On $x$-magnetic Surfaces Generated by Trajectory of $x$-magnetic Curves in Null Cone

Fatma ALMAZ $^1$ and Mihriban ALYAMAÇ KÜLAHCI $^2$

$^{1,2}$ Department of Mathematics, Firat University, 23119 ELAZIĞ/TÜRKİYE

$^1$fb_fat_almaz@hotmail.com, $^2$mihribankulahci@gmail.com

Abstract. In this work, we examine the impact of magnetic fields on the moving particle trajectories by variational approach to the magnetic flow associated with the Killing magnetic field on 2-dimensional lightlike cone $Q^2 \subset E^3_1$. We give some characterizations for $x$-magnetic curve and $x$-magnetic surface of rotation using the Killing magnetic field of this curve in $Q^2$ and we give the different types of axes of rotation, then creates three different types of magnetic surfaces of rotation in 2-dimensional lightlike cone $Q^2 \subset E^3_1$.

Keywords: Killing vector field, lightlike cone, magnetic curve, magnetic surface, $W_x$ trajectory.

2010 MSC No: 53B30, 53B50, 53C80

1 Introduction

Although we know much about the submanifolds of the pseudo-Riemannian space forms, we have very few papers on submanifolds of the pseudo-Riemannian lightlike cone. A simply connected Riemannian manifold of dimension $n \geq 3$ is conformally flat if and only if it can be isometrically immersed as a hypersurface of the lightlike cone and the Killing vector fields define the Killing magnetic fields. Working on the degenerate submanifolds of Lorentzian manifolds with degenerate metric provides us with meaningful relevance between null submanifolds and spacetime. Barros and Romeo proved in [2, 3], that if $(M, g^*)$ has constant curvature, then the magnetic curves corresponding to a Killing magnetic field are centerlines of Kirchhoff elastic rods. V.N. Mishra and at al studied different metrics in different ambient spaces [6, 15, 16, 18, 19, 20]. In [21], the author defined magnetic curves on a Riemannian manifold $(M, g^*)$ according trajectories of charged particles moving on $M$ under the action of a magnetic field $F_x$. In [4], Bozkurt et al. investigated the magnetic flow associated with the Killing magnetic field in a three-dimensional oriented Riemann manifold $(M^3, g^*)$. In [12], the author studied cone curves and investigate the notations of the cone curvature function and also gave some examples of cone curves in Minkowski space. In [6], they characterized the Killing vector fields on a Walker manifold $M^3_f$ aiming to obtain the corresponding Killing magnetic curves and characterized the normal magnetic curves corresponding to some Killing vector fields on $M^3_f$, obtaining their explicit expressions for certain functions $f$. In [7], the authors investigated magnetic curves corresponding to the Killing magnetic field $W$ in the 3-dimensional Minkowski space. In [8], they determined all magnetic curves corresponding to the Killing magnetic fields on the 3-dimensional Euclidean space. In [13], the author gave the representation formulas of the curves in $Q^2$ and $Q^3$ and defined the functions of the cone curves

*Corresponding author. Fatma ALMAZ $^1$ fb_fat_almaz@hotmail.com
2 Preliminaries

Let $E^3_1$ be the 3–dimensional pseudo-Euclidean space with the
\[ g^*(V,W) = (V,W) = v_1w_1 + v_2w_2 - v_3w_3 \]
for all $V = (v_1, v_2, v_3), W = (w_1, w_2, w_3) \in E^3_1$. $E^3_1$ is a flat pseudo-Riemannian manifold of signature (2,1).

Let $M$ be a submanifold of $E^3_1$. If the pseudo-Riemannian metric $g^*$ of $E^3_1$ induces a pseudo-Riemannian metric $\tilde{g}$ (respectively, a Riemannian metric, a degenerate quadratic form) on $M$, then $M$ is called a timelike (respectively, spacelike, degenerate) submanifold of $E^3_1$. The lightlike cone is defined by
\[ Q^2 = \{ x \in E^3_1 : g^*(x,x) = 0 \} \]

Let $E^3_1$ be 3–dimensional Minkowski space and $Q^2$ be the lightlike cone in $E^3_1$. A vector $W \neq 0$ in $E^3_1$ is called spacelike, timelike or lightlike, if $(W,W) > 0, (W,W) < 0$ or $(W,W) = 0$, respectively. A frame field $\{ x, y, z \}$ on $E^3_1$ is called an asymptotic orthonormal frame field, if
\[ (x,x) = (y,y) = (x,\alpha) = (y,\alpha) = 0, (x,y) = (\alpha,\alpha) = 1. \]

We assume that curve $x : I \to Q^2 \subset E^3_1$ is a regular curve in $Q^2$ for $t \in I$. In the following, we always assume that the curve is regular.

