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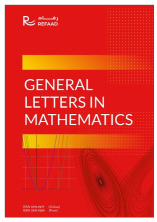

On Chaotic Operators

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Abstract

In this paper, we present new results on chaotic operators on Banach spaces. Also we study the relation between subspace-chaoticity and the direct sum of two operators. In particular, we prove that if the direct sum of two operators is a subspace-chaotic, then both operators are subspace-chaotic, but the converse is not true in general. We show that this is true under necessary conditions.

Keywords: Chaotic operators, subspace-chaotic, Direct sum of Banach spaces, Devenay chaos.
2010 MSC: 47A16, 47B99.

1. Introduction

Let X be an infinite-dimensional separable complex Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. For T in $\mathcal{B}(X)$, x in X, the orbit of the vector x is denoted by $Orb(T, x) = \{T^n x : n \geqslant 0\}$. We say that x is a hypercyclic vector for T if the set $\{T^n x : n \geqslant 0\}$ is dense in X and the operator $T \in \mathcal{B}(X)$ is called a hypercyclic operator if it has a hypercyclic vector.

The definition of hypercyclicity was already studied by G.D. Birkhoff [5] in 1922. Also he proved the first characterization of a hypercyclic operator which is a direct application of the Baire category theorem. In 1929 G.D. Birkhoff [6] gave a historical example of a hypercyclic operator. Later G.R. MacLane [7] found the same phenomenon for the differentiation operator. After all S. Rolewicz in [14] showed that not only non-linear operators but also linear operators can have dense orbits.

A new class of hypercyclic operators are proposed by G. Godefory and J.H. Shapiro when they are proposed to accept Devaney's definition of non-linear chaos as the right definition for linear chaos. In deed, there are many different definitions of chaotic operators from a metric space X to itself (for more details see, for example [2]). In this article we will deal with the most popular definition of chaotic operator, that is Devaney chaos [12]. That is, the linear operator is chaotic if it has a dense orbit, a dense set of periodic points and it is sensitive dependence on initial conditions. This result seems a quit strange because the general view among a wide range in mathematical community that is the chaos phenomena is intimately connected to non-linear phenomena. This is due to linear system behaves as predictable way and the chaos behaves as non-predictable way. The three operators of Birkhoff, Maclane and Rolewicz

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have been chaotic operators. Hypercyclic and chaotic operators have become an interesting and attractive field of research. A lot of excellent results were proposed for hypercyclic operator but until now there are a scarcity of researches related to chaotic operator.

In 2011, Madore and Martnez-Avendano in [3] introduced the concept of subspace-hypercyclic operators. An operator $T \in \mathcal{B}(X)$ is a subspace-hypercyclic for a subspace M or (M-hypercyclic) if there exist $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. Also, They defined a subspace-transitive operators and show that every M-transitive is M-hypercyclic. They proved that the new phenomena like hypercyclicity, is purely infinite-dimensional phenomena. In other word, if T is subspace-hypercyclic operator for M, then M must be infinite-dimensional subspace. In 2012, Talebi and Moosapoor [15] defined subspace-chaotic and subspace-mixing operators which have additional properties than subspace-hypercyclic.

This article is organized as follows. In section 2 we give the basic definitions. In section 3 we present new results related to chaotic operators on Banach spaces. In section 4 the definition of subspace-chaotic operator will be recalled. Also, we will show that if the direct sum of two operators is subspace-chaotic, then both operators are subspace-chaotic, while the converse is true under certain conditions.

2. Preliminaries

A lot of definitions in this section are valid in F-space, (i.e. a complete metrizable topological vector space) and Fréchet space, (i.e. a locally convex F-space). Some readers are uncomfortable with F-spaces and Fréchet spaces, so we will drop these definitions on Banach spaces that are clearly Fréchet spaces. Note that Fréchet space has no isolated points.

Definition 2.1. [8] $T \in \mathcal{B}(X)$ is called topologically transitive if, for any pair of non-empty open sets U, V of X, there exist some $n \geqslant 0$ such that $T^n(U) \cap V \neq \emptyset$.

G. D. Birkhoff introduce a wonderful result by proving that hypercyclicity equivalent with topologically transitivity on separable F-space. this result referred to as Birkhoff's transitivity theorem

Theorem 2.2 (Birkhoff's Transitivity Theorem)**.** *[1] Let X be a separable F-space. The following are equivalents:*

1. T *is hypercyclic;*
2. T *is topologically transitive.*

In that case, the set of all hypercyclic elements is a dense G_δ subset of X.

