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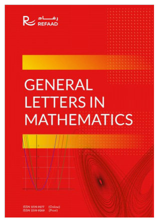

Taylor approximation for solving linear and nonlinear Ill-Posed Volterra equations via an iteration method

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Abstract

In this paper, we present two algorithms for the approximate or exact solution of a class of Volterra integral equations of first kind. As well known, this is an ill posed problem, but we convert it to well-posedness of the second kind Volterra problems, then we apply the variational iteration method. Finally, we present two examples which show the performance and effciency of our method.

Keywords: Linear Volterra integral equation of the first kind -Hammerstien integral equation of the first kind series, variational iteration method.
2010 MSC: 65N20, 34A12, 30K05.

1. Introduction and preliminaries

Two of the most standard forms of linear and nonlinear integral equations of the first kind are:

$$\lambda \int_0^t k\,(t,x)\,\varphi\,(x)\,dx = f\,(t)\,, \tag{1.1}$$

and

$$\lambda \int_0^t k\,(t,x)\,F\,(\varphi\,(x))\,dx = f\,(t)\,, \tag{1.2}$$

here, both, the function $k(t,x)$, and the function $f(t)$ are known, k on the square $0 \leqslant x; t \leqslant 1$, and f on the interval $0 \leqslant t \leqslant 1$. The quantity λ is a given constant parameter. The function φ is to be determined on $[0,1]$ and $F(\varphi(t))$ is a nonlinear function of $\varphi(t)$ (note that we can to take $0 \leqslant t \leqslant 1$, since every interval such as $[a,b]$ can be transformed into this interval by a linear transformation (see in [10])).

As a classical ill-posed problem, the Volterra integral equations of the first kind have been investigated by many references (see, for instance [12, 8, 11, 3]). These equations arise in many scientific applications such as the population dynamics, spread of epidemics, and semi-conductor devices [12]. For sufficiently smooth f and k, we may diferentiate these equations with respect to t to obtain Volterra integral equations

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of the second kind which are known to be well-posed problems. This classical converting is applied in many references (see, for instance [12, 7]).

We recall that in the variational iteration method, it was first proposed in [5] and recently used in the literature for solving both linear and nonlinear problems analytic and numerical. The variational iteration method is based on the general Lagrange's multiplier method is simple and powerful method. Let us take the differential equation

$$L\varphi + N\varphi = g,$$

where L is a linear operator, N is an operator which can be nonlinear, and g is an inhomogeneous term. Using method of successive approximations, we can write the $(n+1)^{th}$ approximation of the solution as the n^{th} plus some corrections. According to the variational iteration method (see, for instance [13, 12, 10, 2, 9, 1, 4]), we can construct a correction functional as follows

$$\varphi_{n+1}(t) = \varphi_n(t) + \int_0^t \lambda(\varepsilon)\left(L\varphi_n(\varepsilon) + N\tilde{\varphi}_n(\varepsilon) - g(\varepsilon)\right)d\varepsilon,\ 0 \leqslant \varepsilon \leqslant t \leqslant 1$$

where the correction functional contains a general Lagrange multiplier λ which can be identified in an optimal way by the variational theory, noting that in this method λ may be a constant or a function, φ_n is the n^{th} approximate solution, and $\tilde{\varphi}_n$ denotes a restricted variation, i.e. $\delta\tilde{\varphi}_n = 0$, where δ is the variational derivative.

In this article, we intend to combine the variational iteration method and Taylor approximation to obtain a systematic and efficient method for solving previous ill-posed Volterra equations (1.1) and (1.2). We will transform these equations to an equivalent integral equations of the second kind defined in $[0, 1]$.

Let $A(t)$ be an operator with derivatives of first order with respect to t in an interval $[0, 1]$ than for $0 < t - h < t < t + h < 1$, with $h \to 0$. The Taylor series given by

$$A(t+h) = A(t) + \frac{h}{1!}A'(t) + O(h), \tag{1.3}$$

$$A(t-h) = A(t) - \frac{h}{1!}A'(t) + O(h), \tag{1.4}$$

where $O(h)$ is an unknown error term of approximation.