Using $x'(s) = \alpha(s)$, from an asymptotic orthonormal frame along the curve $x(s)$ and the cone Frenet formulas of $x(s)$ are given by
\[
\begin{align*}
x'(s) &= \alpha(s) \\
\alpha'(s) &= \kappa(s)(x(s) - y(s)) \\
y'(s) &= -\kappa(s)\alpha(s)
\end{align*}
\]

where the function $\kappa(s)$ is called cone curvature function of the curve $x(s)$. \[12\] The Lorentzian cross-product $\times : E^3_1 \times E^3_1 \to E^3_1$ is defined by the formula
\[ V \times W = \begin{bmatrix} i & j & -k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \]

where $V = (v_1, v_2, v_3), W = (w_1, w_2, w_3) \in E^3_1$. Here $i, j, k$ have usual meaning. We say that this product has similar algebraic properties as the cross product in $E^3$. Hence, it is skew-symmetric and $V \times W$ is orthogonal on both $V$ and $W$. The Lorentz force $\psi$ of a magnetic field $F_*$ on $Q^2$ is defined to be a skew-symmetric operator given by
\[ g^*(\psi(V),W) = F_*(V,W) \]

for all $V, W \in Q^2$. The $\alpha$–magnetic trajectories of $F_*$ are $x$ on $Q^2$ that satisfy the Lorentzian equation
\[ \nabla_{x'} x' = \psi(x'). \]

Furthermore, the mixed product of the vector fields $V, W, Z \in Q^2$ is the defined by
\[ g^*(V \times W, Z) = dv_{VW}^*(V,W,Z), \]

where $dv_{VW}$ denotes a volume on $Q^2$. If $W$ is a Killing vector in $Q^2$ and let $F_{sW} = s_{W\text{vol}_{g^*}}$ be the corresponding Killing magnetic field, here the inner product is indicated by $\iota$. Hence the equation Lorentz force of $F_{sW}$ is
\[ \psi(X) = W \times X \]

for all $X \in Q^2$. Corresponding the Lorentz equation can be written as
\[ \nabla_{x'} x' = \psi(x') = W \times x'. \]

In Minkowski space $E^3_1$, consider the Killing vector field $W = a\partial_x + b\partial_y + c\partial_z$, with $a, b, c \in \mathbb{R}$, the magnetic trajectories $x : I \to Q^2 \subset E^3_1$ determined by $W$ are solutions of the Lorentz equation
\[ x'' = W \times x', \]

\[21\]
Definition 2.1. A one-parameter group of diffeomorphisms of a manifold $M$ is a smooth map $\psi: M \times \mathbb{R} \to M$, such that $\psi_t(x) = \psi(x, t)$, where $\psi_t: M \to M$ is a diffeomorphism, $\psi_0 = \text{id}$, $\psi_{s+t} = \psi_s \circ \psi_t$.

This group is associated with a vector field $V$ given by $\frac{d}{dt} \psi_t(x) = W(x)$, and the group of diffeomorphisms is called the flow of $W$. If a one-parameter group of isometries is generated by a vector field $W$, then this vector field is called a Killing vector field.

3 $x$-magnetic Curves in the Null Cone $Q^2 \subset E_1^3$

In this section, we give some characterizations for $x$-magnetic curve and surface of rotation using the Killing magnetic field of this curve in $Q^2 \subset E_1^3$.

Definition 3.1. Let $x: I \to Q^2 \subset E_1^3$ be a spacelike curve in $Q^2$ and $F\cdot W$ be a magnetic field on $Q^2 \subset E_1^3$. We call the curve $x$ a $x$-magnetic curve if its vector field $W_x$ satisfies the Lorentz force equation $\nabla_{\alpha} x = \psi^x(x) = W_x \times x$.