Definition 2.3. [1] Let X be Banach space and $T \in \mathcal{B}(X)$.

1. A vector $x \in X$ is called a fixed point vector if $Tx = x$.
2. A vector $x \in X$ is called a periodic vector if there exist some $n \in \mathbb{N}$ such that $T^n x = x$. In addition the least number n is called the $period$ of x.
3. The set of all periodic vectors of T denoted by $\mathrm{Per}(T)$.

Definition 2.4. [8] Let (X, d) be a metric space without isolated points. Then a map $T : X \to X$ is said to have sensitive dependence on initial conditions if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exist some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geqslant 0$, $d(T^n x, T^n y) \geqslant \delta$. The number δ is called sensitivity constant for T.

Definition 2.5 (Devaney chaos - preliminary version)**.** [8] Let T be a continuous map acting on some metric space (X, d). The map T is said to be $chaotic$ if

1. T is topologically transitive;
2. T has a dense set of periodic points.
3. T has sensitive dependence on initial conditions.

Banks *et al.* in [10] show that the sensitive dependence condition implied from the first two conditions on a metric space without isolated points.

Theorem 2.6. *[10] Let T be a topologically transitive map acting on some metric space (X, d) without isolated points and has dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

Fortunately, we can drop sensitive dependence from the above definition when the space X is F-space. So in Banach space we can safely drop the third condition.

Now in what follows, X will refer to an infinite-dimintional separable complex Banach space, unless otherwise indicated. The Devaney chaos definition on Banach space will be as follow:

Definition 2.7 (Devaney chaos)**.** An operator $T \in \mathcal{B}(X)$ is chaotic operator if it satisfies the following conditions:

1. T is hypercyclic;
2. T has a dense set of periodic vectors.

3. Main Results

Theorem 3.1. *Let $T \in \mathcal{B}(X)$, $x \in Per(T)$ of period n. Then any vector in $Orb(T, x)$ has period n.*

Proof. Suppose that $x \in Per(T)$ of period n. Pick $y \in Orb(T, x)$ be arbitrary such that $x \neq y$. Then $y = T^k x$ for some $k < n$.

$$T^n y = T^n T^k x = T^k T^n x = T^k x = y.$$

Now we want to show that the period of y is n. In other word, n is the least positive integer satisfies $T^n y = y$. Suppose on the contrary, there is $n_0 < n$ such that

$$T^{n_0} y = y. \tag{3.1}$$

Then $T^k x = y = T^{n_0} y$ implies $(T^k)^n x = (T^{n_0})^n y$. Since the period of x is n and $T^{n_0} y = y$, then $x = T^{kn} x = T^{n_0 n} y = y$ which is a contradiction. So there is no $n_0 < n$ with $T^{n_0} y = y$. By these results we conclude that n is the least positive integer satisfies $T^n y = y$. Hence, the period of y is n. □

Corollary 3.2. *Let $T \in \mathcal{B}(X)$, $x \in Per(T)$ of period n. Then any vector in $Orb(T, x)$ has the same orbit.*

Proof. Suppose that $x \in Per(T)$ of period n and take $y \in Orb(T, x)$ such that $x \neq y$. Then $y = T^k x$ for some $k < n$ and from Theorem 3.1 $T^n y = y$. Let $z \in Orb(T, y)$, then $z = T^{k_0} y$ for some $k_0 \leqslant n$. So, $z = T^{k_0} y = T^{k_0} T^k x = T^{k+k_0} x$. Hence, $z \in Orb(T, x)$ and then $Orb(T, y) \subseteq Orb(T, x)$. On the other hand, let $h \in Orb(T, x)$, then $h = T^m x$ for some $m \leqslant n$. If $m \geqslant k$, $h = T^m x = T^{m-k+k} x = T^{m-k} T^k x = T^{m-k} y \in Orb(T, y)$, however, if $k < m$, use the fact $h = T^n h$ from Theorem 3.1 to get $h = T^n h = T^n T^m x = T^{n+m-k+k} x = T^{n+m-k} T^k x = T^{n+m-k} y \in Orb(T, y)$, because $n + m - k > 0$. In any case we have $Orb(T, x) \subseteq Orb(T, y)$. At the end, $Orb(T, y) = Orb(T, x)$. □

