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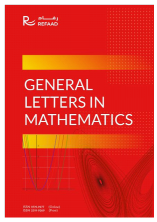


# Two different scenarios when the Collatz Conjecture fails

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**Abstract**

In this article, we construct networks of Collatz sequences such that the initial odd terms of these sequences increase monotonically. We also show how the subsequence of odd numbers in a Collatz sequence can be extended backwards, forever. Convergent sequences cannot contain divergent subsequences. Thus, we conclude that the Collatz Conjecture is false.

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## 1. Introduction

Given a positive integer $A$, construct the sequence $c_i$ as follows:

$$\begin{aligned} c_i &= A && \text{if } i = 0; \\ &= 3c_{i-1} + 1 && \text{if } c_{i-1} \text{ is odd;} \\ &= c_{i-1}/2 && \text{if } c_{i-1} \text{ is even.} \end{aligned}$$

The sequence $c_i$ is called a *Collatz sequence* with *starting number* $A$. The *Collatz Conjecture* says that this sequence will eventually reach the number 1, regardless of which positive integer is chosen initially. The sequence gets in to an infinite cycle of 4, 2, 1 after reaching 1.

*The Collatz sequence of* 911 *is:*

$$\begin{aligned} &911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, \\ &1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, \\ &70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots \end{aligned}$$

For the rest of this article, we will ignore the infinite cycle of $4, 2, 1$, and say that a Collatz sequence *converges to* 1, if it reaches 1. A comprehensive study of the Collatz Conjecture can be found in [3], [4], and [5].

In this article, as we did before in [1], [2], and [? ], we focus on the subsequence of odd numbers of a Collatz sequence. This is because every even number in a Collatz sequence has to reach an odd number

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after a finite number of steps. Observe that the Collatz Conjecture implies that the subsequence of odd numbers of a Collatz sequence converges to 1.

In Section 2, we use networks of Collatz sequences [2] to prove that the Collatz Conjecture fails. In Section 3, we use the notion of Reverse Collatz sequences [1] to give another proof of the collapse of the Collatz Conjecture.

## 2. Networking to prove that the Collatz Conjecture is false.

In this section, we use an array of Collatz sequences to demonstrate how the Collatz Conjecture fails. In [2], we proved the following theorem which showed that the Collatz sequence of an odd number $A$ merges either with the Collatz sequence of $(A-1)/2$ or $2A+1$.

(Theorem 2.1, [2]) Let $N$ be an odd number. Let $n_0 = N$, $m_0 = 2n_0 + 1 = 2N + 1$, and $l_0 = 2m_0 + 1 =$ 4N+3 . Let $n_i, m_i$, and $l_i$ denote the subsequence of odd numbers in the Collatz sequence of $n_0, m_0$, and $l_0$, respectively. Then, for some integer $r$, $n_{r+1} = (3n_r + 1)/2^k$ such that $k > 1$. Let $r$ be the smallest such integer. Then, $m_{r+2} = (3m_{r+1} + 1)/2^j$ for some $j > 1$, and

$$
\begin{aligned}
&m_i = 2n_i + 1, \text{ for } i \leqslant r,\\
&m_{r+1} = 2^k n_{r+1} + 1\\
&l_i = 2m_i + 1 \text{ for } i \leqslant r + 1,\\
&l_{r+2} = 2^j m_{r+2} + 1
\end{aligned}
$$

If $k = 2$, then $m_i = n_i$ for $i > r + 1$. Otherwise, if $k > 2$ then $l_{r+2} = 4m_{r+2} + 1$ and $l_i = m_i$ for $i > r + 2$.

For some integer $u_0$, consider the sequence $u_i = 2u_{i-1} + 1$, for $i \geqslant 1$. Let $v_{j,i}$ denote the sequence of odd integers in the Collatz sequence of $u_i$. Theorem 2 tells us that for every $i > 0$, there is some $r$ such that for $j \leqslant r$, $v_{j,i} = 2v_{j,i-1} + 1$ and $v_{r+1,i} = 2^k v_{r,i-1} + 1$ where $k > 1$. This fact motivates us to construct the array of Collatz sequences in Theorem 2.

Let $A$ be an odd number. Write $A$ in its binary form,

$$
\begin{aligned}
A &= 2^{i_1} + 2^{i_2} + \cdots + 2^{i_m} + 2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1,\\
&\qquad \text{such that } i_1 > i_2 > \cdots > i_m > n + 1.
\end{aligned}
$$

The *tail* of $A$ is defined as $2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1$. We call $n$ the *length of the tail* of $A$ (See [1]).