Theorem 3.2. Let $x(s)$ be a unit speed spacelike $x$-magnetic curve in the $Q^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{x, \alpha, y\}$. The Lorentz force in the Frenet frame are given as follows

$$\psi^x = \begin{bmatrix} 0 & 1 & 0 \\ w_3 & 0 & -1 \\ 0 & -w_3 & 0 \end{bmatrix}$$

where $w_3$ is a function defined by $w_3 = g^*(\psi^x(\alpha), y)$.

Proof 3.3. Let $x(s)$ be a unit speed spacelike $x$-magnetic curve in the $Q^2 \subset E_1^3$ with the asymptotic orthonormal frame $\{x, \alpha, y\}$. From the definition of the magnetic curve and equation (2.1), we know that $\psi^x(x) = \overrightarrow{\alpha}$.

Furthermore, since $\psi^x(\alpha) \in \text{Span}\{x, \alpha, y\}$, we can write

$$\psi^x(\alpha) = A_1 \overrightarrow{x} + B_1 \overrightarrow{\alpha} + C_1 \overrightarrow{y}.$$ 

By using the following equalities

$$A_1 = g^*(\psi^x(\alpha), y) = w_3; B_1 = g^*(\psi^x(\alpha), \alpha) = 0; C_1 = g^*(\psi^x(x), x) = -g^*(\psi^x(x), \alpha) = -g^*(\alpha, \alpha) = -1,$$

we get

$$\psi^x(\alpha) = w_3 \overrightarrow{x} - \overrightarrow{y}.$$ 

Similarly, we can easily obtain

$$\psi^x(y) = -w_3 \overrightarrow{\alpha}.$$ 

Theorem 3.4. Let $x(s)$ be a unit speed spacelike $x$-magnetic curve in the $Q^2 \subset E_1^3$. The curve $x$ is then a $x$-magnetic trajectory of a magnetic vector field $W_x$ if and only if the vector field $W_x$ can be written along the curve $x$ as the following

$$W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}.$$


On $x$-magnetic Surfaces Generated by Trajectory of $x$-magnetic Curves...

**Proof 3.5.** Let $x(s)$ be a unit speed spacelike $x$–magnetic curve with $x$–magnetic trajectory of a magnetic field $W_x$. Using theorem 1 and definition 1, we can easily obtain that

$$W_x(s) = \overrightarrow{x(s)} + \overrightarrow{y(s)}.$$ Conversely, we assume that equation (3) holds. Then we get $\psi^x(x) = W_x \times x$. Therefore the curve $x$ is a $x$–magnetic trajectory of the magnetic vector field $W_x$.

**Theorem 3.6.** Let $x$ be a $x$–magnetic trajectory according to the Killing vector field $W_x = \overrightarrow{x} + \overrightarrow{y}$ in $Q^2 \subset E^3$. Then the curve $x$ can be expressed as follows

$$x(s) = x(0) + cW_x,$$ where $c \in \mathbb{R}_0$.

**Proof 3.7.** We prove the theorem according to the $W_x$. Since $W_x$ and $x(0)$ are linearly independent and $W_x$ is spacelike. We consider $W_x, W_x \times x(0)$ and $W^*$ be linearly independent and satisfy

$$(W_x, W^*) = 0, (W_x, W_x \times x(0)) = 0, (W^*, W_x \times x(0)) = 0.$$ So, we can take

$$W^* = 2x(0) - (W_x, x(0))W_x.$$ We can write

$$x(s) = x(0) + \lambda_x(s)W_x + \mu_x(s)W_x \times x(0) + \rho_x(s)W^*,$$

$$x'(s) = x'(0) + \lambda'_x(s)W_x + \mu'_x(s)W_x \times x(0) + \rho'_x(s)W^*,$$

where $\lambda_x(s), \mu_x(s), \rho_x(s), \lambda'_x(s), \mu'_x(s), \rho'_x(s)$ functions satisfied the following