The presentation and the analysis of our method, based on the idea of Taylor approximation, is the main goal of this paper.

2. Solving linear Volterra integral equation of the first kind

Now, let $A(t)$ be an operator integral defined by

$$A(t) = \int_0^t k(t,x)\,\varphi(x)\,dx,\ 0 \leqslant x \leqslant t \leqslant 1$$

and by using the Taylor series of the first order and Leibnitz rule, we find

$$A(t+h) = A(t) + hk(t,t)\varphi(t) + h\int_0^t \frac{\partial k(t,x)}{\partial t}\varphi(x)\,dx + O(h).$$

Then, the approximate Volterra integral equation of the first kind (1.1) given by

$$f(t) + hk(t,t)\varphi(t) + h\int_0^t \frac{\partial k(t,x)}{\partial t}\varphi(x)\,dx = f(t+h) + O(h),$$

where $k(t,t) \neq 0$ and $\dfrac{\partial k(t,x)}{\partial t} \neq 0$ for $t \in [0,1]$, we obtain the linear Volterra integral equation of the second kind given by

$$\varphi_h(t) + \int_0^t K(t,x)\,\varphi_h(x)\,dx = f_h(t), \quad h \to 0 \tag{2.1}$$

where

$$K(t,x) = \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}, \quad f_h(t) = \frac{f(t+h)-f(t)}{hk(t,t)} \text{ and } \varphi_h = \varphi \text{ if } h \to 0.$$

Substituting $t = 0$ into (2.1) gives the initial condition $\varphi_h(0) = \varphi_h^0$.

Now, using Leibnitz rule to differentiate both sides of (2.1) gives

$$\varphi_h'(t) + K(t,t)\,\varphi_h(t) + \int_0^t \frac{\partial K(t,x)}{\partial t}\varphi_h(x)\,dx = f_h'(t). \tag{2.2}$$

We apply variational iteration method for equation (2.1) and take $\varphi_h(t) = \psi(t)$. According to this method correction functional can be written in the following form

$$\psi_{n+1}(t) = \psi_n(t) + \int_0^t \lambda(\varepsilon)\left(\psi_n'(t) + K(t,t)\,\psi_n(t) + \int_0^\varepsilon \frac{\partial K(t,x)}{\partial t}\psi_h(x)\,dx - f_h'(t)\right)d\varepsilon.$$

Imposing the stationary condition $(\partial\psi_{n+1} = 0)$ on the correction functional, the Lagrange multiplier $\lambda(\varepsilon) = -1$.
As a result, we obtain the following iteration formula

$$\psi_{n+1}(t) = \psi_n(t) - \int_0^t \left(\psi_n'(t) + K(t,t)\,\psi_n(t) + \int_0^\varepsilon \frac{\partial K(t,x)}{\partial t}\psi_h(x)\,dx - f_h'(t)\right)d\varepsilon. \tag{2.3}$$

By starting from $\psi_0(t) = \varphi_h^0(t)$, we can obtain the exact solution or an approximate solution $\varphi_h(t) = \lim_{n\to\infty}\psi_n(t)$ of the equation (2.1).

Moreover, the solution $\varphi_h(t)$ of the equation (2.1) converges to the solution $\varphi(t)$ of Volterra integral equation (1.1) of the first kind as $h \to 0$; or written

$$\varphi(t) = \lim_{h\to 0}\varphi_h(t).$$

Here, we emphasize the two important remarks related to this converting technique for linear Volterra integral equation. First, if we use the Taylor series of the first order and Leibnitz rule, we obtain the Volterra integral equation of the second kind given by

$$\varphi_h(t) + \int_0^t \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}\varphi_h(x)\,dx = \frac{f(t)-f(t+h)}{hk(t,t)}.\ h \to 0$$

Second, if we use Leibnitz rule and the difference of Taylor series between (1.3) and (1.4), we obtain the well-posed Volterra integral equation given by

$$\varphi_h(t) + \int_0^t \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}\varphi_h(x)\,dx = \frac{f(t+h)-f(t-h)}{2hk(t,t)}.\ h \to 0$$

