Explicit Expression for a First Integral for a Class of
Two-dimensional Differential System

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Abstract. In this paper we are interested in studying the existence of a first integral and to the curves which are formed by
the trajectories of the two-dimensional differential systems of the form

\[
\begin{align*}
  x' &= P(x, y) + x \left( \lambda x \exp \left( \frac{M(x, y)}{N(x, y)} \right) + \beta y \exp \left( \frac{R(x, y)}{S(x, y)} \right) \right), \\
  y' &= Q(x, y) + y \left( \lambda x \exp \left( \frac{M(x, y)}{N(x, y)} \right) + \beta y \exp \left( \frac{R(x, y)}{S(x, y)} \right) \right),
\end{align*}
\]

where \( P(x, y), Q(x, y), M(x, y), N(x, y), R(x, y), S(x, y) \) are homogeneous polynomials of degree \( a, a, b, b, c, c \) respectively
and \( \lambda, \beta \in \mathbb{R} \). Concrete examples exhibiting the applicability of our result are introduced.

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1 Introduction

We consider two-dimensional autonomous systems of differential equations of the form

\[
\begin{align*}
  x' &= \frac{dx}{dt} = F(x(t), y(t)), \\
  y' &= \frac{dy}{dt} = G(x(t), y(t)),
\end{align*}
\]

where \( F(x, y) \) and \( G(x, y) \) are real functions. In the qualitative theory of planar dynamical systems [1, 9, 10, 16],
one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [15]. There is a
huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability
and rare are papers concerned by giving them explicitly [2, 4, 7, 13]. There exist three main open problems in the
qualitative theory of real planar differential systems, the distinction between a centre and a focus, the determination
of the number of limit cycles and their distribution, and the determination of its integrability. The importance for
searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain
polynomial or rational first integrals. One of the classical tools in the classification of all trajectories of a dynamical
system is to find first integrals. For more details about first integral see for instance [3, 5, 8, 11, 12, 14], see the

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references quoted in those articles. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1).

System (1) is integrable on an open set \( \Omega \) of \( \mathbb{R}^2 \) if there exists a non constant \( C^1 \) function \( H : \Omega \to \mathbb{R} \), called a first integral of the system on \( \Omega \), which is constant on the trajectories of the system (1) contained in \( \Omega \), i.e. if

\[
\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} f(x,y) + \frac{\partial H(x,y)}{\partial y} g(x,y) = 0 \quad \text{in the points of } \Omega.
\]

Moreover, \( H = h \) is the general solution of this equation, where \( h \) is an arbitrary constant. It is well known that for differential systems defined on the plane \( \mathbb{R}^2 \) the existence of a first integral determines their phase portrait [6].

In this paper we are interested in studying the existence of a first integral and to the curves which are formed by the trajectories of the two-dimensional differential systems of the form

\[
\begin{align*}
x' &= P(x,y) + x \left( \lambda x \exp \left( \frac{M(x,y)}{N(x,y)} \right) + \beta y \exp \left( \frac{R(x,y)}{S(x,y)} \right) \right), \\
y' &= Q(x,y) + y \left( \alpha x \exp \left( \frac{M(x,y)}{N(x,y)} \right) + \gamma y \exp \left( \frac{R(x,y)}{S(x,y)} \right) \right),
\end{align*}
\]

(2)

where \( P(x,y), Q(x,y), M(x,y), N(x,y), R(x,y), S(x,y) \) are homogeneous polynomials of degree \( a, b, c, c \) respectively and \( \lambda, \beta \in \mathbb{R} \).

We define the trigonometric functions

\[
\begin{align*}
f_1(\theta) &= \lambda (\cos \theta) \exp \left( \frac{M(\cos \theta, \sin \theta)}{N(\cos \theta, \sin \theta)} \right) + \beta (\sin \theta) \exp \left( \frac{R(\cos \theta, \sin \theta)}{S(\cos \theta, \sin \theta)} \right), \\
f_2(\theta) &= P(\cos \theta, \sin \theta) \cos \theta + Q(\cos \theta, \sin \theta) \sin \theta, \\
f_3(\theta) &= (\cos \theta) Q(\cos \theta, \sin \theta) - (\sin \theta) P(\cos \theta, \sin \theta).
\end{align*}
\]

2 Main result

Our main result on the integrability of differential system (2) is the following:

