let $ab = 0$ for $a \in N(R)$, $b \in R$. Then there exists $k > 0$ such that $0 = (ab)^k = ab\ ab \dots ab$. Then using [8, Lemma 2.8], $\alpha(a)\ b\ (ab)^{k-1} = 0$, and so $\alpha(a)\ b\ \alpha((ab)^{k-1}) = 0$. Therefore $\alpha(a)\ b\ \alpha(a)\ \alpha(b(ab)^{k-2}) = 0$, and then $\alpha(a)\ b\ \alpha(a)\ b\ \alpha((ab)^{k-2}) = 0$. Continuing this process we have $\alpha(a)b = 0$. Therefore $b\alpha(a) = 0$. Similarly we have R is left $\alpha-$nil reversible, so R is $\alpha-$nil reversible. □

And also by [11, Example 2.10], the converse of this proposition does not hold.

Proposition 2.13. *Let* R *be an* $\alpha-$*compatible ring with an endomorphism* α*. Then the following are equivalent:*
(1) R *is nil reversible;*
(2) R *is* $\alpha-$*nil reversible;*
(3) R *is right* $\alpha-$*nil reversible;*
(4) R *is left* $\alpha-$*nil reversible.*

Proof. (1) ⇒ (2): Assume that R is nil reversible and let $ab = 0$ for $a \in N(R)$, $b \in R$. Then we have $ba = 0$ and $a\alpha(b) = 0$ and so $b\alpha(a) = 0$ and $\alpha(b)a = 0$, since $\alpha(b) \in R$. Thus R is $\alpha-$nil reversible.

(2) ⇒ (3): is trivial.

(3) ⇒ (1): Suppose that R is right $\alpha-$nil reversible and assume that $ab = 0$ for $a \in N(R)$, $b \in R$. Then $b\alpha(a) = 0$ and so $ba = 0$, since R is $\alpha-$compatible. Thus R is nil reversible.

(3) ⇒ (4): Suppose that R is right $\alpha-$nil reversible and assume that $ab = 0$ for $a \in N(R)$, $b \in R$. Then $a\alpha(b) = 0$ since R is $\alpha-$compatible and so $\alpha(b)\alpha(a) = 0$, by assumption we have $\alpha(b)a = 0$. Thus R is left $\alpha-$nil reversible.

The proof (4) ⇒ (1) is similar. □

For an ideal I of R, if $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring R/I.

The class of right $\alpha-$nil reversible is not closed under homomorphic image by the next example:

Example 2.14. *Let* K *be a field and* $R = K\langle a, b\rangle$ *be the free algebra with non-commuting indeterminates* a, b *over* K*. Then* R *is a domain and so it is a reduced ring.*

An automorphism α *of* R *defined by:*

$$a \to b \ \text{ and } b \to a.$$

Thus R *is clearly* $\alpha-$*nil reversible. Now, let* I *be the ideal of* R *generated by:* ab, ab^2 *and* a^3*. Then for* $a + I \in N(R)$, $b + I \in R$ *we have* $(a + I)(b + I) = ab + I = I$ *but,* $(b + I)\ \bar{\alpha}\ (a + I) = b^2 + I \neq I$ *by the construction of* I*. As a result* R/I *is not right* $\bar{\alpha}-$*nil reversible.*

One may question that R is right $\alpha-$nil reversible. if for any right $\alpha-$nil reversible non zero proper ideal I of R, R/I is right $\bar{\alpha}-$nil reversible, where I is considered as a ring without identity. However, the following example removes the possibility.

Example 2.15. *Consider the ring* $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$*, where* K *is a field.*

An endomorphism α *of* R *defined by:* $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$.

Let e_{ij} *be matrix units with* 1 *at the entry* (i, j) *and zero elsewhere and let* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, *we have* $AB = 0$ *but* $B\alpha(A) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq 0$*. Therefore,* R *is not right* $\alpha-$*nil reversible.*

Observe that the only non zero proper ideals of R *are* $I_1 = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ *and* $I_3 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ *It is simple to verify that* I_1, I_2 *and* I_3 *are right* $\alpha-$*nil reversible, and that* $R/I_1 \cong K$ *and* $R/I_2 \cong K$ *are also right* $\bar{\alpha}-$*nil reversible since* R/I_1 *and* R/I_2 *are domain. Now for* $I_3 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$*. One can easily show that* I_3 *is not a reduced ideal. Now consider the quotient ring* $R/I_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I_3 : \ a, c \in K \right\}$ *is reduced and so* α *is an identity map on* R/I_3 *right* $\bar{\alpha}-$*nil reversible. Noticing that* $e_{12} \in N(R)$, $e_{11} \in R$. $e_{12}e_{11} = 0$ *but* $e_{11}\alpha(e_{12}) \neq 0$.