3. Solving Volterra-Hammerstein integral equation of the first kind

In this section, we shall study a nonlinear integral equation. Let $A(t)$ be a nonlinear operator defined by

$$A(t)=\int_0^t k(t,x)\,F[\varphi(x)]\,dx,\ 0\leqslant x\leqslant t\leqslant 1$$

and by using the Taylor series of the first order and Leibnitz rule, we find

$$A(t+h)=A(t)+hk(t,t)F(\varphi(t))+h\int_0^t \frac{\partial k(t,x)}{\partial t}F(\varphi(x))\,dx+O(h),$$

then, the Volterra-Hammerstein integral equation of the rst kind (1.2) given by

$$f(t)+hk(t,t)F(\varphi(t))+h\int_0^t \frac{\partial k(t,x)}{\partial t}F(\varphi(x))\,dx=f(t+h)+O(h).$$

Then, if $k(t,t)\neq 0$ and $\dfrac{\partial k(t,x)}{\partial t}$ we obtain

$$F(\varphi_h(t))+\int_0^t \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}F(\varphi_h(x))\,dx=\frac{f(t+h)-f(t)}{hk(t,t)}, \tag{3.1}$$

where $\varphi_h(t)=\varphi(t)$ if $h\to 0$, After to use the change of variables, we obtain the linear Volterra integral equation of the second kind given by

$$\Psi_h(t)+\int_0^t K(t,x)\,\Psi_h(x)\,dx=\frac{f(t+h)-f(t)}{hk(t,t)}, \tag{3.2}$$

where

$$\Psi_h(t)=F(\varphi_h(t)),\ K(t,x)=\frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}\ \text{and}\ f_h(t)=\frac{f(t+h)-f(t)}{hk(t,t)}.$$

Assuming that $F(\varphi_h(t))$ is invertible, then we can set

$$\varphi_h(t)=F^{-1}(\Psi_h(t)).$$

Substituting $t=0$ into (3.2) gives the initial condition $\Psi_h(0)=F(\varphi(0))=\Psi_h^0$.

Now, using Leibnitz rule to differentiate both sides of (3.2) gives

$$\Psi_h'(t)+K(t,t)\,\Psi_h(t)+\int_0^t \frac{\partial K(t,x)}{\partial t}\Psi_h(x)\,dx=f_h'(t). \tag{3.3}$$

We apply variational iteration method for equation (3.2) and take $\Psi_h(t)=\psi(t)$. According to this method, correction functional can be written in the following form

$$\psi_{n+1}(t)=\psi_n(t)+\int_0^t \lambda(\varepsilon)\left(\psi_n'(t)+K(t,t)\,\psi_n(t)+\int_0^\varepsilon \frac{\partial K(t,x)}{\partial t}\psi_h(x)\,dx-f_h'(t)\right)d\varepsilon.$$

Imposing the stationary condition $(\delta\psi_{n+1}=0)$ on the correction functional, the Lagrange multiplier $\lambda(\varepsilon)=-1$.

As a result, we obtain the following iteration formula

$$\psi_{n+1}(t) = \psi_n(t) + \int_0^t \psi_n'(t) + K(t,t)\psi_n(t) + \int_0^\varepsilon \frac{\partial K(t,x)}{\partial t}\psi_h(x)\,dx - f_h'(t)\,d\varepsilon. \tag{3.4}$$

By starting from $\psi_0(t) = \Psi_h^0(t)$, we can obtain the exact solution or an approximate solution $\Psi_h(t) = \psi(t) = \lim_{n\to\infty}\psi_n(t)$ of the equation (3.2) . Using the transformation

$$\varphi_h(t) = F^{-1}(\Psi_h(t)).$$

Moreover, the solution $\varphi_h(t)$ of equation (3.2) converges to the solution $\varphi(t)$ of Volterra integral equation (1.2) of the first kind as $h \to 0$, or write

$$\varphi(t) = \lim_{h\to 0}\varphi_h(t).$$

Two important remarks related to this converting technique for Volterra-Hammerstein integral equation can be made here. First, if we use the Taylor series of the first order and Leibnitz rule, we obtain the Volterra integral equation of the second kind given by

$$\Psi_h(t) + \int_0^t \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}\Psi_h(x)\,dx = \frac{f(t) - f(t+h)}{hk(t,t)}.\ h \to 0$$