*The tail of* $27 = 2^4 + 2^3 + 2 + 1$ *is* $2 + 1$ *and hence has length* 1*, the tail of* $161 = 2^7 + 2^5 + 1$ *is* 1 *and hence is of length* 0*, and the tail of* $31 = 2^4 + 2^3 + 2^2 + 2 + 1$ *is the entire number* $2^4 + 2^3 + 2^2 + 2 + 1$ *and therefore has length* 4*.*

In [1] we proved the following theorem about the odd numbers in a Collatz sequence. (Theorem 2 [1]) Let $A$ be an odd number and let $n$ denote the length of the tail of $A$. Let $a_i$ denote the sequence of odd numbers in the Collatz sequence of $A$ with $a_0 = A$.

1. If $n \geqslant 1$, then for some $i_1 > i_2 > \cdots > i_m > n + 1$,

$$
a_i = \frac{3a_{i-1} + 1}{2} = \frac{3^i}{2^i}(2^{i_1} + 2^{i_2} + \cdots + 2^{i_m} + 2^{n+1}) - 1, \text{ for } i = 1, \ldots n.
$$

The length of the tail of $a_i$ is $n - i$. Hence the length of the tail of the $n$-th odd number after $A$ is 0.

2. If $n = 0$, then

$$
a_1 = \frac{3A + 1}{2^k}, k \geqslant 2.
$$

**Corollary 2.1.** *Let* $A$ *be an odd number. If* $A \not\equiv 1 \bmod 4$*, then the next odd term in the Collatz sequence of* $A$ *is* $(3A + 1)/2$*.*

*Proof.* Since $A$ is odd and $A \not\equiv 1 \bmod 4$, $A \equiv 3 \bmod 4$. This implies that the tail of $A$ has length greater than zero. Hence the proof follows from Part 1 of Theorem 2.

□

(Theorem 5.1, [2]) For $n \not\equiv 1 \bmod 3$, define a diagonal array as follows. Let $u_0 = 4n+1$ and $u_i = 2u_{i-1}+1$, for $i \geqslant 1$. For $j \geqslant 0$, let $v_{0,j} = u_j$, and for $k \geqslant 1$, let $v_{k,k} = 3v_{k-1,k-1}+2$. Finally, for $j > i$, let $v_{i,j} = 2v_{i,j-1}+1$. We get an array

$$\begin{array}{ccccccccc} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & \dots \\ & v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} & v_{1,5} & v_{1,6} & v_{1,7} & \dots \\ & & v_{2,2} & v_{2,3} & v_{2,4} & v_{2,5} & v_{2,6} & v_{2,7} & \dots \\ & & & v_{3,3} & v_{3,4} & v_{3,5} & v_{3,6} & v_{3,7} & \dots \\ & & & & \dots & \dots & & & \end{array}$$

with the following properties:

1. $u_k \not\equiv 2 \bmod 3$ for all $k \geqslant 0$, whereas, $v_{i,j} \equiv 2 \bmod 3$, if $i > 0$ and $j > 0$.
2. $u_0 \equiv 1 \bmod 4$ and $v_{i,i} \equiv 1 \bmod 4$ for all $i$. $u_i \not\equiv 1 \bmod 4$ for $i > 0$ and $v_{i,j} \not\equiv 1 \bmod 4$ if $i \neq j$.
3. For $j \geqslant 1$, the $j$-th column is the first few odd numbers in the Collatz sequence of $u_j$.
4. For $j \geqslant i > 0$, $v_{i,j} = 3v_{i-1,j-1}+2$.

*Proof.*
1. Since $n \not\equiv 1 \bmod 3$, $u_0 = 4n+1 \not\equiv 2 \bmod 3$. If $u_i \not\equiv 2 \bmod 3$, then $u_{i+1} = 2u_i + 1 \not\equiv 2 \bmod 3$. Consequently, $u_k \not\equiv 2 \bmod 3$ for all $k \geqslant 0$.
   For $i > 0$, $v_{i,i} \equiv 2 \bmod 3$, by definition. If $v_{i,j} \equiv 2 \bmod 3$, then $v_{i,j+1} = 2v_{i,j}+1 \equiv 2 \bmod 3$. Thus, it follows that $v_{i,j} \equiv 2 \bmod 3$, if $i > 0$ and $j > 0$.
2. $u_0 \equiv 1 \bmod 4$ by definition. $v_{i,i} = 3v_{i-1,i-1}+2 \equiv 1 \bmod 4$, if $v_{i-1,i-1} \equiv 1 \bmod 4$. Since $v_{0,0} = u_0$, it follows that $v_{i,i} \equiv 1 \bmod 4$ for $i \geqslant 0$. For $i > 0$, $u_i = 2u_{i-1}+1 \not\equiv 1 \bmod 4$ since $u_{i-1}$ is an odd number. Similarly, $v_{i,j} = 2v_{i,j-1}+1 \not\equiv 1 \bmod 4$, when $i \neq j$.
3. When $j \geqslant 1$, $v_{0,j} = u_j$. We know from Part 2 that $v_{i,j} \equiv 3 \bmod 4$ if $i \neq j$. Therefore, by Corollary 2.1, the odd number that comes after $v_{i,j}$ in the Collatz sequence of $v_{i,j}$ is $(3v_{i,j}+1)/2$. We do not compute the odd numbers that come after $v_{i,i}$. Hence, assume that $j > i$.
   For $k \geqslant 1$, let $z_k$ be defined as follows:

$$z_k = 2^k + 2^{k-1} + 2^{k-2} + \dots + 2 + 1 = \sum_{i=0}^{k} 2^i = \frac{2^{k+1}-1}{2-1} = 2^{k+1}-1.$$

   Then, by definition, for $j > i$, $v_{i,j} = 2^{j-i}v_{ii} + z_{j-i-1}$, and

$$v_{i+1,j} = 2^{j-i-1}v_{i+1,i+1} + z_{j-i-2} = 2^{j-i-1}(3v_{i,i}+2) + z_{j-i-2}.$$

$$\frac{3v_{i,j}+1}{2} = \frac{3 \times 2^{j-i}v_{ii} + 3 \times z_{j-i-1} + 1}{2} = 3 \times 2^{j-i-1}v_{ii} + \frac{3z_{j-i-1}+1}{2}.$$

$$\frac{3z_{j-i-1}+1}{2} = \frac{3(2^{j-i}-1)+1}{2} = 3 \times 2^{j-i-1} - 1 = 2 \times 2^{j-i-1} + (2^{j-i-1}-1) = 2^{j-i} + z_{j-i-2}.$$

   Consequently,

$$\frac{3v_{i,j}+1}{2} = v_{i+1,j}.$$

   This implies $v_{i+1,j}$ is the odd number that comes after $v_{i,j}$ in the Collatz sequence of $v_{i,j}$. Hence, for $j \geqslant 1$, the $j$-th column is the first few odd numbers in the Collatz sequence of $u_j$.

4. For $j \geqslant i > 0$, when $i = j$, we have that $v_{i,j} = 3v_{i-1,j-1} + 2$, by definition. Let $i \neq j$, then $v_{i,j} = 2v_{i,j-1} + 1$. But $v_{i-1,j-1} \not\equiv 1 \bmod 4$, when $i \neq j$. Hence, by Corollary 2.1, $v_{i,j-1} = (3v_{i-1,j-1} + 1)/2$. Consequently, $v_{i,j} = 3v_{i-1,j-1} + 2$.

□

Theorem 2 can be used to construct divergent Collatz sequences as shown in Example 2. Observe that the Collatz sequence of $u_i$ is strictly increasing till $v_{i,i}$. Our choice of $u_0$ and Theorem 2 makes sure this is the case. Check that $v_{i+1,i}$ will be smaller than $v_{i,i}$. But the sequence $v_{i,i}$ increases monotonically and hence is divergent. Thus, we see that as $i$ increases, a subsequence of initial odd terms of the Collatz sequence of $u_i$ is increasing monotonically up to $v_{i,i}$. The length of these subsequences is also monotonically increasing. Convergent sequences cannot contain divergent subsequences. This implies that the Collatz Conjecture is false. Different values of $n$ provide different arrays that lead to different divergent Collatz sequences. In Example 2, we illustrate divergence for $n = 0$ and 3. But this phenomena occurs for all $n$ such that $n \not\equiv 1 \bmod 3$.

*Let $u_i$ and $v_{i,j}$ be defined as in Theorem 2.*

$n = 0:$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_i$ : | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | 1023 | 2047 | 4095 | 8191 |
| $v_{1,i}$ : | | 5 | 11 | 23 | 47 | 95 | 191 | 383 | 767 | 1535 | 3071 | 6143 | 12287 |
| $v_{2,i}$ : | | | 17 | 35 | 71 | 143 | 287 | 575 | 1151 | 2303 | 4607 | 9215 | 18431 |
| $v_{3,i}$ : | | | | 53 | 107 | 215 | 431 | 863 | 1727 | 3455 | 6911 | 13823 | 27647 |
| $v_{4,i}$ : | | | | | 161 | 323 | 647 | 1295 | 2591 | 5183 | 10367 | 20735 | 41471 |
| $v_{5,i}$ | | | | | | 485 | 971 | 1943 | 3887 | 7775 | 15551 | 31103 | 62207 |
| $v_{6,i}$ : | | | | | | | 1457 | 2915 | 5831 | 11663 | 23327 | 46655 | 93311 |
| $v_{7,i}$ : | | | | | | | | 4373 | 8747 | 17495 | 34991 | 69983 | 139967 |