Proposition 2.16. *Let* R *be a ring with an endomorphism* α *and* I *an* $\alpha-$*ideal of* R*. If* R/I *is right* $\bar{\alpha}-$*nil reversible and* I *is* $\alpha-$*rigid as a ring without identity, then* R *is right* $\alpha-$*nil reversible.*

Proof. Assume that R/I is right $\bar{\alpha}-$nil reversible and I is $\alpha-$rigid as a ring without identity. Suppose that $ab = 0$ for $a \in N(R)$, $b \in R$. Then $(a + I)(b + I) = I$ and $a + I \in N(R/I)$, $b + I \in (R/I)$, since R/I is right $\bar{\alpha}-$nil reversible $b\alpha(a) \in I$. Hence $b\alpha(a)\ \alpha(b\alpha(a)) = b\alpha(ab)\ \alpha^2(a) = 0$ and since I is an $\alpha-$rigid ring we have $b\alpha(a) = 0$. So R is right $\alpha-$nil reversible. □

Corollary 2.17. *Assume that* R/I *is nil reversible for some ideal* I *of* R*. If* I *is reduced, then* R *is nil reversible.*

Proof. Let $a \in N(R)$, $b \in R$ with $ab = 0$. This implies $(a + I)\ (b + I) = I$, for $a + I \in N(R/I)$, $b + I \in R/I$. Since R/I is nil reversible, we get $(b + I)\ (a + I) = I$ and so $ba \in I$, $(ba)^2 = baba = 0$, and I is reduced implies $ba = 0$. Thus R is nil reversible. □

Proposition 2.18. *Let* R *be right* $\alpha-$*nil reversible with a monomorphism* α*. If* R *is* NI*, then* R *is* $nil-$*IFP.*

Proof. Suppose that R is NI and right $\alpha-$nil reversible with $ab = 0$ for $a, b \in N(R)$. Then $b\alpha(a) = 0$ and so $0 = b\ \alpha(a)\ \alpha(c) = b\ \alpha(ac)$ for any $c \in R$. Note that $\alpha(ac) \in N(R)$ since $ac \in N(R)$ for any $c \in R$ by hypothesis. Therefore $acb = 0$ for any $c \in R$. So R is $nil-$IFP. □

3. Extensions of right $\alpha-$nil reversible rings

For a ring R with an endomorphism α and $n \geqslant 2$, the corresponding $(a_{ij}) \to (\alpha(a_{ij}))$ induces an endomorphism of $Mat_n(R)$, $U_n(R)$ and $D_n(R)$, respectively. We denote them by $\bar{\alpha}$.

Note that for a reduced ring R, both $U_2(R)$ and $D_2(R)$ are $\bar{\alpha}-$nil reversible for any endomorphism α. However there exists a reduced ring A with an endomorphism α such that $Mat_2(A)$ is not right $\bar{\alpha}-$nil reversible as shown in the next example:

Example 3.1. *An automorphism* α *of* Z_2 *defined by:*

$$0 \to 1 \text{ and } 1 \to 0.$$

Consider $R = Mat_2(Z_2)$*. Now for* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$*, we have* $AB = 0$ *but* $B\alpha(A) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$*. Therefore* $Mat_2(Z_2)$ *is not right* $\bar{\alpha}-$*nil reversible.*

$D_n(R)$ and $U_n(R)$ (for $n \geqslant 2$) need not be right $\bar{\alpha}-$nil reversible, when R is right $\alpha-$nil reversible with an endomorphism α.

Example 3.2. *Take a ring $A = U_2(R)$ over right $\alpha-$nil reversible R and define an endomorphism α of A by:*

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Now for $a = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(D_2(A))$, $b = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in D_2(A)$. *We have* $ab = 0$, *but*

$$b\bar{\alpha}(a) = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Thus $D_2(A)$ is not right $\bar{\alpha}-$nil reversible.

Notice that $Mat_n(R)$, $U_n(R)$ and $D_n(R)$ (for $n \geqslant 3$) are not right $\bar{\alpha}-$nil reversible for any ring R with an endomorphism α, such that $\alpha(1) \neq 0$ (e.g. , α is a monomorphism).

Example 3.3. *Let R be a ring with an endomorphism α, such that $\alpha(1) \neq 0$. Consider the ring $D_3(R)$, let $A = e_{23} \in N(D_3(R))$, $B = e_{12} + e_{13} + e_{23} \in D_3(R)$.*
Then $AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$, *but* $B\bar{\alpha}(A) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha(1) \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. *Thus $D_3(R)$ is not right $\bar{\alpha}-$nil reversible.*
In a similar way, we can show that $D_m(R)$ (for $m \geqslant 4$) is not right $\bar{\alpha}-$nil reversible.

For a field K, some subrings of $D_3(K)$ are right $\bar{\alpha}-$nil reversible rings.