Second, if we use Leibnitz rule and the difference of Taylor series (1.3) and (1.4), we obtain the linear Volterra integral equation of the second kind given by

$$\Psi_h(t) + \int_0^t \frac{\frac{\partial k(t,x)}{\partial t}}{k(t,t)}\Psi_h(x)\,dx = \frac{f(t+h) - f(t-h)}{2hk(t,t)}.\ h \to 0 \text{ and } k(t,t) \neq 0.$$

We note that if we use another iterations method (like, Adomian decomposition method, the modified decomposition method, ...) the resulting systems for solving the Volterra integral equations of the first kind is slightly different from (2.3) and (3.4).

4. Illustrative examples

This method for linear and nonlinear Volterra integral equations will be illustrated by discussing the following examples.

We consider the linear Volterra integral equation of the first kind

$$\int_0^t \left(t - x + 10^2\right)\varphi(x)\,dx = \frac{t^2}{2} + 10^2 t. \tag{4.1}$$

It's equivalent to the Volterra integral equation of the second kind given by

$$\varphi_h(t) + \frac{1}{10^2}\int_0^t \varphi_h(x)\,dx = \frac{2t+h}{2\times 10^2} + 1 + O(h), \tag{4.2}$$

for $t \in [0,1]$ with the boundary condition $\varphi_h(0) \simeq \frac{h}{2}10^{-2} + 1$.

If we pose $\psi(t) = \varphi_h(t)$ and use Leibnitz rule to differentiate both sides of (4.2) the it gives

$$\psi'(t) + 10^{-2}\psi(t) = 10^{-2}.$$

Using the variational iteration method, the iteration formula for equation (4.2) is

$$\psi_{n+1}(t) = \psi_n(t) - \int_0^t \psi_n'(\varepsilon) + \psi_n(\varepsilon) - 10^{-2} d\varepsilon. \tag{4.3}$$

As stated before, we can use the initial condition to select

$$\psi_0(t) = \psi(0) = \frac{h}{2}10^{-2} + 1.$$

Using this selection into (4.3) gives the following successive approximations $(n \geqslant 1)$:

$$\begin{aligned}
\psi_0(t) &= 1 + \frac{h}{2}10^{-2}, \\
\psi_1(t) &= \psi_0(t) - \int_0^t \psi_0'(\varepsilon) + \psi_0(\varepsilon) - 10^{-2} d\varepsilon = 1 + \frac{10^{-2}}{2}h - \frac{(10^{-2})^2}{2}ht, \\
\psi_2(t) &= \psi_1(t) - \int_0^t \psi_1'(\varepsilon) + \psi_1(\varepsilon) - 10^{-2} d\varepsilon = 1 + \frac{10^{-2}}{2}h - \frac{(10^{-2})^2}{2}ht + \frac{(10^{-2})^3}{4}ht^2, \\
\psi_3(t) &= \psi_2(t) - \int_0^t \psi_2'(\varepsilon) + \psi_2(\varepsilon) - 10^{-2} d\varepsilon = 1 + \frac{10^{-2}}{2}h - \frac{(10^{-2})^2}{2}ht + \frac{(10^{-2})^3}{4}ht^2 \\
&\quad - \frac{(10^{-2})^4}{12}ht^3, \\
&\ \ \vdots \\
\psi_n(t) &= \psi_{n-1}(t) - \int_0^t \psi_{n-1}'(\varepsilon) + \psi_{n-1}(\varepsilon) - 10^{-2} d\varepsilon = 1 + \frac{h}{2}\sum_{i=1}^{n+1}(-1)^{i+1}\frac{(10^{-2})^i}{(i-1)!}t^{i-1}.
\end{aligned}$$