$$\lambda_x(0) = 0, \mu_x(0) = 0, \rho_x(0) = 0, \lambda'_x(0) = 0, \mu'_x(0) = 0, \rho'_x(0) = 0,$$ where $s = 0$. The Lorentz equation $x'(s) = W_x \times x(s)$ can be written as

$$x'(0) + \lambda'_x(s)W_x + \mu'_x(s)W_x \times x(0) + \rho'_x(s)W^* = W_x \times x(0) - \mu_x(s)W^* - \rho_x(s)W_x \times x(0),$$ since $x'(0) = W_x \times x(0)$ for $s = 0$, we have

$$0 = \lambda'_x(s)W_x + (\mu'_x(s) + \rho_x(s))W_x \times x(0) + (\mu_x(s) + \rho'_x(s))W^*,$$

which is equivalent to

$$\lambda'_x(s) = 0, \mu'_x(s) + \rho_x(s) = 0, \mu_x(s) + \rho'_x(s) = 0.$$ Solving the previous differential equations and using the initial conditions (3.4), we get

$$\lambda_x(s) = c, \mu_x(s) = 0, \rho_x(s) = 0,$$

where $c \in \mathbb{R}_0$. Hence the curve $x$ is written as $x_x(s) = x(0) + cW_x$.

### 4 Surface of Rotation Generated by The Lorentz force $\psi^x$ with $x$–magnetic curve

**Theorem 4.1.** Let $x$ be a $x$–magnetic trajectory according to the killing vector field $W_x = \frac{\overrightarrow{x}}{w_3}(\overrightarrow{x} + \overrightarrow{y})$ in $Q^2 \subset E^3$. Then the following statements holds i) $\Delta^x(s, t)$ surface of rotation generated by the $\psi^x$ can be expressed as follow

$$\Delta^x(s, t) = \left(\frac{1 + \cosh(\sqrt{2w_3}s)}{2\sqrt{w_3}} + \frac{1 - \cosh(\sqrt{2w_3}s)}{2w_3}\sin(\sqrt{w_3}s), \frac{w_3 - 1}{\sqrt{w_3}}\sin(\sqrt{w_3}s), \frac{w_3(1 - \cosh(\sqrt{2w_3}s))}{2\sqrt{w_3}} + \frac{1 + \cosh(\sqrt{2w_3}s)}{2}(t + b)\right).$$
and the surface of revolution $\Delta^s_1$ with $x_1(t) = (t + b, 0, t + b), b \in \mathbb{R}_0$ and $-\infty < s < \infty, t \in I \subset \mathbb{R}$.  

i) If $w_3 = \text{constant}$, the surface of revolution $\Delta^s_1$ with $x_1(t)$ is minimal surface and the Gaussian and mean curvatures are $K, H = 0$. 

ii) If $w_3 \neq \text{constant}$ for $-\infty < s < \infty, t \in I \subset \mathbb{R}$ and for $s, t = 0$, the Gaussian and mean curvatures of the surface of revolution $\Delta^s_1$ with $x_1(t)$ are 

$$
K = b\sqrt{2}\left(\frac{w_3^2(w_3 - 1) - 2w_3\left(\frac{w_3 - 1}{\sqrt{w_3}}\right)^{\prime}\sqrt{w_3}}{(2(w_3 - 1)^2 - w_3'^2b^2)(1 - w_3^2) - (w_3w_3'b)^2}\right)
$$

$$
H = \frac{-\sqrt{2}}{2}\left(\frac{w_3'(w_3 - 1)(1 - w_3^2)}{(2(w_3 - 1)^2 - w_3'^2b^2)(1 - w_3^2) - (w_3w_3'b)^2} + b\left(\frac{-2\left(\frac{w_3 - 1}{\sqrt{w_3}}\right)\sqrt{w_3w_3'} + w_3'(w_3 - 1)}{w_3'^2b^2}\right)\right)
$$

iii) The condition of minimal surface of revolution $\Delta^s_2$ with $x_2(t)$ is 

$$
w_3''b(w_3 - 1)(w_3'^2b^2 - 2(w_3 - 1)^2w_3) + bw_3'^2(2(w_3 - 1)^2 - w_3'^2b^2)
$$

$$= w_3'^2w_3(w_3 - 1)(1 - w_3^2),$$

for $s, t = 0$.