Proposition 3.4. *Let R be a right $\alpha-$nil reversible ring . Then the following hold:*

(1) *Let* $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in R \right\}$ *be a subring of $D_3(R)$. Then S is right $\bar{\alpha}-$nil reversible.*

(2) *Assume that* $S = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : a, b, c \in R \right\}$ *be a subring of $D_3(R)$. Then S is right $\bar{\alpha}-$nil reversible.*

Proof. (1) Let $A = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(S)$, $B = \begin{pmatrix} u & v & t \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \in S$.
Assume that $AB = 0$, and so $bu = 0$ and $cu = 0$, for $b, c \in N(R)$ and $u \in R$, since R is right $\alpha-$nil reversible, we have $u\alpha(b) = 0$ and $u\alpha(c) = 0$.
Hence $B\bar{\alpha}(A) = \begin{pmatrix} u & v & t \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \bar{\alpha} \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u\alpha(b) & u\alpha(c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. Thus S is right $\bar{\alpha}-$nil reversible.

(2) By using similar method in proof (1), We get S is right $\bar{\alpha}-$nil reversible. □

Given a ring R and an $(R, R)-$bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the next multiplication: $(c_1, m_1)(c_2, m_2) = (c_1c_2, c_1m_2 + m_1c_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} c & m \\ 0 & c \end{pmatrix}$, where $c \in R$ and $m \in M$ and the usual matrix operations are used.

Proposition 3.5. *Let* R *be a ring with an endomorphism* α. *If* $T(R,R)$ *is* $\alpha-$*nil reversible, then* R *is* $\alpha-$*reversible.*

Proof. Let $a,b \in R$ with $ab=0$. Notice that $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N(T(R,R))$, $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in T(R,R)$, we obtain $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $T(R,R)$ is $\alpha-$nil reversible, hence $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \bar{\alpha} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b\alpha(a) \\ 0 & 0 \end{pmatrix} = 0$. This gives $b\alpha(a)=0$. Therefore R is right $\alpha-$reversible.

By using similar method, we get R is left $\alpha-$reversible. Thus R is an $\alpha-$reversible ring. □

The notion of a right $\alpha-$nil reversible ring does not go up to polynomial rings (power series rings) by the following example:

Example 3.6. *Take* Z_2 *as the field of integer modulo 2 and let* $A = Z_2\langle a_0,a_1,a_2,b_0,b_1,b_2,d\rangle$ *be the free algebra of polynomials with zero constant terms in non-commuting indeterminates.* a_0,a_1,a_2,b_0,b_1,b_2 *and* d *over* Z_2, *and an automorphism* δ *of* A *defined by :*

$$a_0,a_1,a_2,b_0,b_1,b_2,d \to b_0,b_1,b_2,a_0,a_1,a_2,d.$$

Suppose that B *be the set of all polynomials with zero constant terms in* A *and consider the ideal* J *of the ring* A *generated by*

$$a_0b_0,\ a_0b_1+a_1b_0,\ a_0b_2+a_1b_1+a_2b_0,\ a_1b_2+a_2b_1,\ a_2b_2,\ a_0pb_0,\ a_2pb_2,\ b_0a_0,\ b_0a_1+b_1a_0,$$

$$b_0a_2+b_1a_1+b_2a_0,\ b_1a_2+b_2a_1,\ b_2a_2,\ b_0pa_0,\ b_2pa_2,\ a_0a_0,\ a_2a_2,\ a_0pa_0,\ a_2pa_2,\ b_0b_0,\ b_2b_2,$$

$$a_0a_1+a_1a_0,\ a_0a_2+a_1a_1+a_2a_0,\ a_1a_2+a_2a_1,\ b_0b_2+b_1b_1+b_2b_0,\ b_1b_2+b_2b_1,\ (a_0+a_1+a_2)p(a_0+a_1+a_2),$$

$$(b_0+b_1+b_2)p(b_0+b_1+b_2),\ (a_0+a_1+a_2)p(b_0+b_1+b_2),\ (b_0+b_1+b_2)p(a_0+a_1+a_2),\ \textit{and}\ p_1p_2p_3p_4.$$

Where $p,p_1,p_2,p_3,p_4 \in B$. *Then it is obvious* $B^4 \subseteq J$. *Set* $R = A/J$. *Since* $\delta(J) \subseteq J$. *we can get an automorphism* α *of* R *by defining* $\alpha(r+J) = \delta(r)+J$ *for* $r \in A$. *For simplicity we identify all element of* A *with its image in* R. *For* $f(x) = a_0+a_1x+a_2x^2 \in N(R[x])$ *since* $B^4 \subseteq J$*and* $g(x) = b_0d+b_1dx+b_2dx^2 \in R[x]$ *we have* $f(x)g(x)=0$ *but* $g(x)\bar{\alpha}(f(x)) = (b_0d+b_1dx+b_2dx^2)(b_0+b_1x+b_2x^2) \neq 0$ *since* $b_0db_1+b_1db_0 \neq 0$. *Thus,* $R[x]$ *is not right* $\bar{\alpha}-$*nil reversible.*

Theorem 3.7. *If* R *is an* IFP *right* $\alpha-$*nil reversible ring, then both* $R[x]$ *and* $R[[x]]$ *are right* $\bar{\alpha}-$*nil reversible.*