**Theorem 1.** Consider a differential system (2), then the following statements hold.

1. If \( f_3(\theta) N(\cos \theta, \sin \theta) S(\cos \theta, \sin \theta) \neq 0 \) and \( a \neq 2 \), then system (2) has the first integral

\[
H(x,y) = (x^2 + y^2)^{\frac{a-2}{2}} \exp \left( \frac{a-2}{2} \int_{0}^{\arctan \frac{y}{x}} A(\omega) d\omega \right) - (a-2) \int_{0}^{\arctan \frac{y}{x}} \exp \left( (a-2) \int_{0}^{\omega} A(\omega) d\omega \right) B(\omega) d\omega,
\]

where \( A(\theta) = \frac{f_2(\theta)}{f_3(\theta)}, B(\theta) = \frac{f_1(\theta)}{f_3(\theta)}, \) and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as

\[
x^2 + y^2 = \left( h \exp \left( (a-2) \int_{0}^{\arctan \frac{y}{x}} A(\omega) d\omega \right) + (a-2) \exp \left( (a-2) \int_{0}^{\arctan \frac{y}{x}} A(\omega) d\omega \right) \int_{0}^{\arctan \frac{y}{x}} \exp \left( (2-a) \int_{0}^{\omega} A(\omega) d\omega \right) B(\omega) d\omega \right)^{\frac{2}{a-2}},
\]

where \( h \in \mathbb{R} \).

2. If \( f_3(\theta) N(\cos \theta, \sin \theta) S(\cos \theta, \sin \theta) \neq 0 \) and \( a = 2 \), then system (2) has the first integral

\[
H(x,y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left( \frac{1}{2} \int_{0}^{\arctan \frac{y}{x}} \left( A(\omega) + B(\omega) \right) d\omega \right),
\]

and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as

\[
(x^2 + y^2)^{\frac{1}{2}} - h \exp \left( \int_{0}^{\arctan \frac{y}{x}} \left( A(\omega) + B(\omega) \right) d\omega \right) = 0,
\]

where \( h \in \mathbb{R} \).

3. If \( f_3(\theta) = 0 \) for all \( \theta \in \mathbb{R} \), then system (2) has the first integral \( H = \frac{y^2}{x^2} \), and the curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as \( y - hx = 0 \), where \( h \in \mathbb{R} \).
Proof. In order to prove our results we write the polynomial differential system (2) in polar coordinates \((r, \theta)\), defined by \(x = r \cos \theta \) and \(y = r \sin \theta\), then system (2) becomes
\[
\begin{cases}
  r' = f_1(\theta) r^2 + f_2(\theta) r^a, \\
  \theta' = f_3(\theta) r^{a-1},
\end{cases}
\]
where the trigonometric functions \(f_1(\theta), f_2(\theta), f_3(\theta)\) are given in introduction, \(r' = \frac{dr}{d \theta}\) and \(\theta' = \frac{d\theta}{d \theta}\).

If \(f_3(\theta) N(\cos \theta, \sin \theta) S(\cos \theta, \sin \theta) \neq 0\) and \(a \neq 2\).

Taking as independent variable the coordinate \(\theta\), this differential system (3) writes
\[
\frac{dr}{d \theta} = A(\theta) r + B(\theta) r^{3-a},
\]
where \(A(\theta) = \frac{f_2(\theta)}{f_3(\theta)}\) and \(B(\theta) = \frac{f_1(\theta)}{f_3(\theta)}\), which is a Bernoulli equation.

By introducing the standard change of variables \(r = r^{a-2}\) we obtain the linear equation
\[
\frac{d\rho}{d \theta} = (a - 2) (A(\theta) \rho + B(\theta)).
\]

The general solution of linear equation (5) is
\[
\rho(\theta) = \exp \left( (a - 2) \int_0^\theta A(\omega) d\omega \right)
\]
\[
\left( \mu + (a - 2) \int_0^\theta \exp \left[ (2 - a) \int_0^w A(\omega) d\omega \right] B(w) dw \right),
\]
where \(\mu \in \mathbb{R}\), which has the first integral
\[
H(x, y) = (x^2 + y^2)^{\frac{a-2}{2}} \exp \left[ (2 - a) \int_0^{\arctan \frac{2}{x}} A(\omega) d\omega \right] -
\]
\[
(a - 2) \int_0^{\arctan \frac{2}{x}} \exp \left[ (2 - a) \int_0^w A(\omega) d\omega \right] B(w) dw.
\]

Let \(\Gamma\) be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let \(h_\Gamma = H(\Gamma)\).