$n = 3:$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_i$ : | 13 | 27 | 55 | 111 | 223 | 447 | 895 | 1791 | 3583 | 7167 | 14335 | 28671 |
| $v_{1,i}$ : | | 41 | 83 | 167 | 335 | 671 | 1343 | 2687 | 5375 | 10751 | 21503 | 43007 |
| $v_{2,i}$ : | | | 125 | 251 | 503 | 1007 | 2015 | 4031 | 8063 | 16127 | 32255 | 64511 |
| $v_{3,i}$ : | | | | 377 | 755 | 1511 | 3023 | 6047 | 12095 | 24191 | 48383 | 96767 |
| $v_{4,i}$ : | | | | | 1133 | 2267 | 4535 | 9071 | 18143 | 36287 | 72575 | 145151 |
| $v_{5,i}$ : | | | | | | 3401 | 6803 | 13607 | 27215 | 54431 | 108863 | 217727 |
| $v_{6,i}$ : | | | | | | | 10205 | 20411 | 40823 | 81647 | 163295 | 326591 |



## 3. Reversing to prove that the Collatz Conjecture is false

In this section, we provide a different proof of how the Collatz Conjecture fails.

Let $A$ be an odd integer. We say $A$ is a *jump*, if $A = 4n + 1$ from some odd number $n$. If $A = 4^i \times P + 4^{i-1} + 4^{i-2} + \cdots + 4 + 1$, such that $i \geqslant 1$ and $P$ is an odd number, then we say $A$ is a *jump from* $P$ *of height* $i$. $13 = 4 \times 3 + 1$ *is a jump from* 3 *of height* 1. $53 = 4 \times 13 + 1 = 4^2 \times 3 + 4 + 1$ *is a jump from* 13 *of height* 1 *and a jump from* 3 *of height* 2.

Jumps are studied in great detail in [1] and [2]. We say two Collatz sequences are *equivalent* if the second odd number occurring in the sequences are same.

*The Collatz sequence of 3 is*

$$3, 10, 5, 16, 8, 4, 2, 1, 1, \ldots$$

*The Collatz sequence of 13 is*

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 1, \ldots$$

*Observe that the two sequences merge at the odd number* 5. *Hence the Collatz sequences of* 3 *and* 13 *are equivalent.*

**Lemma 3.1** (Corollary 2, Section 2, [1]). *Let $A$ be an odd number and let $c_0 = A$ and $c_i = 4c_{i-1} + 1$, that is, $c_i$ are jumps from $A$. Then, for any $i$, the Collatz sequence of $A$ and $c_i$ are equivalent.*

The Reverse Collatz sequence, $r_i$, of a positive integer $A$ was defined in [1] as follows.

$$r_i = \begin{cases} A & \text{if } i = 0; \\ \frac{r_{i-1}-1}{3} & \text{if } r_{i-1} \equiv 1 \bmod 3 \text{ and } r_{i-1} \text{ is even;} \\ 2r_{i-1} & \text{if } r_{i-1} \not\equiv 1 \bmod 3 \text{ and } r_{i-1} \text{ is even;} \\ 2r_{i-1} & \text{if } r_{i-1} \text{ is odd.} \end{cases}$$

We say that a Reverse Collatz sequence *converges* if the subsequence of odd numbers of the sequence converges to a multiple of 3.

*The Reverse Collatz sequence with starting number* 121 *is :*

$$121, 242, 484, 161, 322, 107, 214, 71, 142, 47, 94, 31, 62, 124, 41, 82, 27, 54, 108, 216, \ldots$$

*The Reverse Collatz sequence of* 121 *converges because its subsequence of odd numbers*

$$121, 161, 107, 71, 47, 31, 41, 27$$

*converges to* 27.

Let $p_i$ denote the subsequence of odd numbers in the Reverse Collatz sequence of $A$. Then, in [1], it was proved that, if $p_i \equiv 0 \bmod 3$, then $p_{i+1}$ do not exist. Otherwise, $p_{i+1}$ is the smallest odd number before $p_i$ in any Collatz sequence and

$$p_{i+1} = \begin{cases} \frac{2p_i-1}{3} \text{ if } p_i \equiv 2 \bmod 3 \\ \\ \frac{4p_i-1}{3} \text{ if } p_i \equiv 1 \bmod 3 \end{cases} \tag{3.1}$$

It was conjectured in [1], that, the Reverse Collatz sequence converges to a multiple of 3 for every number greater than one. See [1] and [2] for more details about Reverse Collatz sequences.