ii) $\Delta^s_2(s, t)$ surface of rotation generated by the $\psi^s$ can be expressed as follow 

$$
\Delta^s_2(s, t) = \left(\begin{array}{c}
\sinh(2\sqrt{w_3}s)(t+b) \\
\cosh(2\sqrt{w_3}s)(t+b) \\
-\sqrt{2}\sqrt{w_3}
\end{array}\right),
$$

where $\kappa$ is the curvature of the curve $x$ and $x_2(t) = (t + b, 0, t + b), b \in \mathbb{R}_0$ and $-\infty < s < \infty, t \in I \subset \mathbb{R}$.

ii) the Gaussian and mean curvatures are given by 

$$
K = -\left(\frac{(t+b)^2w_3^3\cosh^4(\sqrt{w_3}s)}{4w_3^2}\right) - \frac{(t+b)^2\sqrt{w_3}\cosh^4(\sqrt{w_3}s)(\sinh(2\sqrt{w_3}s))(\sinh(2\sqrt{w_3}s))}{(w_3 - 2w_3^2 + 1 - 6w_3\sqrt{\sinh(2\sqrt{w_3}s)})},
$$

where $w_3 = \text{constant}$ and 

$$
\xi = (t + b)^2 \left(\begin{array}{c}
\left(\frac{1}{2} - w_3\right)\cosh^2(\sqrt{w_3}s) \\
+ \frac{w_3}{2}\sinh^2(\sqrt{w_3}s)
\end{array}\right) - \frac{(t+b)^2\sqrt{w_3}}{4}\left(\begin{array}{c}
\left(\frac{1}{2} - w_3\right)\sinh^2(\sqrt{w_3}s) + \frac{\cosh(\sqrt{w_3}s) - 3}{4}\right) \\
\left(\frac{1}{2} - w_3 + \frac{1}{2}\right)\sinh^2(\sqrt{w_3}s)
\end{array}\right).
$$

ii) The condition of minimal surface of revolution $\Delta^s_2$ with $x_2(t)$ is 

$$
\cosh(\sqrt{w_3}s) = \frac{3w_3}{1 + w_3 - 2w_3^2}.
$$

**Proof 4.2.** we look for a one parameter group of Lorentz of transformation which fixed all points on the xy-plane. This requires the Killing vector field to satisfy $W_{xy}(x) = x_1(x, y)(x, y)$. Hence, we can use a $3 \times 3$ matrix $\psi^s$. Now, we have the one parameter group of homomorphism $\psi^s(x, y)$ given by $\psi^s_t(x) = \psi^s_t(x)$. So, we obtain $\psi^s_t(x) = e^{s\psi^s_t}$ and calculating the matrix exponential, we have 

$$
\mathbf{H}^{s} = \begin{bmatrix}
\frac{1 + \cosh(\sqrt{w_3}s)}{2} & \frac{\sinh(\sqrt{w_3}s)}{2w_3} & \frac{1 - \cosh(\sqrt{w_3}s)}{2w_3} \\
\frac{\sqrt{w_3}\sinh(\sqrt{w_3}s)}{w_3(1 - \cosh(\sqrt{w_3}s))} & \frac{\sqrt{w_3}\sin(\sqrt{w_3}s)}{w_3} & \frac{-\sinh(\sqrt{w_3}s)}{2w_3} \\
\frac{\sqrt{w_3}\sinh(\sqrt{w_3}s)}{w_3} & \frac{-\sqrt{w_3}\sinh(\sqrt{w_3}s)}{w_3} & \frac{1 + \cosh(\sqrt{w_3}s)}{2}
\end{bmatrix}
$$
where $-\infty < s < \infty$. By the Lorentz force $\psi^x$ rotation for $x$--magnetic curve we mean; rotating a curve using the rotation matrix $(3.1)$, the plane of rotation is given by $W_z(s) = x(s) + y(s)$. Clearly, any point in $Q^2$ can be carried to the $xy$--plane by some notation, so we assume that the $x$ curve lies in $xy$--plane. Hence, one of its parametrations is