Proof. Since $\alpha-$nil reversible are closed under subrings so it suffices to prove that $R[[x]]$ is right $\bar{\alpha}-$nil reversible. Let $f(x) = \sum_{i=0}^{\infty} a_ix^i \in N(R[[x]])$, $g(x) = \sum_{j=0}^{\infty} b_jx^j \in R[[x]]$ with $f(x)g(x)=0$. By [11, Proposition 2.16], $N(R[[x]]) \subseteq N(R)[[x]]$. So $a_i \in N(R)$ for each i. Since $f(x)g(x)=0$ then we get the following equation:

$$a_0b_0 = 0 \qquad (1)$$
$$a_0b_1+a_1b_0 = 0 \qquad (2)$$
$$\vdots$$
$$a_0b_{u-1}+a_1b_{u-2}+...+a_{u-1}b_0 = 0 \qquad (3)$$
$$a_0b_u+a_1b_{u-1}+...+a_ub_0 = 0 \qquad (4)$$

Since R is IFP, $N(R)$ is an ideal of R by [12, Lemma 3.1] Since $a_0b_0=0$, and R is right $\alpha-$nil reversible then $b_0\alpha(a_0)=0$.
If we multiply Eq.(2) from left side by b_0, then it follows that $b_0a_1b_0=0$ and so $a_1b_0=0$ for $a_1 \in N(R)$. Now suppose that u is positive integer such that $a_ib_j=0$ for $a_i \in N(R)$, $b_j \in R$ when $i+j<u$, we show

that $a_i b_j = 0$ when $i + j = u$.
If we multiply the Eq.(4) from left side by b_0, then we have $b_0 a_0 b_u + b_0 a_1 b_{u-1} + ... + b_0 a_u b_0 = 0$ and so by induction hypothesis $a_u b_0 = 0$. Then $a_0 b_u + a_1 b_{u-1} + ... + a_{u-1} b_1 = 0$. Using again the induction hypothesis, $a_0 b_u$, $a_1 b_{u-1}$, $\ldots$, $a_{u-1} b_1 = 0$. Therefore $a_i b_j = 0$ for each i, j . Thus $b_j \alpha^t(a_i) = 0$ for each t, since R is right $\alpha -$ nil reversible. Thus $g\bar{\alpha}(f) = 0$ by [11, Proposition 2.16]. Hence $R[[x]]$ is right $\bar{\alpha} -$ nil reversible. □

Due to Rege and chhawchharia, a ring R called Armendariz [13, Definition 1.1] if whenever the product of any two polynomials in $R[x]$ over R is zero, thus is the product of any pair of coefficients from two polynomials. And it is well-known that any reduced ring is Armendariz, but not conversely. The notion of an Armendariz ring is extended to power series rings over general noncommutative rings. In [14] A ring R is called power-serieswise Armendariz if $a_i b_j = 0$ for all i, j whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=o} b_j x^j \in R[[x]]$ satisfy $f(x)g(x) = 0$. By definition, power serieswise Armendariz rings are Armendariz, but not conversely by [14, Example 2.4].

And a ring R is called skew power-serieswise $\alpha -$ Armendariz [15] if $a_i b_j = 0$ for all i, j whenever $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{\infty} a_i x^i$, $q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\ \alpha]]$. According to [16, Definition 2.1], a ring R with an endomorphism α is called skew power-serieswise Armendariz if $p(x)q(x) = 0$ for all skew power series $p(x) = \sum_{i=0}^{\infty} a_i x^i$, $q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\ \alpha]]$ if and only if $a_i b_j = 0$ for all i, j. And also every skew power-serieswise Armendariz ring is clearly skew power-serieswise $\alpha -$ Armendariz.

Lemma 3.8. *Let R be a skew power-serieswise $\alpha -$ Armendariz ring and α an endomorphism of R. Then:*

(1) *R is nil reversible if and only if R is right $\alpha -$ nil reversible.*

(2) *Let M be one of symbols $[x; \alpha]$, $[x, x^{-1}; \alpha]$, $[[x; \alpha]]$, $[[x, x^{-1}; \alpha]]$, then $N(RM) = N(R)M$, in case α is an automorphism.*

Proof. (1) Assume that R is nil reversible and for $a \in N(R)$, $b \in R$ let $ab = 0$. Then $\alpha(a)b = 0$ by [15, Theorem 3.3(3)] and so $b\alpha(a) = 0$, $\alpha(a) \in N(R)$ since R is nil reversible. Therefore R is right $\alpha -$ nil reversible.

Conversely, assume that R is right $\alpha -$ nil reversible and for $a \in N(R)$, $b \in R$ let $ab = 0$. Then $b\alpha(a) = 0$ and so $ba = 0$ by [15, Theorem 3.3(3)] . Thus R is nil reversible.

(2) It directly follows from [16, Theorem 3.17]. □

For a ring R with endomorphism α, the corresponding $\sum a_i x^i \to \sum \alpha(a_i) x^i$ induces an endomorphism of $R[x; \alpha]$, $R[x, x^{-1}; \alpha]$, $R[[x; \alpha]]$ and $R[[x, x^{-1}; \alpha]]$, respectively. We denote them by $\bar{\alpha}$.