**Lemma 3.2.** *Let $A$ be an odd number such that $A \equiv 2 \mod 3$. Let $u = (A-2)/3$. If $r_i$ denotes subsequence of odd numbers in the Reverse Collatz sequence of $A$ with $r_0 = A$, then $r_1 = 2u + 1$. Consequently,*

$$u \equiv \begin{cases} 0 & \bmod 3, \implies r_1 \equiv 1 \quad \bmod 3 \\ 1 & \bmod 3, \implies r_1 \equiv 0 \quad \bmod 3 \\ 2 & \bmod 3, \implies r_1 \equiv 2 \quad \bmod 3 \end{cases}$$

*Observe that $r_1 < r_0$. Moreover, if $u \equiv 2 \mod 3$, let $t_i$ represent the subsequence of odd numbers in the Reverse Collatz sequence of $u$, then $t_1 = (r_1 - 2)/3$.*

*Proof.* Since $A \equiv 2 \mod 3$, by definition of Reverse Collatz sequence, $r_1 = (2A - 1)/3$. Now

$$2u + 1 = 2\left(\frac{A-2}{3}\right) + 1 = \frac{2A-1}{3}.$$

Therefore, $r_1 = 2u + 1$. Consequently,

$$\frac{r_1 - 2}{3} = \frac{(2u+1) - 2}{3} = \frac{2u - 1}{3}.$$

If $u \equiv 2 \mod 3$, then by definition of Reverse Collatz sequence, $t_1 = (2u - 1)/3$. Thus, $t_1 = (r_1 - 2)/3$. □

Let $A$ be an odd number and let $A \equiv 2 \mod 3$. Define a sequence of odd numbers, $v_{0,j}$, such that, $v_{0,0} = A$, and for $j > 0$, $v_{0,j} = (v_{0,j-1} - 2)/3$. Then, for some integer $n \geqslant 0$, $v_{0,j} \equiv 2 \mod 3$ for $j < n$, and $v_{0,n} \not\equiv 2 \mod 3$. Moreover, there are at least $n+1$ terms in the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,0}$ (by Part 1). Let $v_{i,0}$, $i = 0, \dots, n$, denote the first $n+1$ terms of the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,0}$. Then, for each $i = 1, \dots, n$, we can form an array $v_{i,j} = (v_{i,j-1} - 2)/3$, where $j = 1, \dots, n-i$,

$$\begin{array}{llllllll}
v_{0,0} & v_{0,1} & v_{0,2} & v_{0,3} & \cdots & v_{0,n-2} & v_{0,n-1} & v_{0,n} \\
v_{1,0} & v_{1,1} & v_{1,2} & v_{1,3} & \cdots & v_{1,n-2} & v_{1,n-1} & \\
v_{2,0} & v_{2,1} & v_{2,2} & v_{2,3} & \cdots & v_{2,n-2} & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
v_{n-3,0} & v_{n-3,1} & v_{n-3,2} & v_{n-3,3} & & & & \\
v_{n-2,0} & v_{n-2,1} & v_{n-2,2} & & & & & \\
v_{n-1,0} & v_{n-1,1} & & & & & & \\
v_{n,0} & & & & & & &
\end{array}$$

with the following properties.

1. For each $j = 0, \dots n$, $v_{i,j}$, $i = 0, 1, \dots, n-j$, are the first $n+1-j$ terms of the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,j}$. For $i = 0, \dots, n$, $v_{i,j} \equiv 2 \mod 3$ whenever $j \neq n-i$, and $v_{i,n-i} \not\equiv 2 \mod 3$.
2. For $i > 0$, $v_{i,j} = 2 * v_{i-1,j+1} + 1$. Moreover, if $v_{0,n} \equiv 1 \mod 3$ then,
$$v_{i,n-i} \equiv \begin{cases} 0 \mod 3 & \text{if } i \text{ is odd;} \\ 1 \mod 3 & \text{if } i \text{ is even.} \end{cases}$$
On the other hand, if $v_{0,n} \equiv 0 \mod 3$ then,
$$v_{i,n-i} \equiv \begin{cases} 1 \mod 3 & \text{if } i \text{ is odd;} \\ 0 \mod 3 & \text{if } i \text{ is even.} \end{cases}$$
3. $v_{0,0} = 3^n (v_{0,n} + 1) - 1$ and $v_{n,0} = 2^n (v_{0,n} + 1) - 1$. Moreover, for $i = 1, \dots, n$, $v_{i,0} = 3^{n-i} \times 2^i \times (v_{0,n} + 1) - 1$.