Corollary 3.3. *Let $x, y \in Per(T)$. Then $Orb(T, x) = Orb(T, y)$ or $Orb(T, x) \cap Orb(T, y) = \emptyset$.*

Proof. Let $x, y \in Per(T)$ of period n, m respectively, W.L.O.G. suppose that $n \leqslant m$ and assume that $Orb(T, x) \cap Orb(T, y) \neq \emptyset$. Then there exist $z \in Orb(T, x) \cap Orb(T, y)$. So $z = T^{n_1} x$ and $z = T^{m_1} y$, for some $n_1 \leqslant n$ and $m_1 \leqslant m$.
$x = T^n x = T^{n-n_1+n_1} x = T^{n-n_1} T^{n_1} x = T^{n-n_1} z = T^{n-n_1}(T^{m_1} y) = T^{n-n_1+m_1} y$. Thus $x \in Orb(T, y)$. By Corollary 3.2 x and y have the same orbit. □

It is not so hard to observe that for a linear operator T on a Banach space X, the set of periodic vectors is a subspace.

Proposition 3.4. *Let* $T \in \mathcal{B}(X)$*. If* T *is a chaotic operator, then* $Per(T)$ *can't be closed.*

Proof. Let T be a chaotic operator then T is a hypercyclic operator. We claim that $Per(T)$ isn't closed because when T is chaotic so $X = \overline{Per(T)}$ and if $Per(T)$ is closed then $X = \overline{Per(T)} = Per(T)$. This show that X is a space of periodic vectors. Obviously, every orbit in X is a finite set and hence there is no dense orbit in X which is a contradiction with the hypercyclicity of the operator T. Hence, $Per(T)$ doesn't closed subspace. □

Next, we will show that the subspace of periodic vectors is strongly T-invariant subspace, i.e. $T(Per(T)) = Per(T)$.

Proposition 3.5. *If* $T \in \mathcal{B}(X)$ *is chaotic operator, then* $Per(T)$ *is strongly* T*-invariant subspace of* X*.*

Proof. Let $x \in Per(T)$ then there is $n \in \mathbb{N}$ such that $T^n x = x$. Now, by (3.1), $T^{n-1}x$ is periodic vector. So, $T(T^{n-1}x) = T^n x = x$. So, $x \in T(Per(T)$. Hence, $Per(T) \subseteq T(Per(T))$. On the other hand, if $y \in T(Per(T))$, then there is $x \in Per(T)$ of period n such that $Tx = y$ and $T^n x = x$. So, $y = Tx = TT^n x = T^n Tx = T^n y$. Hence, y is a periodic vector and $T(Per(T)) \subseteq Per(T)$. Therefore, $Per(T)$ is strongly T-invariant subspace of X. □

Theorem 3.6. *Let* $T \in \mathcal{B}(X)$ *be a chaotic operator. Then for each* $x \in X$ *there exist a periodic vector* $p \in X$ *and* $\delta > 0$ *such that* $dist(x, Orb(T, p)) \geqslant \delta$*.*

Proof. Since X is an infinite space and $Per(T)$ is dense in X, then there are infinite periodic vector in X. Let $x \in X$ we can find a periodic vector p such that $x \notin Orb(T, p)$ because $Per(T)$ is infinite subset. Since $Orb(T, p)$ is finite set then it is closed subset of X. Also since $x \notin Orb(T, p)$ then there is $\delta > 0$ such that $dist(x, Orb(T, p) \geqslant \delta$. □

In the following will present a special version of the Banach–Steinhaus theorem that is specific to Banach spaces.

Theorem 3.7 (Banach-Steinhaus Theorem)**.** *[13] Let* X, Y *be Banach spaces and* $T_j : X \to Y, j \in J$ *be operators. If, for every* $x \in X$, $\sup_{j\in J} \|T_j x\| < \infty$*, then* $\sup_{j\in J} \|T_j\| < \infty$.