The variational iteration method admits the use of

$$\varphi_h(t) = \psi(t) = \lim_{n\to\infty}\psi_n(t),$$

that gives the exact solution of the Volterra integral equation (4.1) by

$$\varphi(t) = \lim_{h\to 0}\varphi_h(t) = 1.$$

Use the variational iterationmethod to solve the nonlinear Volterra-Hammerstein integral equation of the first kind

$$\int_0^t (10t - 10x + 6)\log|\varphi(x)|\,dx = 5t^3 + 9t^2. \tag{4.4}$$

It's equivalent to the Volterra integral equation of the second kind given by

$$\log|\varphi_h(t)| + \frac{5}{3}\int_0^t \log|\varphi_h(x)|\,dx = \frac{5}{2}t^2 + \left(3 + \frac{3}{2}h\right)t + \frac{9h + 5h^2}{6} + O(h), \tag{4.5}$$

for $t \in [0,1]$ with the boundary condition $\varphi_h(0) \simeq \frac{9h+5h^2}{6}$.

We first set

$$\varphi_h(t) = \exp(\omega_h(t)),$$

to carry out the nonlinear equation (4.5) to the linear Volterra integral equation

$$\omega_h(t) + \frac{5}{3}\int_0^t \omega_h(x)\,dx = \frac{5}{2}t^2 + \left(3 + \frac{3}{2}h\right)t + \frac{9h+5h^2}{6},\ h \to 0 \tag{4.6}$$

if we pose $\psi(t) = \omega_h(t)$, using Leibnitz rule to di¤erentiate both sides of (4.5) it gives

$$\psi'(t) + \frac{5}{3}\psi(t) = 5t + 3 + \frac{3h}{2},$$

Using the variational iteration method, the iteration formula for equation (4.5) is

$$\psi_{n+1}(t) = \psi_n(t) - \int_0^t \psi_n'(\varepsilon) + \psi_n(\varepsilon) - 5\varepsilon - 3 - \frac{3h}{2}d\varepsilon. \tag{4.7}$$

As stated before, we can use the initial condition to select

$$\psi_0(t) = \psi(0) = \frac{9h+5h^2}{6}$$

Using this selection into (4.7) gives the following successive approximations:

$$\begin{aligned}
\psi_0(t) &= \frac{9h+5h^2}{6},\\
\psi_1(t) &= \psi_0(t) - \int_0^t \psi_0'(\varepsilon) + \psi_0(\varepsilon) - 5\varepsilon - 3 - \frac{3h}{2}d\varepsilon = \frac{9h+5h^2}{6} - \\
&\left(\frac{5\left(9h+5h^2\right)}{18} - 3 - \frac{3h}{2}\right)t + \frac{5}{2}t^2,\\
\psi_2(t) &= \psi_1(t) - \int_0^t \psi_1'(\varepsilon) + \psi_1(\varepsilon) - 5\varepsilon - 3 - \frac{3h}{2}d\varepsilon = \frac{9h+5h^2}{6} - \\
&\left(\frac{5\left(9h+5h^2\right)}{18} - 3 - \frac{3h}{2}\right)t + \left(\frac{5\left(9h+5h^2\right)}{18} - 3 - \frac{3h}{2}\right)t^2 + \frac{25}{182}t^3,\\
&\vdots\\
\psi_n(t) &= \psi_{n-1}(t) - \int_0^t \psi_{n-1}'(\varepsilon) + \psi_{n-1}(\varepsilon) - 10^{-2}d\varepsilon = \cdots.
\end{aligned}$$

The variational iteration method admits the use of

$$\omega_h(t) = \psi(t) = \lim_{n\to\infty}\psi_n(t),$$

that gives the exact solution of linear equation (4.6) by

$$\omega(t) = \lim_{h\to 0}\omega_h(t) = 3t.$$

Finally, the exact solution $\varphi(t)$ of the Volterra-Hammerstein integral equation (4.4) can be obtained by

$$\varphi(t) = \exp(\omega(t)) = \exp(3t).$$

5. Conclusion

In this work, we studied the new technique that converted Volterra integral equation of the first kind to well-posed Volterra integral equation. This technique is combined with a variational iteration method for solving these ill-posedness equations. Two examples have been presented; the method is very useful and reliable for many types of linear and nonlinear Volterra integral equations of the first kind.

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