Theorem 3.9. *Let R be skew power-serieswise $\alpha -$ Armendariz ring with an automorphism α of R. Then the following are equivalent:*

(1) *R is right $\alpha -$ nil reversible;*

(2) *$R[x; \alpha]$ is right $\bar{\alpha} -$ nil reversible;*

(3) *$R[x, x^{-1}; \alpha]$ is right $\bar{\alpha} -$ nil reversible;*

(4) *$R[[x; \alpha]]$ is right $\bar{\alpha} -$ nil reversible;*

(5) *$R[[x, x^{-1}; \alpha]]$ is right $\bar{\alpha} -$ nil reversible.*

Proof. It is sufficient to show that $(1) \Rightarrow (5)$. Suppose that $f(x)g(x) = 0$, for $f(x) = \sum_{i=0}^{\infty} a_i x^i \in N(R[[x, x^{-1}; \alpha]])$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x, x^{-1}; \alpha]]$. Then we get $a_i \in N(R)$, $b_j \in R$ by lemma 3.8(2) and $a_i b_j = 0$ for every i, j. Therefore $b_j a_i = 0$ by lemma 3.8(1) and $b_j \alpha^n(a_i) = 0$ for any non-negative integer n, since R is $\alpha -$ compatible as noted above. This implies $g(x)\ \bar{\alpha}\ (f(x)) = 0$, therefore $R[[x, x^{-1}; \alpha]]$ is right $\bar{\alpha} -$ nil reversible. □

Corollary 3.10. *Let* R *be a power-serieswise Armendariz ring. The following are equivalent:*
(1) R *is nil reversible;*
(2) $R[x]$ *is nil reversible;*
(3) $R[x, x^{-1}]$ *is nil reversible;*
(4) $R[[x]]$ *is nil reversible;*
(5) $R[[x, x^{-1}]]$ *is nil reversible.*

Corollary 3.11. *For an Armendariz ring* R *with an endomorphism* α*, the following are equivalent:*
(1) R *is right* $\alpha-$*nil reversible;*
(2) $R[x]$ *is right* $\bar{\alpha}-$*nil reversible;*
(3) $R[x, x^{-1}]$ *is right* $\bar{\alpha}-$*nil reversible.*

An element a of a ring R is called right regular if $ac = 0$ implies $c = 0$ for $c \in R$. In the same way, left regular is defined and it is regular if it is both left and right regular (and hence non-zero divisor). Let Δ is a multiplicatively closed subset of R consisting of central regular elements. Assume that α be an automorphism of R and suppose $\alpha(s) = s$ for all $s \in \Delta$. Then $\alpha(s^{-1}) = s^{-1}$ in $\Delta^{-1}R$ and the induced map $\bar{\alpha} : \Delta^{-1}R \to \Delta^{-1}R$ defined by $\bar{\alpha}(a^{-1}b) = a^{-1}\alpha(b)$ is also an automorphism.

Proposition 3.12. *Let* R *be a ring with an automorphism* α *and* Δ *denotes a multiplicatively closed subset of* R *consisting of central regular elements and* $\alpha(s) = s$ *for every* $s \in \Delta$*. Then* R *is right* $\alpha-$*nil reversible if and only if* $\Delta^{-1}R$ *is right* $\bar{\alpha}-$*nil reversible.*

Proof. It is sufficient to prove the necessary condition. Suppose that R is right $\alpha-$nil reversible and let $s^{-1}a \in N(\Delta^{-1}R)$ for $s \in \Delta$, $a \in N(R)$. Then $(s^{-1}a)^n = 0$ for some positive integer n. This implies $a^n = 0$. So $a \in N(R)$, let $({s_1}^{-1}b) \in \Delta^{-1}R$, for $s_1 \in \Delta$, $b \in R$ with $(s^{-1}a)({s_1}^{-1}b) = 0$ we have $(s_1 s)^{-1}ab = 0$, $ab = 0$ since R is right $\alpha-$nil reversible $b\alpha(a) = 0$. So $(ss_1)^{-1}b\alpha(a) = 0$ implies $s_1^{-1}s^{-1}b\alpha(a) = 0$. Thus $({s_1}^{-1}b)\ \bar{\alpha}\ (s^{-1}(a)) = 0$. Therefore $\Delta^{-1}R$ is right $\bar{\alpha}-$nil reversible. □

Let R be a ring with an endomorphism α. Remember that the map $R[x] \to R[x]$ (resp., $R[x, x^{-1}] \to R[x, x^{-1}]$) defined by $\sum_{i=0}^{m} a_i x^i \to \sum_{i=0}^{m} \alpha(a_i)x^i$ (resp., $\sum_{i=0}^{\infty} a_i x^i \to \sum_{i=0}^{\infty} \alpha(a_i)x^i$) is an endomorphism of $R[x]$ (resp., $R[x, x^{-1}]$) and clearly the map extends α. We still denote the extended map $R[x] \to R[x]$ (resp., $R[x, x^{-1}] \to R[x, x^{-1}]$) by $\bar{\alpha}$.