Theorem 3.8. *Let* $T \in \mathcal{B}(X)$*. Then the following statements are equivalent*

1. T *has sensitive dependence on initial conditions;*
2. $\sup_{n\geqslant 0} \|T^n\| = \infty$*;*
3. T *has an unbounded orbit.*

Proof. (1⇒2). Since T has sensitive dependence on initial conditions, then there is a sensitivity constant $\delta > 0$ for T. Suppose that $\sup_{n\geqslant 0} \|T^n\| \leqslant M$ for some $M > 0$. Let $\varepsilon = \frac{\delta}{2M}$ and $x \in X$ be arbitrary. Then there exit $y \in X$ with $\|x - y\| < \varepsilon$ such that $\|T^{n_0}x - T^{n_0}y\| \geqslant \delta$ for some $n_0 \in \mathbb{N}$. For each $n \in \mathbb{N}$ we have $\|T^n x - T^n y\| = \|T^n(x - y)\| \leqslant \|T^n\|\|x - y\| < M\varepsilon = \frac{\delta}{2}$. In particular, $\delta \leqslant \|T^{n_0}x - T^{n_0}y\| < \frac{\delta}{2}$ which is a contradiction. Hence, $\sup_{n\geqslant 0} \|T^n\| = \infty$.

To prove (2⇒3) it suffices to apply Banach-Steinhaus theorem. That is if $\sup_{n\geqslant 0} \|T^n\| = \infty$, then there is $x \in X$, and $n \in \mathbb{N}$ such that $\sup_{n\geqslant 0} \|T^n x\| = \infty$. Hence, T has an unbounded orbit.

(3⇒1), suppose that T has unbounded orbit, then there is $y \in X$ such that $Orb(T, y)$ is unbounded orbit. So $\sup_{n\geqslant 0} \|T^n y\| \to \infty$ as $n \to \infty$. Let $\varepsilon > 0$ be given and $x \in X$. We want to show that there is a vector $y' \in X$ with $\|x - y'\| < \varepsilon$ and for some $n \in \mathbb{N}$ $\|T^n x - T^n y'\| \to \infty$. We can find $\lambda \in \mathbb{C}$ such that $\|\lambda y\| < \varepsilon$. Since y has unbounded orbit then λy has unbounded orbit. $\|x - (x + \lambda y)\| = \|\lambda y\| < \varepsilon$ and $\|T^n x - T^n(x + \lambda y)\| = \|T^n x - T^n x - T^n \lambda y\| = \|T^n \lambda y\| \to \infty$ as $n \to \infty$. Hence, for every $x \in X$ there is $x + \lambda y$ such that $\|x - (x + \lambda y)\| < \varepsilon$ and for some $n \in \mathbb{N}$ $\|T^n x - T^n(x + \lambda y)\| \to \infty$. □

Next we are going to show that every convergent sequence in $Per(T)$ with finite periods will converges to a periodic vector. On the other hand, every convergent sequence in $Per(T)$ with infinite periods will converges to a non-periodic vector.

Theorem 3.9. *Let $T \in \mathcal{B}(X)$, $(x_n)_{n\in\mathbb{N}}$ be a sequence of periodic vectors which is converges, d_n be the period of the vector x_n and $d := \sup\{d_i : d_i$ the period of $x_i\}$. If $d < \infty$, then $(x_n)_n$ will converges to a periodic vector and if $d = \infty$, then $(x_n)_n$ converges to a non-periodic vector.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $Per(T)$ such that $x_n \to x$. Suppose that d_n and d as defined above. Assume that $d < \infty$, then we can take l to be the least common multiple of d_i's and then $T^l x_n = x_n$ for all n. Also, since T is bounded then we have $T^l x_n \to T^l x$. But the limit in X is unique, so $T^l x = x$. Hence x is periodic vector. On the other hand, If $d = \infty$, then there is no common multiple of the periods of the vectors x_n's. Thus, $T^k x_n \neq x_n$ for all $k \in \mathbb{N}$. Now, assume that x is a periodic vector with period s. Since $d = \infty$, then there is a vector in $(x_n)_n$ with period bigger than s. Also, since $x_n \to x$, then every ball a round x contains a vector of $(x_n)_n$ with periods bigger than s. We can construct a subsequence (x'_n) of (x_n) that contains vectors with periods bigger than s. Obviously, $x'_n \to x$ which is a contradiction. Hence x must be a non-periodic vector. □

Corollary 3.10. *Let $T \in \mathcal{B}(X)$ be a chaotic operator. Then the periods of the vectors haven't an upper bound.*

Proof. Let T be a chaotic operator. Then $Per(T)$ is dense in X and $Per(T) \neq \overline{Per(T)}$, so there is a vector $y \in \overline{Per(T)} \setminus Per(T)$ and a sequence $(y_n)_{n\in\mathbb{N}}$ in $Per(T)$ such that $y_n \to y$. For every n define d_n be the period of y_n. Let $d := \sup\{d_i : d_i$ the period of $y_i\}$. Since $y \notin Per(T)$, then $d = \infty$, so, the periods of vectors in (y_n) haven't an upper bound. Therefore, the periods of the vectors in X haven't an upper bound. □