The curves \(H = h\) with \(h \in \mathbb{R}\), which are formed by trajectories of the differential system (2), in cartesian coordinates are written as
\[
x^2 + y^2 = h \exp \left( (a - 2) \int_0^{\arctan \frac{2}{x}} A(\omega) d\omega \right) +
\]
\[
(a - 2) \int_0^{\arctan \frac{2}{x}} \exp \left( (2 - a) \int_0^w A(\omega) d\omega \right) B(w) dw.
\]

where \(h \in \mathbb{R}\).

Hence statement (1) of Theorem 1 is proved.
Suppose now that \(f_3(\theta) N(\cos \theta, \sin \theta) S(\cos \theta, \sin \theta) \neq 0\) and \(a = 2\).

Taking as independent variable the coordinate \(\theta\), this differential system (3) writes
\[
\frac{dr}{d \theta} = (A(\theta) + B(\theta)) r.
\]

The general solution of equation (6) is
\[
r(\theta) = \mu \exp \left( \int_0^\theta (A(\omega) + B(\omega)) d\omega \right),
\]
where $\mu \in \mathbb{R}$, which has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{3}{2}} \exp \left( -\int_{0}^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) \, d\omega \right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in cartesian coordinates are written as

$$(x^2 + y^2)^{\frac{1}{2}} - h \exp \left( \int_{0}^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) \, d\omega \right) = 0,$$

where $h \in \mathbb{R}$.

Hence statement (2) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in cartesian coordinates are written as $y = hx$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories.

This completes the proof of statement (3) of Theorem 1.

\[
\square
\]

### 3 Examples

The following examples are given to illustrate our result.

**Example 1** If we take $\lambda = 1$, $\beta = -2$, $P(x, y) = 2x - 3y$, $Q(x, y) = 3x + 2y$. $M(x, y) = x^2 + 2y^2$, $N(x, y) = x^2 + y^2$, $R(x, y) = x^4 + 3x^2y^2 + y^4$ and $S(x, y) = x^4 + 2x^2y^2 + y^4$, then system (2) reads

$$\begin{align*}
\begin{cases}
x' = 2x - 3y + x \left( x \exp \left( \frac{x^2 + 2y^2}{x^2 + y^2} \right) - 2y \exp \left( \frac{x^4 + 3x^2y^2 + y^4}{x^2 + 2x^2y^2 + y^4} \right) \right), \\
y' = 3x + 2y + y \left( x \exp \left( \frac{x^2 + 2y^2}{x^2 + y^2} \right) - 2y \exp \left( \frac{x^4 + 3x^2y^2 + y^4}{x^2 + 2x^2y^2 + y^4} \right) \right),
\end{cases}
\end{align*}$$

(7)

the differential system (7) in polar coordinates $(r, \theta)$ becomes

$$\begin{align*}
\begin{cases}
r' = \left( (\cos \theta) \exp \left( 1 + \sin^2 \theta \right) - 2 (\sin \theta) \exp \left( \frac{3}{8} - \frac{1}{8} \cos 4\theta \right) \right) r^2 + 2r, \\
\theta' = 3,
\end{cases}
\end{align*}$$

here $f_1(\theta) = (\cos \theta) \exp \left( 1 + \sin^2 \theta \right) - 2 (\sin \theta) \exp \left( \frac{3}{8} - \frac{1}{8} \cos 4\theta \right)$, $f_2(\theta) = 2$ and $f_3(\theta) = 3$, it is the case (1) of the Theorem 1.

The differential system (7) has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left( \frac{2}{3} \arctan \frac{y}{x} \right) - \int_{0}^{\arctan \frac{y}{x}} \exp \left( \frac{2}{3} w \right) B(w) \, dw,$$

where

$$B(w) = \frac{(\cos w) \exp \left( 1 + \sin^2 w \right) - 2 (\sin w) \exp \left( \frac{3}{8} - \frac{1}{8} \cos 4w \right)}{3}.$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (7), in cartesian coordinates are written as

$$x^2 + y^2 = \left[ \left( h + \int_{0}^{\arctan \frac{y}{x}} \exp \left( \frac{2}{3} w \right) B(w) \, dw \right) \exp \left( \frac{-2}{3} \arctan \frac{y}{x} \right) \right]^{-2},$$

where $h \in \mathbb{R}$.