Corollary 3.13. *Let* R *be a ring with an endomorphism* α*, such that* $\alpha(1) = 1$*. Then* $R[x]$ *is right* $\bar{\alpha}-$*nil reversible if and only if* $R[x, x^{-1}]$ *is right* $\bar{\alpha}-$*nil reversible.*

Proof. Suppose that $R[x]$ is right $\bar{\alpha}-$nil reversible. Take $\Delta = \{1, x, x^2, x^3, ...\} \subseteq R[x]$. Clearly, Δ is multiplicatively closed under subset consisting of central regulars of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$ and $\bar{\alpha}(x) = x$ since $\alpha(1) = 1$. So $R[x, x^{-1}]$ is right $\bar{\alpha}-$nil reversible.
The proof of the converse is trivial. □

Proposition 3.14. *For a ring* R *with an endomorphism* α*,* $R[x]$ *is right* $\bar{\alpha}-$*nil reversible if and only if* $(\Delta^{-1}R)[x]$ *is right* $\bar{\alpha}-$*nil reversible.*

Proof. For the necessity, let $R[x]$ be right $\bar{\alpha}-$nil reversible, $f(x) = \sum_{i=0}^{n} s_i^{-1}a_i x^i \in N(\Delta^{-1}R)[x]$ and $g(x) = \sum_{j=0}^{m} t_j^{-1}b_j x^j \in (\Delta^{-1}R)[x]$ such that $f(x)g(x) = 0$. Suppose that $s = s_0 s_1 ... s_n$ and $t = t_0 t_1 ... t_m$. Then $f_1(x) = sf(x)$ is nilpotent of $R[x]$, $g_1(x) = tg(x) \in R[x]$, and $f_1(x)g_1(x) = 0$ implies $g_1(x)\bar{\alpha}(f_1(x)) = 0$. Since $R[x]$ is right $\bar{\alpha}-$nil reversible then $g(x)\bar{\alpha}(f(x)) = 0$. Thus $(\Delta^{-1}R)[x]$ is right $\bar{\alpha}-$nil reversible. □

A ring Ris called right Ore if for given $a, b \in R$ with b is regular, there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known fact that R is a right Ore ring if and only if the classical right quotient ting $Q(R)$ of R exists. Let R be a ring with the classical right quotient ring $Q(R)$. Then each automorphism α of R extends to $Q(R)$ by setting $\bar{\alpha}(ab^{-1}) = \alpha(a)(\alpha(b))^{-1}$ for $a, b \in R$, considering that $\alpha(b)$ is regular for each regular element $b \in R$.

Theorem 3.15. *Let R be a right Ore ring with the classical right quotient ring $Q(R)$ of R and α an automorphism of R. If $Q(R)$ is an NI ring, then R is right $\alpha -$ nil reversible if and only if $Q(R)$ is right $\bar{\alpha} -$ nil reversible.*

Proof. It is enough to prove the necessity. Suppose that $Q(R)$ be an NI ring and R be a right $\alpha -$ nil reversible ring. Then R is NI by [17, Lemma 2.1]. Let $AB = 0$ for $A = ab^{-1} \in N(Q(R))$, $B = cd^{-1} \in Q(R)$ where $a, b, c, d \in R$ with b, d regular. Set I_1 and I_2 be the ideals of $Q(R)$generated by A in $N(Q(R))$ and B in $Q(R)$ respectively. Then I_1 is nil ideal with $a = Ab \in I_1$ and I_2 is an ideal with $c = Bd \in I_2$, and so $a \in N(R)$, $c \in R$. Since Ris right Ore, there exists $c_1, b_1 \in R$ with b_1 regular such that $bc_1 = cb_1$ and $c_1{b_1}^{-1} = b^{-1}c$. Here note that $c_1 \in R$. In fact, $bc_1 = cb_1 \in I_2$ and so $c_1 = b^{-1}(bc_1) \in I_2$. And from $0 = AB = ab^{-1}cd^{-1} = ac_1{b_1}^{-1}d^{-1}$, we have $0 = ac_1$ and so $c_1\alpha(a) = 0$. Then

$0 = c_1\alpha(a) = c_1\alpha(a)\alpha(b) = c_1\alpha(ab)$

$\Rightarrow 0 = \alpha(ab)\alpha(c_1) = \alpha(abc_1)$(since $ab \in N(R),\ c_1 \in R$)

$\Rightarrow 0 = abc_1 = acb_1 \Rightarrow 0 = ac = c\alpha(a)$ (since $a \in N(R), c \in R$).