Theorem 3.11. *Let $T \in \mathcal{B}(X)$ be a chaotic operator and $M \subset Per(T)$ such that $\overline{Per(T) \setminus M} \neq X$. Then M is an infinite subset.*

Proof. Let T be a chaotic operator and $M = \{x_1, x_2, \dots, x_n\} \subset Per(T)$ with $\overline{Per(T) \setminus \{x_1, x_2, \dots, x_n\}} \neq X$. Since X is Banach space, then X hasn't isolated vectors.

$$\begin{aligned} X = \overline{Per(T)} &= \overline{Per(T) \setminus M \cup M} \\ &= \overline{Per(T) \setminus M} \cup \overline{M} \\ &= \overline{Per(T) \setminus M} \cup M, \text{ because } M \text{ is finite and then closed.} \end{aligned}$$

Since $X \neq \overline{Per(T) \setminus M}$, then there is a non-empty set $K \subseteq M$ such that $K \cap \overline{Per(T) \setminus M} = \emptyset$. This show that the elements in K are isolated vectors which is a contradiction. Hence, M must be infinite. □

Corollary 3.12. *Let $T \in \mathcal{B}(X)$ be a chaotic operator and x be a periodic vector. Then $X = \overline{Per(T) \setminus \{x\}}$*

Theorem 3.13. *[16] Let $T \in \mathcal{B}(X)$, Then the following statements are equivalent:*

1. *T is a hypercyclic operator.*
2. *For every non-empty open sets $\mathcal{U}$ and $\mathcal{V}$ in X, there is $n \geqslant 0$ such that $T^n(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$*
3. *For each $x, y \in X$, there are sequences (x_k) in X, and (n_k) in $\mathbb{N}$ such that $x_k \to x$ and $T^{n_k} x_k \to y$.*
4. *For each $x, y \in X$, and each neighbourhood $\mathcal{W}$ for zero in X, there is z in X, and n in $\mathbb{N}$ such that $x - z \in \mathcal{W}$ and $T^n z - y \in \mathcal{W}$.*

Proposition 3.14. *Let $T \in \mathcal{B}(X)$ be a chaotic operator. For each $x, y \in X$, there is a sequence of periodic vectors (z_k) in X, and (n_k) in $\mathbb{N}$ such that $z_k \to x$ and $T^{n_k} z_k \to y$.*

Proof. Let $x, y \in X$ and $\varepsilon > 0$. By Theorem 3.13 there are sequences (x_k) in X and (n_k)in $\mathbb{N}$ such that $x_k \to x$ and $T^{n_k} x_k \to y$. Since T is a bounded operator then $T^{n_k} x_k \to T^{n_k} x$. Since T is chaotic operator, then there is a sequence of periodic vectors (z_k) in X, such that $z_k \to x$ and then $T^{n_k} z_k \to T^{n_k} x$. Now for sufficiently large k we get $\|T^{n_m} z_m - T^{n_m} x_m\| < \frac{\varepsilon}{2}$ and $\|T^{n_m} x_m - y\| < \frac{\varepsilon}{2}$, for $m > k$. Hence, $\|T^{n_m} z_m - y\| \leqslant \|T^{n_m} z_m - T^{n_m} x_m\| + \|T^{n_m} x_m - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for $m > k$. □

Let $T \in \mathcal{B}(X)$ and $Per(T)$ is closed and dense in X we conclude that $X = \bigcup_{x \in Per(T)} Orb(T, x)$. Since the different orbits of periodic vectors are finite and disjoints sets, then all the orbits under T is closed and then X can be separate into disjoint closed sets.