$$x_1(t) = (t + b, 0, t + b), b \in \mathbb{R}_0.$$

Hence, the surface of revolution $w_z \Delta^2_t$ around $W_z$ can be parametrized as follow:

$$\Delta^2_t(s, t) = \Pi^2(s) \times \begin{bmatrix} t + b \\ 0 \\ t + b \end{bmatrix}$$

$$\Delta^2_t(s, t) = \left( \begin{array}{c}
\left( \frac{1 + \cosh(\sqrt{w_3} s)}{2} \right) + \frac{1 - \cosh(\sqrt{w_3} s)}{2}(t + b), \\
\frac{w_3(1 - \cosh(\sqrt{w_3} s))}{2} + \frac{1 + \cosh(\sqrt{w_3} s)}{2}(t + b),
\end{array} \right).$$

where $-\infty < s < \infty, t \in I$. We examined the surface of revolution $\Delta^2_t$ for $s \neq 0$. For the gaussian and mean curvatures, we have

$$E = (t + b)^2(1 - w_3)^2 \sinh^2(\sqrt{2w_3} s)(\frac{1}{4w_3} - \frac{w_3}{2})$$

$$+ (1 - w_3)^2 \cosh^2(\sqrt{w_3} s)$$

$$F = \left( \phi^2 \frac{(w_3 - 1) \sqrt{2w_3}}{2} \left( 1 - \frac{1}{w_3} \right) \sinh(\sqrt{2w_3} s)(t + b) \right)$$

$$G = \phi^2 \frac{(w_3 - 1)}{2}; N = 0, M = 0, L = 0;$$

$$n_{A^2_t} = \left( -(w_3 - 1) \cos(\sqrt{w_3} s) \phi, 0, - (w_3 - 1) \cos(\sqrt{w_3} s) \frac{\phi}{\sqrt{w_3}} \right)$$

where $(\frac{w_3 + 1}{2})^2 + \frac{(w_3 - 1)}{2} \cosh(\sqrt{2w_3} s) = \phi$.

Thus, this results in the first fundamental form is $L_{A^2_t} = EG - F^2 \neq 0$, for $s \neq 0, K, H = 0$. Hence, we say that the surface of revolution $\Delta^2_t$ with $x_1(t)$ is minimal surface.

For $s, t = 0$ and the function $w_3(s)$, we have

$$E = 2(1 - w_3)^2 \phi^2 - b^2 \phi^2,$$

$$F = -bw_3 w_3^3; G = (1 - w_3^2),$$

$$N = b \sqrt{2} \left( -2w_3 \sqrt{w_3} \left( \frac{w_3 - 1}{\sqrt{w_3}} \right) + w_3^3 (w_3 - 1) \right),$$

$$M = 0; \ L = \sqrt{2}(w_3 - 1) w_3^3;$$

$$n_{A^2_t} = \left( -\sqrt{2}w_3 (w_3 - 1), -w_3^3, -\sqrt{2} (w_3 - 1) \right),$$

using the previous equations, we can write

$i_1)$ If $w_3$ is constant, since the Gaussian and mean curvatures are $K, H = 0$, the surface of revolution $\Delta^2_t$ with $x_1(t)$ is minimal surface.

$i_2)$ If $w_3 \neq$ constant, for $s, t = 0$ the Gaussian and mean curvatures of the surface of revolution $\Delta^2_t$ with $x_1(t)$ are

$$K = b \sqrt{2} \left( \frac{w_3^3 (w_3 - 1) - 2w_3 \left( \frac{w_3 - 1}{\sqrt{w_3}} \right) \sqrt{w_3}}{(2(w_3 - 1)^2 - w_3^2 b^2)(1 - w_3^2) - (w_3 w_3^3 b)^2} \right)$$

$$H = \left( \frac{2(w_3 - 1)^2 - w_3^2 b^2}{2(w_3 - 1)^2 - w_3^2 b^2} \right) \left( \frac{w_3^3 (w_3 - 1)(1 - w_3^2)}{2} + b \left( \frac{2(w_3 - 1)^2 - w_3^2 b^2}{2(w_3 - 1)^2 - w_3^2 b^2} \right) \right).$$
The condition of minimal surface of revolution $\Delta_1^x$ with $x_1(t)$ is

$$w_3^2 b(w_3 - 1) (w_3^2 b^2 - 2(w_3 - 1)^2 w_3) + b w_3^2 (2(w_3 - 1)^2 - w_3^2 b^2) = w_3^2 w_3 (w_3 - 1)(1 - w_3^2).$$