And now for $a \in N(R), d \in R$ with d regular, there exists $a_1 \in N(R)$, $d_1 \in R$ with d_1 regular such that $da_1 = \alpha(a)d_1$ where $\alpha(a) \in N(R)$ and $a_1{d_1}^{-1} = d^{-1}\alpha(a)$ by the same calculation as above. Then $a_1 = d^{-1}\alpha(a)d_1 \in N(R)$ (because $\alpha(a) \in N(R)$) and

$0 = c\alpha(a) = c\alpha(a)d_1 = \alpha(a)d_1\alpha(c) = da_1\alpha(c)$ (since $\alpha(a)d_1, da_1 \in N(R),\ c \in R$).

$0 = a_1\alpha(c) = \alpha(c)\alpha(a_1) = \alpha(ca_1)$ (since d is regular and $a_1 \in N(R),\ \alpha(c) \in R$).

$ca_1 = 0$ (since α is an automorphism).

$B\bar{\alpha}(A) = cd^{-1}\bar{\alpha}(ab^{-1}) = c\left(d^{-1}\alpha(a)\right)\alpha(b)^{-1} = ca_1{d_1}^{-1}\alpha(b)^{-1} = 0.$

We obtaining that $Q(R)$ is right $\bar{\alpha} -$ nil reversible. □

Let R be a ring and α a monomorphism of R. Now we consider the Jordan's construction of an over-ring of R by α (for more details, see [18]). Let $A(R, \alpha)$ be the subset $\{x^{-i}rx^i : r \in R \text{ and } i \geqslant 0\}$of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Notice that for $j \geqslant 0$, $x^jr = \alpha^j(r)x^j$ implies $rx^{-j} = x^{-j}\alpha^j(r)$ for $r \in R$. This yields that we have $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$ for each $j \geqslant 0$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the natural operations that follow: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j)$ $=x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $i, j \geqslant 0$ and $r, s \in R$.

Note that $A(R, \alpha)$ is an over-ring of R, and $\bar{\alpha} : A(R, \alpha) \to A(R, \alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$ is an automorphism of $A(R, \alpha)$. Jordan demonstrated, with the use of left localization of the skew polynomial $R[x; \alpha]$ with respect to the set of powers of x, that for any pair (R, α), such an extension $A(R, \alpha)$ always exists in [18]. This ring $A(R, \alpha)$ is usually called the Jordan extension of R by α.

Proposition 3.16. *Let R be a ring with a monomorphism α, then R is right $\alpha -$ nil reversible if and only if the Jordan extension $A = A(R, \alpha)$ of R by α is right $\bar{\alpha} -$ nil reversible.*

Proof. If R is right $\alpha -$ nil reversible, then so is any subring S with $\alpha(S) \subseteq S$. Hence it is sufficient to show the necessity.

Assume that R is right $\alpha -$ nil reversible and $ab = 0$ for $a = x^{-i}r_1x^i \in N(A)$, $b = x^{-j}r_2x^j \in A$, for $i, j \geqslant 0$. Then $r_1 \in N(R)$, $r_2 \in R$ obviously. From $ab = 0$, we have $\alpha^j(r_1)\ \alpha^i(r_2) = 0$ and so $0 = \alpha^i(r_2)\alpha(\alpha^j(r_1)) = \alpha^i(r_2)\alpha^{j+1}(r_1)$ by assumption. Thus, $b\bar{\alpha}(a) = (x^{-j}r_2x^j)\ \bar{\alpha}\ (x^{-i}r_1x^i) = (x^{-j}r_2x^j)(x^{-i}\alpha(r_1)x^i)$
$= x^{-(j+i)}\alpha^i(r_2)\alpha^j(\alpha(r_1))x^{i+j} = x^{-(j+i)}\alpha^i(r_2)\alpha^{j+1}(r_1))x^{i+j} = 0$. Therefore Jordan extension $A(R, \alpha)$ is right $\bar{\alpha} -$ nil reversible. □

Let A be an algebra (not necessarily with identity) over a commutative ring C. According to Dorroh [19] The Dorroh extension of A by C is the ring denoted by $D(A, C) = \{(a, s) : a \in A,\ s \in C\}$ with the operations $(a_1, s_1) + (a_2, s_2) = (a_1 + a_2, s_1 + s_2)$ and $(a_1, s_1)\ (a_2, s_2) = (a_1a_2 + s_1a_2 + s_2a_1, s_1s_2)$ for all $a_i \in A$ and $s_i \in C$, $i = 1, 2$. For an $C-$endomorphism α of A and the Dorroh extension D of A by C, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(a, s) = (\alpha(a), s)$ is an $C-$algebrahomomorphism.

Theorem 3.17. *Let R be an algebra with identity over a commutative reduced ring Z, with a $Z-$endomorphism α. Then R is right $\alpha-$nilreversible if and only if the Dorroh extension D of R by Z is right $\bar{\alpha}-$nil reversible.*

Proof. It is easy to notice that $N(D) = (N(R), 0)$, since Z is a commutative reduced ring. Let $(a, 0) \in N(D\ (R, Z))$ and $(s, m) \in D\ (R, Z)$ with $(0, 0) = (a, 0)\ (s, m) = (a\ (s + m), 0)$. Thus $a\ (s + m) = 0$, $a \in N(R)$. Since R is right $\alpha-$nil reversible, we get $s + m \in Z$, $(s + m)\ \alpha(a) = 0$. So $(s, m)\ \bar{\alpha}\ (a, 0) = 0$. So $D\,(R, Z)$ is right $\bar{\alpha}-$nil reversible. □

4. Further notes

In this section, we introduce some examples and theorems of nil reversible rings.