Theorem 3.15. *Let $T \in \mathcal{B}(X)$ be hypercyclic and not chaotic. Then every vector with dense orbit lies out $\overline{Per(T)}$*

Proof. Let $T \in \mathcal{B}(X)$ be hypercyclic and not chaotic. Then $\overline{Per(T)} \neq X$. Since T is hypercyclic then there is a vector $x \in X$ such that $Orb(T, x)$ is dense in X. Clearly, $x \notin Per(T)$. Hence, $x \in \partial Per(T)$ or $x \in X \backslash \overline{Per(T)}$. If $x \in \partial Per(T) \subseteq \overline{Per(T)}$, then there is a sequence $(x_n) \subseteq Per(T)$ such that $x_n \to x$. Since T is bounded, then $Tx_n \to Tx$. In general $T^k x_n \to T^k x$ for every $k \in \mathbb{N}$. By Proposition 3.5, for every k, $(T^k x_n)$ are sequences in $Per(T)$. Thus, for every $k \in \mathbb{N}$, $T^k x$ are limits points for sequences from $Per(T)$. Thus $Orb(T, x) = \{x, Tx, T^2 x, \dots\}$ is contained in $\overline{Per(T)}$. This shows that $\overline{Per(T)}$ is dense in X, that contradict with our first result. Therefore, the vector x and it's orbit lies in $X \backslash \overline{Per(T)}$. □

4. Subspace Chaotic Operators

In this section we will deal with the case when the space X has no dense orbit and has a non-trivial subspace with dense orbit. This phenomena is called subspace-hypercyclicity or for simply (M-hypercyclicity) for subspace M

Definition 4.1. [15] Let M be non-trivial subspace in X. An operator $T \in \mathcal{B}(X)$ is called subspace-hypercyclic operator for M or simply (M-hypercyclic) if there exist a vector $x \in X$ such that $Orb(T, x) \cap M$ is dense in M. Such a vector x is called an M-hypercyclic vector for T.

The subspace M in the definition maybe closed or not, but when the subspace is closed the results will be more interesting. In the following we will prove that if the operator T is M-hypercyclic then T is $\overline{M}$-hypercyclic.

Proposition 4.2. *Let $T \in \mathcal{B}(X)$ and let M be a non-trivial subspace in X. If T is M-hypercyclic then T is $\overline{M}$-hypercyclic.*

Proof. If the subspace M is closed, then there is no thing to prove. Now suppose that M is a subspace of X which is not closed and T is M-hypercyclic, i.e, $\overline{Orb(T, x) \cap M} = M$, then $\overline{\overline{Orb(T, x) \cap M}} = \overline{M}$. It is well known that $Orb(T, x) \cap M \subset Orb(T, x) \cap \overline{M} \subseteq \overline{Orb(T, x) \cap \overline{M}}$, so we have $\overline{\overline{Orb(T, x) \cap M}} \subseteq \overline{\overline{Orb(T, x) \cap \overline{M}}}$. Hence, $\overline{M} = \overline{\overline{Orb(T, x) \cap M}} \subseteq \overline{Orb(T, x) \cap \overline{M}}$. It is well known that, $\overline{Orb(T, x) \cap \overline{M}} \subseteq \overline{M}$ which implies, $\overline{Orb(T, x) \cap \overline{M}} = \overline{M}$. This shows that T is $\overline{M}$-hypercyclic. □

From now we will not worry about assuming that the subspace M is closed.

In what follows, we will use the notation of direct sum of two Banach spaces that defined as following: Let X and Y are two Banach spaces, then the direct sum of X and Y is defined as follow:

$$X \oplus Y = \{(x, y) : x \in X, y \in Y\}$$

and the norm $\|(x, y)\|^2 = \|x\|^2 + \|y\|^2$ on $X \oplus Y$ make the space $X \oplus Y$ is Banach space. For more details of direct sum of Banach spaces see [9].

Next, the readers should be convenient with definition and properties of the projection map P on a subspace in Hilbert space. So we recall the definition of the orthogonal projection of a Hilbert space onto a closed subspace.

Definition 4.3. [4] Let $\mathcal{H}$ be Hilbert space, and M be a closed subspace of $\mathcal{H}$ so that $\mathcal{H} = M \oplus M^{\perp}$. The orthogonal projection is the bounded linear operator $P : \mathcal{H} \to \mathcal{H}$, such that for each $x = m \oplus m' \in \mathcal{H}$, where, $m \in M$ and $m' \in M^{\perp}$, $P(x) = m$.