*Proof.*

1. By Lemma 3.2, since $v_{0,1} \equiv 2 \mod 3$, $v_{1,0} \equiv 2 \mod 3$, and $v_{1,1}$ is the next odd term in the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,1}$. Now, because $v_{1,0} \equiv 2 \mod 3$, $v_{2,0}$ is well defined as the next term in the Reverse Collatz sequence of $v_{1,0}$, by Equation 3.1. Since, $v_{0,2} \equiv 2 \mod 3$, we apply Lemma 3.2, again, to conclude that $v_{1,1} \equiv 2 \mod 3$, and $v_{1,2}$ is the next odd term in the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,2}$.
Applying this argument, repeatedly, we derive that for each $j = 1, \dots, n-1$, $v_{1,j}$ is the first odd term that comes after $v_{0,j}$ in the Reverse Collatz sequence of $v_{0,j}$. We also get that $v_{1,j} \equiv 2 \mod 3$, for $j = 1, \dots, n-2$. Since $v_{0,n} \not\equiv 2 \mod 3$, $v_{1,n-1} \not\equiv 2 \mod 3$, by Lemma 3.2.
Now, since we established that $v_{1,1} \equiv 2 \mod 3$, we will repeat the above argument to derive that for each $j = 1, \dots, n-2$, $v_{2,j}$ is the first odd term that comes after $v_{1,j}$ in the Reverse Collatz sequence of $v_{1,j}$. We also get that $v_{2,j} \equiv 2 \mod 3$, for $j = 1, \dots, n-3$, and $v_{2,n-2} \not\equiv 2 \mod 3$.
Continuing thus, we get for each $j = 0, \dots n$, $v_{i,j}$, $i = 0, 1, \dots, n-j$, are the first $n+1-j$ terms of the subsequence of odd numbers in the Reverse Collatz sequence of $v_{0,j}$. For $i = 0, \dots, n$, $v_{i,j} \equiv 2 \mod 3$ whenever $j \neq n-i$, and $v_{i,n-i} \not\equiv 2 \mod 3$.
2. This result follows from Lemma 3.2.

3. Rewriting $v_{i,j} = (v_{i,j-1} - 2)/3$, we get $v_{i,j-1} = 3v_{i,j} + 2$. Thus,

$$v_{i,0} = 3v_{i,1} + 2 = 3(3(v_{i,2} + 2) + 2 = 3^2 v_{i,2} + 3 \times 2 + 2.$$

Continuing thus we get $v_{i,0} = 3^{n-i} \times v_{i,n-i} + 2 \times \sum_{s=0}^{n-i-1} 3^s$. Since

$$2 \times \sum_{s=0}^{n-i-1} 3^s = 3^{n-i} - 1,$$

we get

$$v_{i,0} = 3^{n-i}(v_{i,n-i} + 1) - 1.$$

By Part 2, $v_{i,j} = 2 * v_{i-1,j+1} + 1$, for $i > 0$. Therefore,

$$v_{i,n-i} = 2 * v_{i-1,n-i+1} + 1 = 2 * (2 * v_{i-2,n-i+2} + 1) + 1$$

Continuing thus we get

$$v_{i,n-i} = 2^i \times v_{0,n} + \sum_{s=0}^{i-1} 2^s.$$

Substituting $\sum_{s=0}^{i-1} 2^s = 2^i - 1$, we get $v_{i,n-i} = 2^i(v_{0,n} + 1) - 1$. In particular, $v_{0,0} = 3^n\,(v_{0,n} + 1) - 1$ and $v_{n,0} = 2^n(v_{0,n} + 1) - 1$.
Since $v_{0,0} = 3^n\,(v_{0,n} + 1) - 1$ and $v_{0,0} \equiv 2 \mod 3$, $v_{1,0} = 3^{n-1} \times 2 \times (v_{0,n} + 1) - 1$, by Equation 3.1. By Part 1, $v_{1,0} \equiv 2 \mod 3$. Therefore, again, by Equation 3.1, we derive $v_{2,0} = 3^{n-2} \times 2^2 \times (v_{0,n} + 1) - 1$. Repeating this argument, we get, for $i = 1, \ldots, n$, $v_{i,0} = 3^{n-i} \times 2^i \times (v_{0,n} + 1) - 1$.