Let be $x$ curve as $x_2(t) = (0, t + b, 0), b \in \mathbb{R}_0$. Hence, using the axis of rotation $x_2(t)$, the surface of revolution $W_x \Delta_2^x$ around $W_x$ can be parametrized as follow:

$$\Delta_2^x(s, t) = \Pi^x(s) \times \begin{bmatrix} t + b \\ 0 \end{bmatrix},$$

$$\Delta_2^x(s, t) = \left( \frac{\sinh(\sqrt{2w_3})(t+b)}{3\cosh(\sqrt{2w_3})(t+b)} - \sqrt{(t+b)\sqrt{2w_3}} \sinh(\sqrt{2w_3}) \right).$$

For the gaussian and mean curvatures, we have

$$E = (t + b)^2 \left( \frac{1}{2} - w_3^2 \right) \cosh^2(\sqrt{2w_3}) + \frac{w_3}{2} \sinh^2(\sqrt{2w_3}),$$

$$F = \frac{(t + b) \sqrt{2w_3}}{4} \left( \frac{1}{2w_3} - w_3 + \frac{1}{2} \right) \sinh(2\sqrt{2w_3}) - 3 \sinh(\sqrt{2w_3}),$$

$$G = \frac{1}{4} \left( \frac{1}{w_3} - 2w_3 \right) \sinh^2(\sqrt{2w_3}) + \frac{(\cosh(\sqrt{2w_3}) - 3)^2}{4},$$

$$M = -\frac{\sqrt{2w_3}(t+b)}{2} \cosh^2(\sqrt{2w_3}), N = 0, L = 0$$

$$n_{\Delta_2^x} = \left( \frac{w_3(t+b)}{2} (1 - 3 \cosh(\sqrt{2w_3})), 0, -\frac{\sqrt{2(t+b)}}{4} (1 - 3 \cosh(\sqrt{2w_3})) \right)$$

this results in the first fundamental form and the second fundamental form are given as

$$I_{\Delta_2^x} = \xi; II_{\Delta_2^x} = -\frac{w_3^2(t+b)^2}{2} \cosh^4(\sqrt{2w_3}).$$

Therefore, the Gaussian and mean curvatures are given by

$$K = \frac{-(t + b)^2 w_3^2 \cosh^4(\sqrt{2w_3})}{2\xi},$$

$$H = \frac{-\left( \frac{(t + b) \sqrt{2w_3}}{4} \cosh^2(\sqrt{2w_3})(\sinh(2\sqrt{2w_3})) \right)}{w_3 - 2w_3^2 + 1 - 6w_3 \cosh(\sqrt{2w_3})},$$

where

$$\xi = (t + b)^2 \left( \frac{1}{2} - w_3^2 \right) \cosh^2(\sqrt{2w_3}) + \frac{w_3}{2} \sinh^2(\sqrt{2w_3}),$$

$$\cdot \left( \frac{1}{w_3} - 2w_3 \right) \sinh^2(\sqrt{2w_3}) + \frac{(\cosh(\sqrt{2w_3}) - 3)^2}{4} - \left( \frac{(t+b) \sqrt{2w_3}}{4} \left( \frac{1}{2w_3} - w_3 + \frac{1}{2} \right) \sinh(2\sqrt{2w_3}) \right)^2. $$

Hence, we say that the surface of revolution $\Delta_2^x$ with $x_2(t)$ is minimal surface for $\cosh(\sqrt{2w_3}) = \frac{3w_3}{1 + w_3 - 2w_3^2}$. 
On $x$-magnetic Surfaces Generated by Trajectory of $x$-magnetic Curves ...

![Image of surfaces $\Delta_f^x$ and $\Delta_f^e$](image)

Figure 1: Graphics of surfaces $\Delta_f^x$ and $\Delta_f^e$

References