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and let M be non-trivial closed subspace in $\mathcal{H}$ and let $P : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection operator on M. If $\mathrm{Orb}(T, x) \cap M$ is dense in M, then $P(\mathrm{Orb}(T, x)) \cap M$ is dense in M.*

Proof.
Suppose that $\mathrm{Orb}(T, x) \cap M$ is dense in M, let $y \in M$ then there exist a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathrm{Orb}(T, x) \cap M$ that converges to y. Clearly, $(y_k)_{k\in\mathbb{N}}$ in $\mathrm{Orb}(T, x)$ and $(y_k)_{n\in\mathbb{N}}$ in M which implies that for every k, $y_k = T^{n_k}x$ for some $n_k \in \mathbb{N}$ and $P(y_k) = y_k$. Thus for every $k \in \mathbb{N}$ we have, $y_k = P(y_k) = P(T^{n_k}x) \in P(\mathrm{Orb}(T, x))$. Hence, $(y_k)_{k\in\mathbb{N}}$ in $P(\mathrm{Orb}(T, x)) \cap M$ that converges to y. Therefore, $P(\mathrm{Orb}(T, x)) \cap M$ is dense in M. □

Definition 4.5. [15] Let M be non-trivial subspace in X. An operator $T \in \mathcal{B}(X)$ is called subspace-chaotic operator for M or simply (M-chaotic) it satisfies the following conditions:

1. T is M-hypercyclic;
2. T has a dense set of periodic vectors in M.

Theorem 4.6. *[11] Let M_1 and M_2 be closed subspaces of X and $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-hypercyclic, then T_1 is M_1-hypercyclic and T_2 is M_2-hypercyclic.*

Theorem 4.7. *Let M_1 and M_2 be closed subspaces of X and $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-chaotic, then T_1 is M_1-chaotic and T_2 is M_2-chaotic.*

Proof. By Theorem (4.6) we have T_1 is M_1-hypercyclic and T_2 is M_2-hypercyclic. Now we want to show that T_1 and T_2 have dense sets of periodic vectors. Let $\varepsilon > 0$ and $x \in M_1$, $y \in M_2$. Since $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-chaotic, then there is a periodic vector $(a, b) \in M_1 \oplus M_2$ of period n such that $\|(x, y) - (a, b)\|^2 < \varepsilon^2$. It follows that, $\|(x - a, y - b)\|^2 < \varepsilon^2$ and then $\|x - a\|^2 + \|y - b\|^2 < \varepsilon^2$. Then $\|x - a\| < \varepsilon$ and $\|y - b\| < \varepsilon$. Since, $(a, b) = (T_1 \oplus T_2)^n(a, b) = (T_1^n a, T_2^n b)$, then a and b are periodic vectors in M_1 and M_2 respectively. Hence, $\mathrm{Per}(T_1)$ is dense in M_1 and $\mathrm{Per}(T_2)$ is dense set in M_2. Therefore, T_1 is M_1-chaotic and T_2 is M_2-chaotic. □

Definition 4.8. [15] Let $T \in \mathcal{B}(\mathcal{H})$ and let M be a closed non-zero subspace of X. We say T is subspace mixing or (M-mixing), if for all non-empty sets $U \subseteq M$, $V \subseteq M$ both relatively open, there exists a positive integer N such that $T^n(U) \cap V$ is non-empty for all $n > N$.

Theorem 4.9. *[11] If T_1 is M_1-hypercyclic and T_2 is M_2-hypercyclic, and at least one of them is subspace-mixing, then $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-hypercyclic.*

Theorem 4.10. *If T_1 is M_1-chaotic and T_2 are M_2-chaotic, and at least one of them is subspace-mixing, then $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-chaotic.*

Proof. By Theorem (4.9) we have $T_1 \oplus T_2$ is$(M_1 \oplus M_2)$-hypercyclic. Now we want to show that $\mathrm{Per}(T_1 \oplus T_2)$ is dense in $M_1 \oplus M_2$. Let $\varepsilon > 0$ and $(x, y) \in M_1 \oplus M_2$, where, $x \in M_1$ and $y \in M_2$. Then there is a periodic vector $a \in M_1$ and periodic vector $b \in M_2$ such that $\|x - a\|^2 < \frac{\varepsilon^2}{2}$ and $\|y - b\|^2 < \frac{\varepsilon^2}{2}$. It follows that, $\|(x, y) - (a, b)\|^2 = \|(x - a, y - b)\|^2 = \|x - a\|^2 + \|y - b\|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$. Thus, $\|(x, y) - (a, b)\| < \varepsilon$. Clearly, $(a, b) \in \mathrm{Per}(T_1 \oplus T_2)$. Hence, $\mathrm{Per}(T_1 \oplus T_2)$ is dense in $M_1 \oplus M_2$. Therefore, $T_1 \oplus T_2$ is $(M_1 \oplus M_2)$-chaotic. □

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