□

*In this example, we apply Theorem 3 to* $A = 2429 \equiv 2 \mod 3$. *Here,* $n = 5$, $v_{0,5} = 9$, *The columns are the first odd terms in the Reverse Collatz sequence of* $v_{0,j}$. *Observe that* $v_{0,5} = 9, v_{2,3} = 39, v_{4,1} = 159$ *are* $\equiv 0 \mod 3$ *and* $v_{1,4} = 19$, $v_{3,2} = 79$, *and* $v_{5,0} = 319$ *are* $\equiv 1 \mod 3$. $v_{0,0} = 2429 = 3^5 \times 10 - 1$, *and* $v_{5,0} = 2^5 \times 10 - 1 = 319$.
*-1in.5in*

| | | | | | | |
|---|---|---|---|---|---|---|
| $v_{0,j}$ : | 2429 | 809 | 269 | 89 | 29 | 9 |
| $v_{1,j}$ : | 1619 | 539 | 179 | 59 | 19 | |
| $v_{2,j}$ : | 1079 | 359 | 119 | 39 | | |
| $v_{3,j}$ : | 719 | 239 | 79 | | | |
| $v_{4,j}$ : | 479 | 159 | | | | |
| $v_{5,j}$ : | 319 | | | | | |

**Lemma 3.3.** *Let* $A \equiv 1 \mod 3$ *be an odd number. Then we can write* $A = 3^n B + 1$ *such that* $B$ *is not divisible by 3. If* $r_i$ *denotes the subsequence of odd numbers in the Reverse Collatz sequence of* $A$, *then, for* $i = 0, \ldots n$, $r_i = 4^i \times 3^{n-i} \times B + 1$. *Thus,* $r_n = 4^n \times B + 1$. *For* $i = 0, \ldots n-1$, $r_i \equiv 1 \mod 3$, *and*

$$r_n \equiv \begin{cases} 2 \mod 3, & \text{if } B \equiv 1 \mod 3, \\ 0 \mod 3, & \text{if } B \equiv 2 \mod 3. \end{cases}$$

*Observe that* $r_i > r_{i-1}$ *for* $i = 1, \ldots, n$.

*Proof.* Since, $A \equiv 1 \mod 3$, by Equation 3.1, we get $r_1 = (4A - 1)/3 = 4 \times 3^{n-1}B + 1$. If $n > 1$ then $r_1 \equiv 1 \mod 3$. Again, by Equation 3.1, $r_2 = 4^2 \times 3^{n-2}B + 1$. If $n > 2$ then $r_2 \equiv 1 \mod 3$. Continuing this argument, we get, for $i = 0, \ldots n-1$, $r_i = 4^i \times 3^{n-i} \times B + 1$, such that, $r_i \equiv 1 \mod 3$. Now since $r_{n-1} \equiv 1 \mod 3$, we get $r_n = 4^n \times B + 1$. Since $B \not\equiv 0 \mod 3$, $r_n \not\equiv 1 \mod 3$. In fact, $r_n \equiv 2 \mod 3$, if $B \equiv 1 \mod 3$, and $r_n \equiv 0 \mod 3$, if $B \equiv 2 \mod 3$. □

*We apply Lemma 3.3 to* $A = 91$*. We can write* $A = 3^2 \times 10 + 1$*. Let* $r_i$ *denote the subsequence of odd numbers in the Reverse Collatz sequence of* $A$*. Then, -1in.5in*

$$\begin{aligned} r_0 &= 91 = 3^2 \times 10 + 1 \\ r_1 &= 121 = 4 \times 3 \times 10 + 1 \\ r_2 &= 161 = 4^2 \times 10 + 1 \equiv 2 \mod 3. \end{aligned}$$

*Since* $10 \equiv 1 \mod 3$, $r_2 \equiv 2 \mod 3$.

*In this example, we demonstrate the convergence of the Reverse Collatz sequence of* 2429*. By Part 3 of Theorem 3,* $v_{0,0} = 3^n (v_{0,n} + 1) - 1$*. Rewriting, we get* $(v_{0,0} + 1)/3^n = v_{0,n} + 1$*. Since* $(2429 + 1)/3^5 = 10$*, and* $10 \not\equiv 0 \mod 3$*, we get* $v_{0,5} = 9$*. By Part 2 of Theorem 3, Since* $9 \equiv 0 \mod 3$*,* $v_{5,0} \equiv 1 \mod 3$*. Now* $v_{5,0} = 319 = 3 \times 106 + 1$*, therefore.* $v_{6,0} = 4 \times 106 + 1 = 425$*, by Lemma 3.3. Also, since,* $106 \equiv 1 \mod 3$*,* $v_{6,0} \equiv 2 \mod 3$*. Since* $426/3 = 142$*, we get* $v_{6,0} = 3 \times 142 - 1$*, by Part 3 of Theorem 3. Since* $142 \equiv 1 \mod 3$*,* $v_{7,0} \equiv 2 \mod 3$*, by Lemma 3.3. Continuing this argument, we see that subsequence of odd integers of the Reverse Collatz sequence of* 2429 *fluctuates between numbers that are* $\equiv 2 \mod 3$ *and* $1 \mod 3$*, till it reaches* $111 \equiv 0 \mod 3$*.*