The Armendariz property of rings and the nil reversibility do not imply each other as shown in the next example:

Example 4.1. (1) *Let R be a reduced and consider the ring $A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$.*

Then A is Armendariz by [20, Proposition 2]. However, A is not nil reversible. For $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(A)$, $b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in A$, we get $ab = 0$ but $ba \neq 0$.

(2) *Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in Z_4 \right\}$ since R is commutative, then R is nil reversible.*

For $f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R[x]$, we get $f(x)g(x) = 0$ but $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$. Thus R is not Armendariz.

The rings $H_{(x,y)}(R)$ [21]: Let R be a ring and let x, y be in the center of R. Let

$$H_{(x,y)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in Mat_3(R) \mid a, c, d, e, f \in R,\ a - d = xc,\ d - f = ye \right\}.$$

Then $H_{(x,y)}(R)$ is a subring of $Mat_3(R)$. Note that any element A of $H_{(x,y)}(R)$ has the form

$$\begin{pmatrix} xc + ye + f & 0 & 0 \\ c & ye + f & e \\ 0 & 0 & f \end{pmatrix}.$$

Theorem 4.2. *For a ring R, The following holds:*

(1) *If R is a reduced ring, then $H_{(0,0)}(R)$ is nil reversible.*

(2) *If R is reduced, then $H_{(0,1)}(R)$ is nil reversible.*

(3) *If R is reduced, then $H_{(1,0)}(R)$ is nil reversible.*

Proof. (1) Suppose that R is a reduced ring and $A = \begin{pmatrix} a & 0 & 0 \\ c & a & e \\ 0 & 0 & a \end{pmatrix} \in N(H_{(0,0)}(R))$ be nilpotent. By [21, Lemma 4.1] , a is nilpotent. By assumption, $a = 0$. Let $B = \begin{pmatrix} s & 0 & 0 \\ t & s & v \\ 0 & 0 & s \end{pmatrix} \in H_{(0,0)}(R)$ with $AB = 0$ implies

$cs = 0$ and $es = 0$. Since R is reduced then this implies $sc = 0$ and $se = 0$. Then $BA = \begin{pmatrix} 0 & 0 & 0 \\ sc & 0 & se \\ 0 & 0 & 0 \end{pmatrix} = 0$. Hence $H_{(0,0)}(R)$ is nil reversible.

(2) Assume that R is a reduced ring and let $A = \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(H_{(0,1)}(R))$ and $B = \begin{pmatrix} e+f & 0 & 0 \\ a & e+f & e \\ 0 & 0 & f \end{pmatrix} \in H_{(0,1)}(R)$, with $AB = 0$ implies $c\,(e+f) = 0$ and $(e+f)\,c = 0$. Then $BA = 0$. thus $H_{(0,1)}(R)$ is nil reversible.

(3) Let R be a reduced ring, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} \in N(H_{(1,0)}(R))$ and $B = \begin{pmatrix} f+c & 0 & 0 \\ c & f & d \\ 0 & 0 & f \end{pmatrix} \in H_{(1,0)}(R)$, with $AB = 0$ implies $ef = 0$ and $fe = 0$. Then $BA = 0$. Therefore $H_{(1,0)}(R)$ is nil reversible. □

The condition of R being reduced in theorem 4.2. is not superfluous, as in the next example:

Example 4.3. *Let K be a field and $R = K\langle a, b\rangle$ be the free algebra with non-commuting indeterminates a, b over K. Let I be the ideal of R generated by ab and a^2. Assume that ring $\bar{R} = R/I$. Identify a and b with their images in $\bar{R}$. Then $\overline{ab} = 0$, but $\overline{ba} \neq 0$.*

Note that $\bar{R}$ is not reduced. Consider the ring $A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \bar{R} \right\}$. *Let* $X = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(A)$, $Y = \begin{pmatrix} b & 1 & 1 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \in A$. *Then $XY = 0$ but $YX \neq 0$. Therefore A is not nil reversible.*

Proposition 4.4. *Let R be a ring and I a reversible ideal of R. Then R/I is a nil reversible ring.*

Proof. Let $\bar{a}$ indicate the image of $a \in R$ in R/I under the natural homomorphism from R onto R/I. Suppose that $\bar{a} \in N(R/I)$, $\bar{b} \in R/I$ with $\bar{a}\bar{b} = 0$. Then $ab \in I$ and so $ba \in I$ since I is reversible. Therefore $\bar{b}\bar{a} = 0$. It means that R/I is nil reversible. □

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