**Example 2** If we take $\lambda = 1$, $\beta = -2$, $P(x, y) = 5x^2 + 2xy$, $Q(x, y) = -2xy + 5y^2$. $M(x, y) = x^2 + 2y^2$, $N(x, y) = x^2 + y^2$, $R(x, y) = y$ and $S(x, y) = x$, then system (2) reads

$$\begin{align*}
\begin{cases}
x' = 5x^2 + 2xy + x \left( x \exp \left( \frac{x^2 + 2y^2}{x^2 + y^2} \right) - 2y \exp \left( \frac{y}{2} \right) \right), \\
y' = -2xy + 5y^2 + y \left( x \exp \left( \frac{x^2 + 2y^2}{x^2 + y^2} \right) - 2y \exp \left( \frac{y}{2} \right) \right),
\end{cases}
\end{align*}$$

(8)
the differential system (8) in polar coordinates \((r, \theta)\) becomes

\[
\begin{cases}
    r' = (7 \cos^3 \theta + 3 \sin^3 \theta - 2 (\sin \theta) (-1 + \exp \tan \theta) + (\cos \theta) (-2 + \exp (1 + \sin^2 \theta))) r^2, \\
    \theta' = (3 \cos \theta \sin^2 \theta - 7 \cos^2 \theta \sin \theta) r,
\end{cases}
\]

then the differential system (8) has the first integral

\[
H(x, y) = (x^2 + y^2)^\frac{3}{2} \exp \left(-\frac{\arctan \frac{x}{y}}{3} (A(\omega) + B(\omega)) d\omega \right)
\]

where \(A(\omega) + B(\omega) = \frac{7 \cos^3 \omega + 3 \sin^3 \omega - 2 (\sin \omega) (-1 + \exp \tan \omega) + (\cos \omega) (-2 + \exp (1 + \sin^2 \omega))}{3 \cos \omega \sin^2 \omega - 7 \cos^2 \omega \sin \omega}\), it is the case (2) of the Theorem 1.

The curves \(H = h\) with \(h \in \mathbb{R}\), which are formed by trajectories of the differential system (8), in cartesian coordinates are written as

\[
(x^2 + y^2)^\frac{3}{2} - h \exp \left(\arctan \frac{x}{y} (A(\omega) + B(\omega)) d\omega \right) = 0,
\]

where \(h \in \mathbb{R}\).

**Example 3** If we take \(\lambda = 1, \beta = -2, P(x, y) = x^3 + xy^2, Q(x, y) = y^3 + yx^2, M(x, y) = x^2 + 2y^2, N(x, y) = x^2 + y^2, R(x, y) = x^4 + 3x^2y^2 + y^4\) and \(S(x, y) = x^4 + 2x^2y^2 + y^4\), then system (2) reads

\[
\begin{cases}
    x' = x^3 + xy^2 + x \exp \left(\frac{x^2 + 2y^2}{x^2 + y^2}\right) - 2y \exp \left(\frac{x^4 + 3x^2y^2 + y^4}{x^4 + 2x^2y^2 + y^4}\right), \\
    y' = y^3 + yx^2 + y \exp \left(\frac{x^2 + 2y^2}{x^2 + y^2}\right) - 2y \exp \left(\frac{x^4 + 3x^2y^2 + y^4}{x^4 + 2x^2y^2 + y^4}\right),
\end{cases}
\]

the differential system (9) in polar coordinates \((r, \theta)\) becomes

\[
\begin{cases}
    r' = r^3 + ((\cos \theta) \exp (1 + \sin^2 \theta) - 2 (\sin \theta) \exp \left(\frac{\theta}{\pi} - \frac{1}{2} \cos 4\theta\right)) r^2, \\
    \theta' = 0,
\end{cases}
\]

it is the case (3) of the Theorem 1, then from system (10) it follows that \(\theta' = 0\). Hence, \(\frac{\theta}{\pi}\) is a first integral of the system, then curves which are formed by the trajectories of the differential system (9), in cartesian coordinates are written as \(y - h x = 0\), where \(h \in \mathbb{R}\), since all straight lines through the origin are formed by trajectories.

### 4 Conclusion

The elementary method used in this paper seems to be fruitful to investigate planar differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories, this is one of the classical tools in the classification of all trajectories of dynamical systems.

### References