*-1in.5in*

$$\begin{aligned} v_{0,0} &= 2429 = 3^5 \times 10 - 1 \\ v_{1,0} &= 1619 = 3^4 \times 2 \times 10 - 1 \\ v_{2,0} &= 1079 = 3^3 \times 2^2 \times 10 - 1 \\ v_{3,0} &= 719 = 3^2 \times 2^3 \times 10 - 1 \\ v_{4,0} &= 479 = 3 \times 2^4 \times 10 - 1 \\ v_{5,0} &= 319 = 2^5 \times 10 - 1 = 3 \times 106 + 1 \\ v_{6,0} &= 425 = 4 \times 106 + 1 = 3 \times 142 - 1 \\ v_{7,0} &= 283 = 4 \times 142 - 1 = 3 \times 94 - 1 \\ v_{8,0} &= 377 = 4 \times 94 - 1 = 3^3 \times 14 - 1 \\ v_{9,0} &= 251 = 3^2 \times 2 \times 14 - 1 \\ v_{10,0} &= 167 = 3 \times 2^2 \times 14 - 1 \\ v_{11,0} &= 111 = 2^3 \times 14 - 1 \end{aligned}$$

Thus, we see that the odd numbers of a Reverse Collatz sequence, keep alternating between numbers that are congruent to $1 \mod 3$ and $2 \mod 3$, till it reaches a number that is divisible by 3. Which also means the sequence increases and decreases at regular intervals. Does this sequence converge? Or does it alternate forever? We cannot answer this question yet.

A Reverse Collatz sequence will continue till it reaches a number $A$ that is a multiple of 3. Now if $A$ is a multiple of 3, then $4A + 1 \equiv 1 \mod 3$. So the Reverse Collatz sequence of $4A + 1$ is non trivial. Moreover, the Collatz sequences of $A$ and $4A + 1$ are equivalent. Thus, a Collatz sequence can be extended backwards forever using jumps as in Example 3. A sequence containing infinite terms is divergent.

*A Collatz sequence can be extended backwards forever using jumps!*

$$\begin{array}{rrrrr}
 & & & & 204729 \\
 & & & & 153547 \\
 & & & & 230321 \\
 & & & & 4\times 43185+1=172741 \\
 & & & & \Uparrow \\
 & & & & 43185 \\
 & & & 8097 & \Rightarrow 4\times 8097+1=32389 \\
 & & & 6073 & \\
 & & & 4555 & \\
 & & & 6833 & \\
 & & 1281 & \Rightarrow 4\times 1281+1=5125 & \\
 & & 961 & & \\
 & & 721 & & \\
 & & 4\times 135+1=541 & & \\
 & & \Uparrow & & \\
 & & 135 & & \\
 & & 203 & & \\
 & & 305 & & \\
 & & 4\times 57+1=229 & & \\
 & & \Uparrow & & \\
 & & 57 & & \\
 & & 43 & & \\
 & & 65 & & \\
 & & 49 & & \\
 & 9 & \Rightarrow 4\times 9+1=37 & & \\
 & 7 & & & \\
 & 11 & & & \\
 & 17 & & & \\
 & 4\times 3+1=13 & & & \\
 & \Uparrow & & & \\
 & 3 & & & \\
1 & \Rightarrow 4\times 1+1=5 & & &
\end{array}$$

For any odd integer $A$, there are infinite Collatz sequences that do not converge. *Proof.* Given an odd integer $A$, consider the sequence of jumps $b_i = 4b_{i-1}+1$ with $b_0 = A$. This is an infinite sequence with equivalent Collatz sequences by Lemma 3.1. If for any $i$, $b_i \equiv 0 \bmod 3$, then $b_{i+1} \equiv 1 \bmod 3$, $b_{i+2} \equiv 2 \bmod 3$, and $b_{i+3} \equiv 0 \bmod 3$. Which implies there are infinite jumps for any number $A$ which are not multiples of 3. The reverse Collatz sequences of these jumps are non trivial. Hence, there are infinite ways to go backwards. Moreover, as in Example 3, these sequences can be extended backwards infinitely. All these sequences have infinite terms and hence are divergent. □

By Theorem 3, the Collatz Conjecture is false. End of story.



## References


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